Fractional Laplacian with supercritical killings

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Abstract

In this paper, we study Feynman-Kac semigroups of symmetric α-stable processes with supercritical killing potentials belonging to a large class of functions containing functions of the form $b|x|^{-\beta}$, where $b > 0$ and $\beta > \alpha$. We obtain two-sided estimates on the densities $p(t, x, y)$ of these semigroups for all $t > 0$, along with estimates for the corresponding Green functions.

Keywords: fractional Laplacian, symmetric stable process, Feynman-Kac semigroup, heat kernel, Green function

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1 Introduction

Suppose $Y = \{Y_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d\}$ is a Markov process on $\mathbb{R}^d$ and κ is a Borel function on $\mathbb{R}^d$. Under some conditions on κ, the operators

$$P_t^\kappa f(x) = \mathbb{E}_x \left[\exp\left(-\int_0^t \kappa(Y_s)ds \right) f(Y_t) \right], \quad t \geq 0,$$

form a semigroup. The semigroup $(P_t^\kappa)_{t\geq0}$ is called a Feynman-Kac semigroup or a Schrödinger semigroup, with potential κ.

If L is the generator of Y, then the generator of $(P_t^\kappa)_{t\geq0}$ is $L - \kappa$. Feynman-Kac semigroups are very important in probability, statistical physics and other fields. They have been intensively studied, see, for instance, [17,27] and the references therein for early contributions, and [11,12,14,16] and the references therein for recent contributions. If κ is a non-negative function, then $(P_t^\kappa)_{t\geq0}$ is a (sub)-Markov semigroup, and so there is a Markov process associated with it. In this case, we say that κ is a killing potential.

A function κ is said to belong to the Kato class of Y if

$$\lim_{t\to0} \sup_{x\in\mathbb{R}^d} \mathbb{E}_x \int_0^t |\kappa(Y_s)|ds = 0.$$

For equivalent analytic characterizations of the Kato class of Brownian motion, we refer to [17,27]. A potential belonging to the Kato class is small, in the sense of quadratic forms, compared to the generator L of Y, so $L - \kappa$ can be regarded as a small (or subcritical) perturbation of L. For various Hunt processes, it is well known that Kato class perturbations preserve short time heat

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kernel estimates, see, for instance, [5, 18, 29] for the cases of Gaussian heat kernel estimates, [28, 31] for α-stable-like heat kernel estimates, and [21] for more general heat kernel estimates.

In this paper, we will concentrate on Feynman-Kac semigroups with killing potentials belonging to a certain class, containing functions of the form $\kappa(x) = b|x|^{-\beta}$ with $b, \beta > 0$ as examples, in the case when Y is an isotropic α-stable process (with generator $-(-\Delta)^{\alpha/2}$), $\alpha \in (0, 2)$. Potentials of the form $\kappa(x) = b|x|^{-\beta}$, particularly the case $\beta = \alpha$, are very important and have been studied intensively, see, for instance, [1, 3, 4, 6, 7, 23] and the references therein.

In the remainder of this paper, we always assume that Y is an isotropic α-stable process with generator $-(-\Delta)^{\alpha/2}$. The Dirichlet form of Y is $(\mathcal{E}, D(\mathcal{E}))$, where

$$D(\mathcal{E}) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dxdy < \infty \right\}$$

and

$$\mathcal{E}(u, u) = \frac{1}{2} \mathcal{A}(d, -\alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dxdy, \quad u \in D(\mathcal{E})$$

with

$$\mathcal{A}(d, -\alpha) = \frac{\alpha 2^{\alpha-1} \Gamma((d + \alpha)/2)}{\pi^{d/2} \Gamma(1 - \alpha)}.$$

The process Y has a transition density $q(t, x, y)$ such that

$$C^{-1} \widetilde{q}(t, x, y) \le q(t, x, y) \le C \widetilde{q}(t, x, y), \qquad (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \tag{1.1}$$

for some $C = C(d, \alpha) > 1$, where

$$\widetilde{q}(t, x, y) := t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x - y|} \right)^{d+\alpha}. \tag{1.2}$$

When $\beta \in (0, \alpha)$, the function $\kappa(x) = b|x|^{-\beta}$ with $b \in \mathbb{R}$ belongs to the Kato class of Y. In this case, the corresponding Feynman-Kac semigroup $(P^\kappa_t)_{t \ge 0}$ has a density $p^\kappa(t, x, y)$ and, for small t, $p^\kappa(t, x, y)$ is comparable with $q(t, x, y)$. More precisely, for any $T > 0$, there exists a constant $C = C(d, \alpha, b, \beta, T) > 1$ such that

$$C^{-1} \widetilde{q}(t, x, y) \le q(t, x, y) \le C \widetilde{q}(t, x, y), \qquad (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

See [28, Theorem 3.4].

The function $\kappa(x) = b|x|^{-\alpha}$ does not belong to the Kato class of Y. According to the fractional Hardy inequality, the quadratic form below

$$\mathcal{E}(u, u) + b \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^\alpha} dx, \quad u \in C_c^\infty(\mathbb{R}^d)$$

is non-negative definite if and only if

$$b \ge -b^*_{d,\alpha} \quad \text{where} \quad b^*_{d,\alpha} := \frac{2^\alpha \Gamma((d + \alpha)/4))^2}{\Gamma((d - \alpha)/4))^2}.$$

In this sense, $\kappa(x) = b|x|^{-\alpha}$ is a critical potential. In this case, the effective state space of the Markov process associated with $(P_t^\kappa)_{t\ge 0}$ is $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$. When $d > \alpha$ and $b \ge -b^*_{d,\alpha}$, according to [7, Theorem 1.1], [16, Theorem 3.9] and [20, Theorem 1.1], the corresponding Feynman-Kac semigroup $(P_t^\kappa)_{t\ge 0}$ has a density $p^\kappa(t,x,y)$ satisfying the following estimates: there exists $C = C(d,\alpha,b,\alpha) > 1$ such that for all $(t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$,

$$C^{-1}\left(1\wedge\frac{|x|}{t^{1/\alpha}}\right)^\delta\left(1\wedge\frac{|y|}{t^{1/\alpha}}\right)^\delta\widetilde{q}(t,x,y) \le p^\kappa(t,x,y) \le C\left(1\wedge\frac{|x|}{t^{1/\alpha}}\right)^\delta\left(1\wedge\frac{|y|}{t^{1/\alpha}}\right)^\delta\widetilde{q}(t,x,y),$$

where $\delta \in [-(d-\alpha)/2, \alpha)$ is uniquely determined by the one-to-one map

$$\delta \mapsto -\frac{2^\alpha\Gamma((\alpha-\delta)/2)\,\Gamma((d+\delta)/2)}{\Gamma(-\delta/2)\,\Gamma((d+\delta-\alpha)/2)} = b$$

from $[-(d-\alpha)/2,\alpha)$ to $[-b^*_{d,\alpha},\infty)$.

When $\beta > \alpha$, the function $\kappa(x) = b|x|^{-\beta}$ is said to be supercritical. Note that if $b < 0$, the quadratic form

$$\mathcal{E}(u,u) + b\int_{\mathbb{R}^d}\frac{u^2(x)}{|x|^\beta}dx, \quad u \in C_c^\infty(\mathbb{R}^d)$$

is not non-negative definite. Thus, we will concentrate on the case $b > 0$. The purpose of this paper is to obtain two-sided estimates on the density of the corresponding Feynman-Kac semigroup In fact, we will deal with a general class of supercritical potentials which includes $\kappa(x) = b|x|^{-\beta}$, with $\beta > \alpha$ and $b > 0$, as examples. We state here our main result in the particular case of $\kappa(x) = b|x|^{-\beta}$ with $\beta > \alpha$ and $b > 0$. We will use $p^{\beta,b}(t,x,y)$ to denote the density of the corresponding Feynman-Kac semigroup and use $G^{\beta,b}(x,y)$ to denote the corresponding Green function defined as $G^{\beta,b}(x,y) = \int_0^\infty p^{\beta,b}(t,x,y)\,dt$. For $a,b\in\mathbb{R}$, we use the usual notation $a\wedge b := \min\{a,b\}$ and $a\vee b := \max\{a,b\}$. For two non-negative functions f and g, the notation $f \asymp g$ means that there exists a constant $c > 1$ such that $c^{-1}g(x) \le f(x) \le cg(x)$ in the common domain of definition of f and g. In this paper, we will use the following notation

$$\operatorname{Log} r := \log(e-1+r), \quad r \ge 0. \tag{1.3}$$

Theorem 1.1. *(i)* **(Small time estimates)** *Let $T > 0$. There exist constants $\lambda_1 = \lambda_1(\beta,b) > 0$, $\lambda_2 = \lambda_2(\beta,b) > 0$ and $C = C(d,\alpha,\beta,b,T) > 1$ such that for all $t \in (0,T]$ and $x,y \in \mathbb{R}^d_0$,*

$$\begin{aligned} p^{\beta,b}(t,x,y) \le{}& C\left(1\wedge\frac{|x|^\beta}{t}\right)\left(1\wedge\frac{|y|^\beta}{t}\right) \\ &\times\left[e^{-\lambda_1 t/(|x|\vee|y|)^\beta}t^{-d/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} + t^{-(d+2\alpha-2\beta)/\beta}\left(1\wedge\frac{t^{1/\beta}}{|x-y|}\right)^{d+2\alpha}\right] \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} p^{\beta,b}(t,x,y) \ge{}& C^{-1}\left(1\wedge\frac{|x|^\beta}{t}\right)\left(1\wedge\frac{|y|^\beta}{t}\right) \\ &\times\left[e^{-\lambda_2 t/(|x|\vee|y|)^\beta}t^{-d/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} + t^{-(d+2\alpha-2\beta)/\beta}\left(1\wedge\frac{t^{1/\beta}}{|x-y|}\right)^{d+2\alpha}\right]. \end{aligned} \tag{1.5}$$

(ii) **(Large time estimates)** *There exist comparison constants depending only on d, α, β and b such that the following estimates hold for all $t \in [2, \infty)$ and $x, y \in \mathbb{R}^d_0$: (1) If $d > \alpha$, then*

$$p^{\beta,b}(t,x,y) \asymp (1 \wedge |x|^\beta)(1 \wedge |y|^\beta)\, t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha};$$

(2) if $d = 1 < \alpha$, then

$$p^{\beta,b}(t,x,y) \asymp \left(1 \wedge \frac{|x|^\beta \wedge |x|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right)\left(1 \wedge \frac{|y|^\beta \wedge |y|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right) t^{-1/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{1+\alpha};$$

and (3) if $d = 1 = \alpha$, then

$$p^{\beta,b}(t,x,y) \asymp \left(1 \wedge \frac{|x|^\beta \wedge \mathrm{Log}\,|x|}{\mathrm{Log}\, t}\right)\left(1 \wedge \frac{|y|^\beta \wedge \mathrm{Log}\,|y|}{\mathrm{Log}\, t}\right) t^{-1} \left(1 \wedge \frac{t}{|x-y|}\right)^{2}.$$

(iii) **(Green function estimate)** *There exist comparison constants depending only on d, α, β and b such that for all $x, y \in \mathbb{R}^d_0$, (1) if $d > \alpha$, then*

$$G^{\beta,b}(x,y) \asymp \left(1 \wedge \frac{(|x| \wedge 1)^\beta}{(|x-y| \wedge 1)^\alpha}\right)\left(1 \wedge \frac{(|y| \wedge 1)^\beta}{(|x-y| \wedge 1)^\alpha}\right) \frac{1}{|x-y|^{d-\alpha}};$$

(2) if $d = 1 < \alpha$, then

$$G^{\beta,b}(x,y) \asymp \left(1 \wedge \frac{(|x| \wedge 1)^\beta}{(|x-y| \wedge 1)^\alpha}\right)\left(1 \wedge \frac{(|y| \wedge 1)^\beta}{(|x-y| \wedge 1)^\alpha}\right)\left(1 \wedge \frac{|x| \vee 1}{|x-y| \vee 1}\right)^{\alpha-1}\left(1 \wedge \frac{|y| \vee 1}{|x-y| \vee 1}\right)^{\alpha-1}$$
$$\times \left((|x|^\beta \wedge |x|^\alpha) \vee |x-y|^\alpha\right)^{(\alpha-1)/(2\alpha)}\left((|y|^\beta \wedge |y|^\alpha) \vee |x-y|^\alpha\right)^{(\alpha-1)/(2\alpha)};$$

and (3) if $d = 1 = \alpha$, then

$$G^{\beta,b}(x,y) \asymp \left(1 \wedge \frac{(|x| \wedge 1)^\beta}{|x-y| \wedge 1}\right)\left(1 \wedge \frac{(|y| \wedge 1)^\beta}{|x-y| \wedge 1}\right)\left(1 \wedge \frac{\mathrm{Log}\,|x|}{\mathrm{Log}\,|x-y|}\right)^{1/2}\left(1 \wedge \frac{\mathrm{Log}\,|y|}{\mathrm{Log}\,|x-y|}\right)^{1/2}$$
$$\times \left[\mathrm{Log}\left(\frac{|x|^\beta \wedge |x|}{|x-y| \wedge 1}\right)\mathrm{Log}\left(\frac{|y|^\beta \wedge |y|}{|x-y| \wedge 1}\right)\right]^{1/2}.$$

Remark 1.2. Both terms in the square brackets on the right hand sides of (1.4) and (1.5) are needed. In some region of $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, one term dominates and otherwise the other term dominates.

It is natural to study heat kernel estimates of Feynman-Kac semigroups of Brownian motion with supercritical killing potentials of the $\kappa(x) = b|x|^{-\beta}$ with $\beta > 2$. We believe that the corresponding heat kernel estimates are drastically different. One reason for this is that, according to [24–26], the density $p^{2,b}(t,x,y)$ of the Feynman-Kac semigroup corresponding to the generator $\Delta - b|x|^{-2}$ admits the following estimates: There exist positive constants $c_i, i = 1,2,3,4$, such that for all $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c_1 \left(1 \wedge \frac{|x|}{\sqrt{t}}\right)^\delta \left(1 \wedge \frac{|y|}{\sqrt{t}}\right)^\delta t^{-d/2} \exp\left(-c_2 \frac{|x-y|^2}{t}\right)$$

$$\leq p^{2,b}(t,x,y) \leq c_3 \left(1 \wedge \frac{|x|}{\sqrt{t}}\right)^{\delta} \left(1 \wedge \frac{|y|}{\sqrt{t}}\right)^{\delta} t^{-d/2} \exp\left(-c_4 \frac{|x-y|^2}{t}\right),$$

where $\delta = \sqrt{(d-2)^2/4 + b} - (d-2)/2$. Another reason is that, according to [23, Lemma 2.1], if $\beta > 2$, then

$$u(x) = |x|^{-\frac{d}{2}+1} K_{(d-2)/(\beta-2)} \left(\frac{2}{\beta - 2} \sqrt{b} |x|^{-\frac{\beta-2}{2}} \right)$$

is a solution of

$$\Delta u - b|x|^{-\beta} u = 0,$$

where $K_{(d-2)/(\beta-2)}$ is the modified Bessel function of the second kind. It is known that

$$\lim_{r\to\infty} K_{(d-2)/(\beta-2)}(r)/(r^{-1/2}e^{-r}) = (\pi/2)^{1/2}.$$

See [2, 9.7.2]. Using this asymptotic property of $K_{(d-2)/(\beta-2)}$, one can easily check that the function u decays exponentially at the origin. Due to the two reasons above, we believe that the density of the Feynman-Kac semigroup corresponding to the generator $\Delta - b|x|^{-\beta}$, $\beta > 2$, decays to 0 exponentially at the origin, instead of algebraically as in Theorem 1.1. Because of this, heat kernel estimates of Feynman-Kac semigroups of Brownian motion with supercritical killing potentials of the $\kappa(x) = b|x|^{-\beta}$ with $\beta > 2$ are more delicate. We intend to tackle this in a separate project.

The approach of this paper is probabilistic. The probabilistic representation of the Feynman-Kac semigroup, the Lévy system formula, and the sharp two-sided Dirichlet heat estimates for the fractional Laplacian in balls and exterior balls obtained in [8,13] play essential roles. To help us get the behavior of the heat kernel near the origin, we need to construct appropriate barrier functions, see Section 3.

Now we introduce the class of potentials that we will deal with in this paper.

For $\beta_2 \geq \beta_1 > \alpha$ and $\Lambda \geq 1$, let $\mathcal{C}_\alpha(\beta_1, \beta_2, \Lambda)$ be the family of all strictly increasing continuous functions $\psi : (0,\infty) \to (0,\infty)$ with $\psi(1) = 1$ satisfying the following property:

$$\Lambda^{-1} \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\psi(R)}{\psi(r)} \leq \Lambda \left(\frac{R}{r}\right)^{\beta_2} \quad \text{for all } 0 < r \leq R. \tag{1.6}$$

Definition 1.3. Let $\psi \in \mathcal{C}_\alpha(\beta_1, \beta_2, \Lambda)$ and κ be a non-negative Borel function on $\mathbb{R}^d_0$.
(i) We say that κ belongs to the class $\mathcal{K}^0_\alpha(\psi, \Lambda)$ if

$$\frac{\Lambda^{-1}}{\psi(|x|)} \leq \kappa(x) \leq \frac{\Lambda}{\psi(|x|)} \quad \text{for all } x \in \mathbb{R}^d_0 \text{ with } |x| \leq 1 \tag{1.7}$$

and

$$\sup_{x \in \mathbb{R}^d \setminus B(0,1)} \kappa(x) \leq \Lambda. \tag{1.8}$$

(ii) We say that κ belongs to the class $\mathcal{K}_\alpha(\psi, \Lambda)$ if (1.7) holds and

$$\kappa(x) \leq \frac{\Lambda}{\psi(|x|)} \quad \text{for all } x \in \mathbb{R}^d_0 \text{ with } |x| \geq 1. \tag{1.9}$$

The inclusion $\mathcal{K}_\alpha(\psi,\Lambda)\subset\mathcal{K}^0_\alpha(\psi,\Lambda)$ is obvious.

In this paper, we consider heat kernel estimates for non-local operators of the form

$$L^\kappa_\alpha := -(-\Delta)^{\alpha/2}-\kappa(x)$$

when $\kappa\in\mathcal{K}^0_\alpha(\psi,\Lambda)$ or $\kappa\in\mathcal{K}_\alpha(\psi,\Lambda)$. It is obvious that the function $\kappa(x)=b|x|^{-\beta}$, with $\beta>\alpha$ and $b>0$, belongs to $\mathcal{K}_\alpha(r^\beta, b\vee b^{-1})$.

The rest of this paper is organized as follows. In Section 2, we collect some preliminary results. In Section 3, we construct appropriate barrier functions and prove some survival probability estimates. Small time heat kernel estimates are proved in Section 4. In Section 5, we prove a key proposition needed to get large time heat kernel estimates. Large time heat kernel estimates are proved in Section 6. In Section 7, we prove the Green function estimates. In the Appendix, we prove a lemma which is used in getting the large time heat kernel upper bound in the case $d=1=\alpha$. We believe this result is of independent interest.

2 Preliminaries

In the remainder of this paper, we assume that $\beta_2\ge\beta_1>\alpha$ and $\Lambda\ge1$ are given constants, $\psi\in\mathcal{C}_\alpha(\beta_1,\beta_2,\Lambda)$ and $\kappa\in\mathcal{K}^0_\alpha(\psi,\Lambda)$.

Recall that $Y=\{Y_t, t\ge0;\mathbb{P}_x, x\in\mathbb{R}^d\}$ is an isotropic α-stable process on $\mathbb{R}^d$ with generator $-(-\Delta)^{\alpha/2}$ and the density $q(t,x,y)$ of Y satisfies (1.1) with $\widetilde{q}$ defined by (1.2). From (1.1), one sees that there exists $C=C(d,\alpha)>1$ such that for all $t,s>0$ and $x,y\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\widetilde{q}(t,x,z)dz\le C\int_{\mathbb{R}^d}q(t,x,z)dz=C \tag{2.1}$$

and

$$\int_{\mathbb{R}^d}\widetilde{q}(t,x,z)\widetilde{q}(s,z,y)dz\le C^2\int_{\mathbb{R}^d}q(t,x,z)q(s,z,y)dz=C^2q(t+s,x,y)\le C^3\widetilde{q}(t+s,x,y). \tag{2.2}$$

Note that

$$\widetilde{q}(t,x,y)\asymp\widetilde{q}(t,0,y)\quad\text{for all } t>0 \text{ and } x,y\in\mathbb{R}^d \text{ with } |x|\le 2t^{1/\alpha}. \tag{2.3}$$

Indeed, if $|y|<4t^{1/\alpha}$, then $\widetilde{q}(t,x,y)\asymp t^{-d/\alpha}\asymp\widetilde{q}(t,0,y)$ and if $|y|\ge4t^{1/\alpha}$, then $\widetilde{q}(t,x,y)=t|y-x|^{-d-\alpha}\asymp t|y|^{-d-\alpha}\asymp\widetilde{q}(t,0,y)$. We also note that

$$2^{-1}\widetilde{q}(t,x,y)\le\widetilde{q}(s,x,y)\le2^{d/\alpha}\widetilde{q}(t,x,y)\quad\text{for all } t>0,\ t/2\le s\le t \text{ and } x,y\in\mathbb{R}^d. \tag{2.4}$$

For an open subset U of $\mathbb{R}^d$, let $\tau^Y_U:=\inf\{t>0: Y_t\notin U\}$ be the first exit time from U for Y. The killed process Y^U on U is defined by $Y^U_t=Y_t$ if $t<\tau^Y_U$ and $Y^U_t=\partial$ if $t\ge\tau^Y_U$, where ∂ is the cemetery point. It is known that the process Y^U has a jointly continuous transition density $q^U:(0,\infty)\times U\times U\to(0,\infty)$ such that $q^U(t,x,y)\le q(t,x,y)$ for all $t>0$ and $x,y\in U$.

Define for $t\ge0$, $x\in\mathbb{R}^d$ and a non-negative Borel function f on $\mathbb{R}^d$,

$$P^\kappa_t f(x)=\mathbb{E}_x\left[\exp\left(-\int_0^t\kappa(Y_s)ds\right)f(Y_t)\right]. \tag{2.5}$$

Since κ is non-negative, we can extend $(P_t^\kappa)_{t\geq 0}$ to a strongly continuous Markov semigroup on $L^2(\mathbb{R}^d)$. Let X^κ be the Markov process on $\mathbb{R}^d$ associated with the semigroup $(P_t^\kappa)_{t\geq 0}$. When $d \geq \alpha$, singletons are polar, and so $\mathbb{P}_x(\tau^Y_{\mathbb{R}^d_0} = \infty) = 1$. So in this case, we can regard X^κ as a Markov process on $\mathbb{R}^d_0$. Now let us consider the case $d = 1 < \alpha$. In this case, since $\beta_1 > \alpha > 1$, by (1.7) and (1.6) (with $\psi(1) = 1$), we have for all $\delta \in (0,1)$,

$$\int_{-\delta}^{\delta} \kappa(y)dy \geq \frac{1}{\Lambda}\int_{-\delta}^{\delta} \frac{dy}{\psi(|y|)} \geq \frac{1}{\Lambda^2}\int_{-\delta}^{\delta}\frac{dy}{|y|^{\beta_1}} = \infty.$$

Hence, it follows from [32, Lemma 1.6] that

$$\mathbb{P}_0\left(\int_0^t \kappa(Y_s)ds = \infty \text{ for all } t>0\right) = 1.$$

Combining the above with the strong Markov property, we get that for any $x \in \mathbb{R}_0$,

$$\mathbb{P}_x\left(\int_0^t \kappa(Y_s)ds = \infty \text{ for all } t > \tau^Y_{\mathbb{R}^d_0}\right) = 1.$$

Thus, in both cases, it holds that for any $t > 0$ and $x \in \mathbb{R}^d_0$,

$$P_t^\kappa f(x) = \mathbb{E}_x\left[\exp\left(-\int_0^t \kappa(Y_s)ds\right) f(Y_t) : t < \tau^Y_{\mathbb{R}^d_0}\right] = \mathbb{E}_x\left[\exp\left(-\int_0^t \kappa(Y_s^{\mathbb{R}^d_0})ds\right) f(Y_t^{\mathbb{R}^d_0})\right].$$

Therefore, X^κ can be considered as a process on $\mathbb{R}^d_0$ and it can be obtained as follows: first kill the process Y upon exiting $\mathbb{R}^d_0$ and then kill the resulting process using the killing potential κ. From now on, we always regard X^κ as a process on $\mathbb{R}^d_0$.

For an open subset U of $\mathbb{R}^d_0$, we denote by $\tau_U^\kappa = \tau_U^{X^\kappa}$ the first exit time from U for X^κ and by $X^{\kappa,U}$ the killed process of X^κ on U. Let U be an open subset of $\mathbb{R}^d_0$ such that $\overline{U} \subset \mathbb{R}^d_0$. By (1.7) and (1.8), we have $\sup_{x\in U}\kappa(x) < \infty$, and therefore,

$$\lim_{t\to 0}\sup_{x\in U}\left|\int_0^t\int_U q^U(s,x,y)\kappa(y)dyds\right| \leq \lim_{t\to 0} t\sup_{y\in U}\kappa(y) = 0.$$

Hence, by general theory, the semigroup $(P_t^{\kappa,U})_{t\geq 0}$ of $X^{\kappa,U}$ can be represented by the following Feynman-Kac formula: For any $t\geq 0$, $x\in U$ and any non-negative Borel function f on U,

$$P_t^{\kappa,U} f(x) = \mathbb{E}_x\left[\exp\left(-\int_0^t \kappa(Y_s^U)ds\right) f(Y_t^U)\right]. \tag{2.6}$$

We refer to [14, Section 1.2] for more details. Define $p_0^{\kappa,U}(t,x,y) := q^U(t,x,y)$ and, for $k\geq 1$,

$$p_k^{\kappa,U}(t,x,y) = -\int_0^t\int_U q^U(s,x,z)p_{k-1}^{\kappa,U}(t-s,z,y)\kappa(z)dzds. \tag{2.7}$$

Set $p^{\kappa,U}(t,x,y) := \sum_{k=0}^\infty p_k^{\kappa,U}(t,x,y)$. According to [14, Theorem 3.4], $p^{\kappa,U}(t,x,y)$ is jointly continuous on $(0,\infty)\times U\times U$ and is the transition density for $P_t^{\kappa,U}$. Define for $t>0$ and $x,y\in\mathbb{R}^d_0$,

$$p^\kappa(t,x,y) := \lim_{n\to\infty} p^{\kappa, B(0,1/n)^c}(t,x,y).$$

Using the monotone convergence theorem, one sees that $p^\kappa(t,x,y)$ is the transition density of X^κ. Consequently, we have $p^\kappa(t,x,y)=p^\kappa(t,y,x)\le q(t,x,y)$ for all $t>0$ and $x,y\in\mathbb{R}^d_0$. Moreover, as an increasing limit of continuous functions, for every fixed $t>0$ and $z\in\mathbb{R}^d_0$, the maps $x\mapsto p^\kappa(t,x,z)$ and $y\mapsto p^\kappa(t,z,y)$ are lower semi-continuous.

2.1 Some properties of Y

The following result is well known, see [10, Theorem 5.1] for the corresponding result for Lévy-type processes.

Proposition 2.1. *There exists $C=C(d,\alpha)>0$ such that for all $x\in\mathbb{R}^d$, $t>0$ and $R>0$,*

$$\mathbb{P}_x(\tau^Y_{B(x,R)}\le t)\le CtR^{-\alpha}.$$

For an open set $E\subset\mathbb{R}^d$ and $x\in E$, we let

$$\delta_E(x):=\inf\{|x-y|:y\in E^c\}.$$

Proposition 2.2. *For any $k\ge 1$, there exist comparison constants depending only on d,α and k such that for any $z\in\mathbb{R}^d$, $R>0$, $0<t\le(kR)^\alpha$ and $x,y\in B(z,R)$,*

$$\frac{q^{B(z,R)}(t,x,y)}{\widetilde{q}(t,x,y)}\asymp\left(1\wedge\frac{\delta_{B(z,R)}(x)^{\alpha/2}}{t^{1/2}}\right)\left(1\wedge\frac{\delta_{B(z,R)}(y)^{\alpha/2}}{t^{1/2}}\right).$$

Proof. Without loss of generality, by the translation-invariance of Y, we assume that $z=0$. By the scaling property of Y, we have

$$q^{B(0,R)}(t,x,y)=R^{-d}q^{B(0,1)}(t/R^\alpha,x/R,y/R). \tag{2.8}$$

Applying [13, Thoerem 1.1(i)] with $T=k^\alpha$, since $\delta_{B(0,1)}(w/R)=\delta_{B(0,R)}(w)/R$ for all $w\in B(0,R)$, we obtain

$$\begin{aligned}
&q^{B(0,1)}(t/R^\alpha,x/R,y/R)\\
&\asymp\left(1\wedge\frac{\delta_{B(0,1)}(x/R)^{\alpha/2}}{(t/R^\alpha)^{1/2}}\right)\left(1\wedge\frac{\delta_{B(0,1)}(y/R)^{\alpha/2}}{(t/R^\alpha)^{1/2}}\right)(t/R^\alpha)^{-d/\alpha}\left(1\wedge\frac{(t/R^\alpha)^{1/\alpha}}{|x-y|/R}\right)^{d+\alpha}\\
&=R^d\left(1\wedge\frac{\delta_{B(z,R)}(x)^{\alpha/2}}{t^{1/2}}\right)\left(1\wedge\frac{\delta_{B(z,R)}(y)^{\alpha/2}}{t^{1/2}}\right)\widetilde{q}(t,x,y),
\end{aligned}$$

where the comparison constants above depend only on d,α and k. Combining this with (2.8), we get the desired result. □

As a consequence of Proposition 2.2, we obtain

Corollary 2.3. *For any $k\ge 1$, there exists $C=C(d,\alpha,k)>0$ such that for all $x\in\mathbb{R}^d$, $R>0$, $0<t\le(kR)^\alpha$ and $y\in B(x,t^{1/\alpha}/(2k))$,*

$$q^{B(x,R)}(t,x,y)\ge Ct^{-d/\alpha}.$$

Similar to Proposition 2.2, using the translation-invariance and the scaling property of Y, we get the following results from [8, Theorem 3 and Corollary 2].

Proposition 2.4. *There are comparability constants depending only on d and α such that the following estimates hold for all $z \in \mathbb{R}^d$, $R > 0$, $t > 0$ and $x, y \in B(z,R)^c$.*
(i) If $d > \alpha$, then

$$\frac{q^{B(z,R)^c}(t,x,y)}{\widetilde{q}(t,x,y)} \asymp \left(1 \wedge \frac{(\delta_{B(z,R)^c}(x) \wedge R)^{\alpha/2}}{(t \wedge R^\alpha)^{1/2}}\right)\left(1 \wedge \frac{(\delta_{B(z,R)^c}(y) \wedge R)^{\alpha/2}}{(t \wedge R^\alpha)^{1/2}}\right).$$

(ii) If $d = 1 < \alpha$, then

$$\frac{q^{B(z,R)^c}(t,x,y)}{\widetilde{q}(t,x,y)} \asymp \left(1 \wedge \frac{\delta_{B(z,R)^c}(x)^{\alpha-1}(\delta_{B(z,R)^c}(x) \wedge R)^{(2-\alpha)/2}}{t^{(\alpha-1)/\alpha}(t \wedge R^\alpha)^{(2-\alpha)/(2\alpha)}}\right) \times \left(1 \wedge \frac{\delta_{B(z,R)^c}(y)^{\alpha-1}(\delta_{B(z,R)^c}(y) \wedge R)^{(2-\alpha)/2}}{t^{(\alpha-1)/\alpha}(t \wedge R^\alpha)^{(2-\alpha)/(2\alpha)}}\right).$$

(iii) If $d = 1 = \alpha$, then

$$\frac{q^{B(z,R)^c}(t,x,y)}{\widetilde{q}(t,x,y)} \asymp \left(1 \wedge \frac{(\delta_{B(z,R)^c}(x) \wedge R)^{1/2}\,\mathrm{Log}\,(\delta_{B(z,R)^c}(x)/R)}{(t \wedge R)^{1/2}\,\mathrm{Log}\,(t/R)}\right) \times \left(1 \wedge \frac{(\delta_{B(z,R)^c}(y) \wedge R)^{1/2}\,\mathrm{Log}\,(\delta_{B(z,R)^c}(y)/R)}{(t \wedge R)^{1/2}\,\mathrm{Log}\,(t/R)}\right).$$

Proof. Since the proofs are similar, we only present the proof for (iii). Suppose that $d = 1 = \alpha$. Without loss of generality, by the translation-invariance of Y, we assume that $z = 0$. Using the scaling property of Y in the first line below and [8, Corollary 2] in the second, since $\delta_{B(0,1)^c}(w/R) = \delta_{B(0,R)^c}(w)/R$ for all $w \in B(0,R)^c$, we get that for all $R > 0$, $t > 0$ and $x, y \in B(0,R)^c$,

$$\begin{aligned} q^{B(0,R)^c}(t,x,y) &= R^{-1} q^{B(0,1)^c}(t/R, x/R, y/R) \\ &\asymp \left(1 \wedge \frac{\log(1 + (\delta_{B(0,R)^c}(x)/R)^{1/2})}{\log(1 + (t/R)^{1/2})}\right)\left(1 \wedge \frac{\log(1 + (\delta_{B(0,R)^c}(y)/R)^{1/2})}{\log(1 + (t/R)^{1/2})}\right)\widetilde{q}(t,x,y). \end{aligned} \tag{2.9}$$

Note that for all $a > 0$ and $r > 0$,

$$\begin{aligned} \log(1 + (a/r)^{1/2}) &\asymp \begin{cases} (a/r)^{1/2} & \text{if } a/r \le 1, \\ \mathrm{Log}\,(a/r) & \text{if } a/r > 1 \end{cases} \\ &\asymp r^{-1/2}(a \wedge r)^{1/2}\mathrm{Log}\,(a/r). \end{aligned}$$

Hence, for all $R > 0$, $t > 0$ and $w \in B(0,R)^c$, we have

$$1 \wedge \frac{\log(1 + (\delta_{B(0,R)^c}(w)/R)^{1/2})}{\log(1 + (t/R)^{1/2})} \asymp 1 \wedge \frac{(\delta_{B(0,R)^c}(w) \wedge R)^{1/2}\,\mathrm{Log}\,(\delta_{B(0,R)^c}(w)/R)}{(t \wedge R)^{1/2}\,\mathrm{Log}\,(t/R)}.$$

Combining this with (2.9), we arrive at the result. □

2.2 Preliminary estimates for X^κ

For $x \in \mathbb{R}^d_0$ and a Borel subset A of $\mathbb{R}^d_0 \cup \{\partial\}$, define

$$N(x, A) = \mathcal{A}(d, -\alpha) \int_{A \cap \mathbb{R}^d_0} |x - y|^{-d-\alpha} dy + \kappa(x) 1_A(\partial).$$

Then (N, t) is a Lévy system for X^κ (cf. [19, Theorem 5.3.1] and the argument on [15, p. 40]), that is, for any $x \in \mathbb{R}^d_0$, non-negative Borel function f on $\mathbb{R}^d_0 \times (\mathbb{R}^d_0 \cup \{\partial\})$ vanishing on $\{(x, x) : x \in \mathbb{R}^d_0\} \cup \{(x, \partial) : x \in \mathbb{R}^d_0\}$, and any stopping time τ,

$$\mathbb{E}_x\Big[\sum_{s \le \tau} f(X^\kappa_{s-}, X^\kappa_s)\Big] = \mathbb{E}_x\Big[\int_0^\tau \int_{\mathbb{R}^d_0} f(X^\kappa_s, y) N(X^\kappa_s, dy)\, ds\Big]. \tag{2.10}$$

Proposition 2.5. *For any $a \ge 1$, there exists $C = C(d, \alpha, \Lambda, a) > 0$ such that for all $R > 0$, $0 < r \le 1 \wedge (R/4)$ and $0 < t \le a\psi(r)$,*

$$p^{\kappa, B(0,R)\setminus B(0,r)}(t, x, y) \ge C\widetilde{q}(t, x, y) \quad \text{for all } x, y \in B(0, R/2) \setminus B(0, 2r).$$

Proof. By (1.7) and (1.8), we have $\sup_{z \in B(0,R)\setminus B(0,r)} \kappa(z) \le \Lambda/\psi(r)$. Hence, from (2.6), we see that for all $0 < t \le a\psi(r)$ and $x, y \in B(0, R/2) \setminus B(0, 2r)$,

$$p^{\kappa, B(0,R)\setminus B(0,r)}(t, x, y) \ge e^{-\Lambda t/\psi(r)} q^{B(0,R)\setminus B(0,r)}(t, x, y) \ge e^{-\Lambda a} q^{B(0,R)\setminus B(0,r)}(t, x, y). \tag{2.11}$$

Set $r_t := t^{1/\alpha}/(2\Lambda^{1/\alpha} a^{1/\alpha})$. We consider two separate cases.

Case 1: $|x - y| < r_t$. Note that by (1.6), since $\psi(1) = 1$, $r \le 1$ and $\beta_1 > \alpha$,

$$t^{1/\alpha} \le a^{1/\alpha}\psi(r)^{1/\alpha} \le \Lambda^{1/\alpha} a^{1/\alpha} \psi(1)^{1/\alpha} r^{\beta_1/\alpha} \le \Lambda^{1/\alpha} a^{1/\alpha} r.$$

Hence, since $B(x, r) \subset B(0, R) \setminus B(0, r)$, using Corollary 2.3 (with $k = \Lambda^{1/\alpha} a^{1/\alpha}$), we obtain

$$q^{B(0,R)\setminus B(0,r)}(t, x, y) \ge q^{B(x,r)}(t, x, y) \ge c_1 t^{-d/\alpha}. \tag{2.12}$$

Case 2: $|x - y| \ge r_t$. Using the strong Markov property, (2.12) and (2.10), we get that

$$\begin{aligned} q^{B(0,R)\setminus B(0,r)}(t, x, y) &\ge \mathbb{E}_x\Big[q^{B(0,R)\setminus B(0,r)}(t - \tau^Y_{B(x, r_t/2)}, Y_{\tau^Y_{B(x,r_t/2)}}, y) : \\ &\qquad \tau^Y_{B(x,r_t/2)} < t/2,\, Y_{\tau^Y_{B(x,r_t/2)}} \in B(y, (t/2)^{1/\alpha}/(4\Lambda^{1/\alpha} a^{1/\alpha}))\Big] \\ &\ge c_1 t^{-d/\alpha} \mathbb{P}_x\left(\tau^Y_{B(x,r_t/2)} < t/2,\, Y_{\tau^Y_{B(x,r_t/2)}} \in B(y, (t/2)^{1/\alpha}/(4\Lambda^{1/\alpha} a^{1/\alpha}))\right) \\ &= c_1 t^{-d/\alpha} \mathbb{E}_x\left[\int_0^{\tau^Y_{B(x,r_t/2)} \wedge (t/2)} \int_{B(y, (t/2)^{1/\alpha}/(4\Lambda^{1/\alpha} a^{1/\alpha}))} \frac{\mathcal{A}(d, -\alpha)}{|Y_s - w|^{d+\alpha}}\, dw ds\right]. \end{aligned} \tag{2.13}$$

For all $z \in B(x, r_t/2)$ and $w \in B(y, (t/2)^{1/\alpha}/(4\Lambda^{1/\alpha} a^{1/\alpha}))$, we have

$$|z - w| < |x - y| + r_t/2 + r_t/2 \le 2|x - y|. \tag{2.14}$$

Besides, by Proposition 2.1, there exists $\varepsilon = \varepsilon(d,\alpha) \in (0,1)$ such that

$$\mathbb{P}_x(\tau^Y_{B(x,r_t/2)} \geq (\varepsilon r_t/2)^\alpha) = 1 - \mathbb{P}_x(\tau^Y_{B(x,r_t/2)} < (\varepsilon r_t/2)^\alpha) \geq 1 - c_2\varepsilon^{1/\alpha} \geq 1/2.$$

It follows that

$$\begin{aligned}\mathbb{E}_x[\tau^Y_{B(x,r_t/2)} \wedge (t/2)] &\geq ((\varepsilon r_t/2)^\alpha \wedge (t/2))\,\mathbb{P}_x(\tau^Y_{B(x,r_t/2)} \geq (\varepsilon r_t/2)^\alpha) \\ &\geq 2^{-1}\left((2^{-2\alpha}\Lambda^{-1}a^{-1}\varepsilon^\alpha) \wedge 2^{-1}\right)t. \qquad (2.15)\end{aligned}$$

Using (2.14) and (2.15), we get from (2.13) that

$$\begin{aligned}q^{B(0,R)\setminus B(0,r)}(t,x,y) &\geq \frac{c_1(t/2)^{-d/\alpha}\mathbb{E}_x[\tau^Y_{B(x,r_t/2)} \wedge (t/2)]}{(2|x-y|)^{d+\alpha}} \int_{B(y,(t/2)^{1/\alpha}/(4\Lambda^{1/\alpha}a^{1/\alpha}))} dw \\ &\geq c_3 t|x-y|^{-d-\alpha}. \qquad (2.16)\end{aligned}$$

Now, combining (2.11) with (2.12) and (2.16), we arrive at the result. □

Lemma 2.6. *There exists $C = C(d,\alpha,\Lambda) > 0$ such that for all $x \in \mathbb{R}^d_0$ with $|x| \leq 1$, $0 < t \leq \psi(|x|/2)$ and $y \in B(x, t^{1/\alpha}/(2\Lambda^{1/\alpha}))$,*

$$p^{\kappa,B(x,|x|/2)}(t,x,y) \geq Ct^{-d/\alpha}.$$

Proof. By (1.7) and (1.8),

$$\sup_{z\in B(x,|x|/2)} t\kappa(z) \leq \Lambda\psi(|x|/2)/\psi(|x|/2) = \Lambda.$$

Hence, from (2.6), we obtain

$$p^{\kappa,B(x,|x|/2)}(t,x,y) \geq e^{-\Lambda}q^{B(x,|x|/2)}(t,x,y). \qquad (2.17)$$

By (1.6), since $\psi(1) = 1$, $\beta_1 > \alpha$ and $|x| \leq 1$, it holds that

$$t \leq \Lambda\psi(1)(|x|/2)^{\beta_1} \leq \Lambda(|x|/2)^\alpha.$$

Applying Corollary 2.3 (with $k = \Lambda^{1/\alpha}$), we get that $q^{B(x,|x|/2)}(t,x,y) \geq c_1 t^{-d/\alpha}$ for some constant $c_1 = c_1(d,\alpha,\Lambda) > 0$. Combining this with (2.17), we get the desired result. □

Lemma 2.7. *There exists $C = C(d,\alpha,\Lambda) \in (0,1)$ such that for all $x \in \mathbb{R}^d_0$ with $|x| \leq 1$,*

$$\mathbb{P}_x\big(\tau^\kappa_{B(x,|x|/2)} \geq \psi(|x|/2)\big) \geq C.$$

Proof. By Lemma 2.6, we obtain

$$\begin{aligned}&\mathbb{P}_x\big(\tau^\kappa_{B(x,|x|/2)} \geq \psi(|x|/2)\big) \\ &= \int_{B(x,|x|/2)} p^{\kappa,B(x,|x|/2)}(\psi(|x|/2),x,y)\,dy \geq c_1\psi(|x|/2)^{-d/\alpha}\int_{B(x,\,\psi(|x|/2)^{1/\alpha}/(2\Lambda^{1/\alpha}))} dy = c_2.\end{aligned}$$

□

Using Markov's inequality, from Lemma 2.7, we deduce the following corollary.

Corollary 2.8. *There exists $C = C(d,\alpha,\Lambda) \in (0,1)$ such that for all $x \in \mathbb{R}^d_0$ with $|x| \leq 1$,*

$$\mathbb{E}_x\big[\tau^\kappa_{B(x,|x|/2)}\big] \geq C\psi(|x|/2).$$

3 Survival probability estimates

In this section, we continue to assume that $\psi \in \mathcal{C}_\alpha(\beta_1, \beta_2, \Lambda)$ and $\kappa \in \mathcal{K}_\alpha^0(\psi, \Lambda)$, where $\beta_2 \geq \beta_1 > \alpha$ and $\Lambda \geq 1$. Define a function $H : (0, \infty) \to (0, \infty)$ by

$$H(r) = \frac{2}{r^2} \int_0^r \int_0^s \psi(u) du ds.$$

Note that H is twice differentiable on $(0, \infty)$.

Lemma 3.1. *There exists $C_0 = C_0(\beta_2, \Lambda) > 1$ such that for all $r > 0$,*

$$\begin{aligned} C_0^{-1}\psi(r) \leq H(r) \leq \psi(r), \\ |rH'(r)| \leq 2C_0 H(r), \\ |r^2 H''(r)| \leq 6C_0 H(r). \end{aligned} \tag{3.1}$$

Proof. Let $r > 0$. Since ψ is increasing, we have

$$H(r) \leq \frac{2\psi(r)}{r^2} \int_0^r \int_0^s du ds = \psi(r).$$

On the other hand, using (1.6), we get that

$$H(r) \geq \frac{2}{r^2} \int_{r/2}^r \int_{s/2}^s \psi(u) du ds \geq \frac{2\psi(r/4)}{r^2} \int_{r/2}^r \int_{s/2}^s du ds \geq \frac{3\psi(r/4)}{8} \geq \frac{3\psi(r)}{2^{2\beta_2+3}\Lambda}.$$

Hence, (3.1) holds with $C_0 = 2^{2\beta_2+3}\Lambda/3$. Observe that, since ψ is increasing,

$$\int_0^r \int_0^s \psi(u) du ds \leq r \int_0^r \psi(u) du \leq r^2 \psi(r). \tag{3.2}$$

Using this in the first and second inequalities below and (3.1) in the third, we obtain

$$|rH'(r)| = \left| \frac{2}{r} \int_0^r \psi(u) du - \frac{4}{r^2} \int_0^r \int_0^s \psi(u) du ds \right| \leq \frac{2}{r} \int_0^r \psi(u) du \leq 2\psi(r) \leq 2C_0 H(r).$$

Similarly, using (3.2) and (3.1), we also get that

$$\begin{aligned} r^2 H''(r) &= \frac{12}{r^2} \int_0^r \int_0^s \psi(u) du ds - \frac{8}{r} \int_0^r \psi(u) du + 2\psi(r) \\ &\leq \frac{4}{r} \int_0^r \psi(u) du + 2\psi(r) \leq 6\psi(r) \leq 6C_0 H(r) \end{aligned}$$

and $r^2 H''(r) \geq -8\psi(r) + 2\psi(r) \geq -6C_0 H(r)$. The proof is complete. □

Let $C_0 > 1$ be the constant in Lemma 3.1. The fractional Laplacian can be written as

$$-(-\Delta)^{\alpha/2} f(x) = \mathcal{A}(d, -\alpha)\, p.v. \int_{\mathbb{R}^d} \frac{f(y) - f(x)}{|y - x|^{d+\alpha}} dy,$$

whenever the above principal value integral makes sense. Let $A_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ denote the hypervolume of the unit sphere in $\mathbb{R}^d$. In the remainder of this paper, we let

$$\varepsilon_0 := \frac{1}{8} \wedge \Big[\frac{2-\alpha}{2^{\alpha-\beta_1+10}(1+2d)3^{\beta_1}\mathcal{A}(d,-\alpha)C_0^2\Lambda^2 A_{d-1}} \wedge \frac{\alpha}{2^{\alpha+2\beta_1+4}\mathcal{A}(d,-\alpha)C_0\Lambda^2 A_{d-1}} \wedge \frac{\beta_1-\alpha}{2^{d-\alpha+2\beta_1+4}\mathcal{A}(d,-\alpha)C_0\Lambda^2 A_{d-1}} \wedge \frac{\alpha}{2^{d-\alpha+2\beta_1+4}\mathcal{A}(d,-\alpha)C_0\Lambda^2 A_{d-1}} \Big]^{1/(\beta_1-\alpha)} \tag{3.3}$$

and

$$\delta_0 := \frac{\alpha\varepsilon_0^\alpha}{2^{d-\alpha+4}\mathcal{A}(d,-\alpha)C_0\Lambda A_{d-1}}.$$

Note that

$$\frac{\Lambda(4\varepsilon_0)^{\beta_1}}{\delta_0} = \frac{2^{d-\alpha+2\beta_1+4}\mathcal{A}(d,-\alpha)C_0\Lambda^2 A_{d-1}\varepsilon_0^{\beta_1-\alpha}}{\alpha} \le 1. \tag{3.4}$$

Further, for all $R \in (0,1]$ and $x \in B(0, 4\varepsilon_0 R)$, by (3.1), (1.6) (with $\psi(1) = 1$) and (3.4), we have

$$H(|x|) \le \psi(|x|) \le \Lambda|x|^{\beta_1} \le \delta_0 R^{\beta_1} \le \delta_0 R^\alpha. \tag{3.5}$$

For each $R \in (0,1]$, let $\phi_R : \mathbb{R}^d \to [0,\infty)$ be an element of $C^2(\mathbb{R}^d_0)$ satisfying the following properties:

(1) $\phi_R(x) = H(|x|)$ for $x \in B(0, 4\varepsilon_0 R) \setminus \{0\}$.
(2) $H(\varepsilon_0 R) \le \phi_R(x) \le \delta_0 R^\alpha$ for $x \in B(0,R) \setminus B(0, 4\varepsilon_0 R)$.
(3) $\phi_R(x) = \delta_0 R^\alpha$ for $x \in B(0, 2^{1/d}R) \setminus B(0,R)$.
(4) $0 \le \phi_R(x) \le \delta_0 R^\alpha$ for $x \in B(0, 4^{1/d}R) \setminus B(0, 2^{1/d}R)$.
(5) $\phi_R(x) = 0$ for $x \in B(0, 4^{1/d}R)^c$.

Lemma 3.2. *For all $R \in (0,1]$ and $x \in \mathbb{R}^d_0$ with $|x| < \varepsilon_0 R$, it holds that*

$$-(-\Delta)^{\alpha/2}\phi_R(x) \le \frac{1}{4C_0\Lambda}.$$

Proof. Let $R \in (0,1]$ and $x = (x_1, \cdots, x_d) \in B(0, \varepsilon_0 R) \setminus \{0\}$. Observe that

$$\begin{aligned}
&-(-\Delta)^{\alpha/2}\phi_R(x) \\
&= \mathcal{A}(d,-\alpha)\Big[p.v.\int_{B(x,|x|/2)} \frac{H(|y|)-H(|x|)}{|y-x|^{d+\alpha}}dy + \int_{B(x,3|x|)\setminus B(x,|x|/2)} \frac{H(|y|)-H(|x|)}{|y-x|^{d+\alpha}}dy \\
&\qquad + \int_{B(0,4\varepsilon_0 R)\setminus B(x,3|x|)} \frac{\phi_R(y)-H(|x|)}{|y-x|^{d+\alpha}}dy + \int_{B(0,4^{1/d}R)\setminus B(0,4\varepsilon_0 R)} \frac{\phi_R(y)-H(|x|)}{|y-x|^{d+\alpha}}dy \\
&\qquad + \int_{B(0,4^{1/d}R)^c} \frac{-H(|x|)}{|y-x|^{d+\alpha}}dy \Big] \\
&=: \mathcal{A}(d,-\alpha)(I_1+I_2+I_3+I_4+I_5).
\end{aligned}$$

By symmetry, we have

$$I_1 = \lim_{\varepsilon\to 0}\Big(\int_{B(x,|x|/2)\setminus B(x,\varepsilon)} \frac{H(|y|)-H(|x|)-H'(|x|)|x|^{-1}x\cdot(y-x)}{|y-x|^{d+\alpha}}dy \Big).$$

For any $y = (y_1, \cdots, y_d) \in B(x, |x|/2)$, using Taylor's theorem in the first inequality below, Lemma 3.1 in the second, the inequality $\sum_{1\le i<j\le d} |x_i - y_i||x_j - y_j| \le d \sum_{1\le i\le d} |x_i - y_i|^2 = d|x-y|^2$ in the third, and (1.6) and $\psi(1) = 1$ in the fourth, we obtain

$$
\begin{aligned}
&|H(|y|) - H(|x|) - H'(|x|)|x|^{-1} x \cdot (y - x)| \\
&\le \sup_{z=(z_1,\cdots,z_d)\in\{x+s(y-x):s\in[0,1]\}} \Bigg| \frac{1}{2} \sum_{1\le i\le d} \left(H''(|z|) \frac{z_i^2}{|z|^2} + H'(|z|) \frac{|z|^2 - z_i^2}{|z|^3} \right) |x_i - y_i|^2 \\
&\qquad\qquad + \sum_{1\le i<j\le d} \left(H''(|z|) \frac{z_i z_j}{|z|^2} - H'(|z|) \frac{z_i z_j}{|z|^3} \right) |x_i - y_i||x_j - y_j| \Bigg| \\
&\le \sup_{z\in\mathbb{R}^d:|x|/2\le|z|\le3|x|/2} \left[\sum_{1\le i\le d} \frac{4C_0\psi(|z|)}{|z|^2} |x_i - y_i|^2 + \sum_{1\le i<j\le d} \frac{8C_0\psi(|z|)}{|z|^2} |x_i - y_i||x_j - y_j| \right] \\
&\le \sup_{z\in\mathbb{R}^d:|x|/2\le|z|\le3|x|/2} \left[\frac{(4+8d)C_0\psi(|z|)}{|z|^2} \right] |x-y|^2 \\
&\le \sup_{z\in\mathbb{R}^d:|x|/2\le|z|\le3|x|/2} \left[\frac{(4+8d)C_0\Lambda|z|^{\beta_1}}{|z|^2} \right] |x-y|^2 \\
&\le 16(1+2d)(3/2)^{\beta_1} C_0 \Lambda |x|^{\beta_1-2} |x-y|^2.
\end{aligned}
$$

Thus, since $|x| \le \varepsilon_0$ and $\beta_1 > \alpha$, it holds that

$$
\begin{aligned}
|I_1| &\le 16(1+2d)(3/2)^{\beta_1} C_0 \Lambda |x|^{\beta_1-2} \int_{B(x,|x|/2)} \frac{dy}{|y-x|^{d+\alpha-2}} \\
&= \frac{16(1+2d)(3/2)^{\beta_1} C_0 \Lambda A_{d-1} |x|^{\beta_1-2} (|x|/2)^{2-\alpha}}{2-\alpha} \\
&\le \frac{16(1+2d)3^{\beta_1} C_0 \Lambda A_{d-1} \varepsilon_0^{\beta_1-\alpha}}{2^{2-\alpha+\beta_1}(2-\alpha)} \le \frac{1}{16\mathcal{A}(d,-\alpha)C_0\Lambda}.
\end{aligned}
$$

For I_2, using (3.5) and the facts that $|x| \le \varepsilon_0$ and $\beta_1 > \alpha$, we obtain

$$
|I_2| \le \int_{B(x,3|x|)\setminus B(x,|x|/2)} \frac{\Lambda(4|x|)^{\beta_1}}{|y-x|^{d+\alpha}} dy \le \frac{4^{\beta_1}\Lambda A_{d-1}|x|^{\beta_1}}{\alpha(|x|/2)^\alpha} \le \frac{2^{\alpha+2\beta_1}\Lambda A_{d-1}\varepsilon_0^{\beta_1-\alpha}}{\alpha} \le \frac{1}{16\mathcal{A}(d,-\alpha)C_0\Lambda}.
$$

Note that for $y \in B(x, 3|x|)^c$, we have $|y| \ge 2|x|$. Hence,

$$
|y - x| \ge |y|/2 \quad \text{for} \quad y \in B(x, 3|x|)^c. \tag{3.6}
$$

Further, for any $y \in B(0, 4\varepsilon_0 R) \setminus B(x, 3|x|)$, since $|y| \ge 2|x|$, by (3.5), it holds that

$$
|\phi_R(y) - H(|x|)| \le H(|y|) \vee H(|x|) \le \Lambda |y|^{\beta_1}. \tag{3.7}
$$

Using (3.6) and (3.7), we obtain

$$
|I_3| \le 2^{d+\alpha}\Lambda \int_{B(0,4\varepsilon_0 R)\setminus B(x,3|x|)} \frac{dy}{|y|^{d+\alpha-\beta_1}} dy \le \frac{2^{d+\alpha}\Lambda A_{d-1}(4\varepsilon_0 R)^{\beta_1-\alpha}}{\beta_1 - \alpha} \le \frac{1}{16\mathcal{A}(d,-\alpha)C_0\Lambda}.
$$

For I_4, we note that by (3.5) and (3.4), since $\beta_1 > \alpha$ and $R \le 1$,

$$H(|x|) \le \Lambda |x|^{\beta_1} \le \Lambda \varepsilon_0^{\beta_1} R^\alpha \le \delta_0 R^\alpha. \tag{3.8}$$

Therefore, using (3.6), we get that

$$|I_4| \le 2^{d+\alpha}\delta_0 R^\alpha \int_{B(0,4^{1/d}R)\setminus B(0,4\varepsilon_0 R)} \frac{dy}{|y|^{d+\alpha}} \le \frac{2^{d+\alpha}\delta_0 A_{d-1}}{\alpha(4\varepsilon_0)^\alpha} = \frac{1}{16\mathcal{A}(d,-\alpha)C_0\Lambda}.$$

For I_5, by (3.6) and (3.8), we have

$$0 \ge I_5 \ge -2^{d+\alpha}\delta_0 R^\alpha \int_{B(0,4^{1/d}R)^c} \frac{dy}{|y|^{d+\alpha}} = -\frac{2^{d+\alpha-2\alpha/d}\delta_0 A_{d-1}}{\alpha}.$$

Combining the estimates for I_1, I_2, I_3, I_4 and I_5 above, we deduce that $-(-\Delta)^{\alpha/2}\phi_R(x)$ is well-defined and that

$$-(-\Delta)^{\alpha/2}\phi_R(x) \le c_{d,\alpha}(I_1 + I_2 + I_3 + I_4) \le \frac{1}{4C_0\Lambda}.$$

The proof is complete. □

Corollary 3.3. *For all $R \in (0,1]$ and $x \in \mathbb{R}^d_0$ with $|x| < \varepsilon_0 R$, we have*

$$L_\alpha^{\kappa/2}\phi_R(x) \le -\frac{1}{4C_0\Lambda}.$$

Proof. Using Lemma 3.2, (1.7) and (3.1), we get that

$$L_\alpha^{\kappa/2}\phi_R(x) = -(-\Delta)^{\alpha/2}\phi_R(x) - \frac{1}{2}\kappa(x)H(|x|) \le \frac{1}{4C_0\Lambda} - \frac{\psi(|x|)}{2C_0\Lambda\psi(|x|)} = -\frac{1}{4C_0\Lambda}.$$

□

Under the assumptions of this section, conditions [22, **(H1)**–**(H4)** and (1.2)–(1.3)] hold. Thus, by [22, Theorem 4.8], we have the following Dynkin-type theorem for $L_\alpha^{\kappa/2}$. Note that, although [22] imposes a stronger condition on κ (see (1.8) therein), [22, Theorem 4.8] only relies on the local boundedness of κ, which is satisfied in our context since $\kappa \in \mathcal{K}_\alpha^0(\psi,\Lambda)$.

Theorem 3.4. *Let U be a relatively compact subset of $\mathbb{R}^d_0$. For any non-negative function u defined on $\mathbb{R}^d_0$ satisfying $u \in C^2(\overline{U})$ and any $x \in U$,*

$$\mathbb{E}_x\Big[u(X^{\kappa/2}_{\tau_U^{\kappa/2}})\Big] = u(x) + \mathbb{E}_x \int_0^{\tau_U^{\kappa/2}} L_\alpha^{\kappa/2} u(X_s^{\kappa/2})ds.$$

Proposition 3.5. *There exists $C = C(d,\alpha,\beta_1,\beta_2,\Lambda) > 0$ such that for all $R \in (0,1]$ and $x \in \mathbb{R}^d_0$ with $|x| < \varepsilon_0 R$,*

$$\mathbb{E}_x\big[\tau^\kappa_{B(0,\varepsilon_0 R)}\big] \le \mathbb{E}_x\big[\tau^{\kappa/2}_{B(0,\varepsilon_0 R)}\big] \le C\psi(|x|). \tag{3.9}$$

Proof. Let $R \in (0,1]$. The first inequality in (3.9) is obvious. We now prove the second inequality. Take $\varepsilon \in (0, \varepsilon_0 R)$. For any $x \in B(0,\varepsilon_0 R) \setminus \{0\}$, using Theorem 3.4, Corollary 3.3 and (3.1), since $\phi_R(x) = H(|x|)$ by definition, we obtain

$$\mathbb{E}_x\Big[\phi_R\big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)\setminus\overline{B(0,\varepsilon)}}}\big)\Big] = \phi_R(x) + \mathbb{E}_x \int_0^{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)\setminus\overline{B(0,\varepsilon)}}} L^{\kappa/2}_\alpha u(X^{\kappa/2}_s) ds \le H(|x|) \le \psi(|x|). \tag{3.10}$$

On the other hand, by (2.10), we have

$$\begin{aligned}\mathbb{E}_x\Big[\phi_R\big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)\setminus\overline{B(0,\varepsilon)}}}\big)\Big] &\ge \mathbb{E}_x\left[\int_0^{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)\setminus\overline{B(0,\varepsilon)}}} \int_{B(0,2^{1/d}R)\setminus B(0,R)} \frac{\mathcal{A}(d,-\alpha)\phi_R(y)}{|X^{\kappa/2}_s - y|^{d+\alpha}}\, dy\, ds\right] \\ &\ge \frac{\mathcal{A}(d,-\alpha)\delta_0 R^\alpha}{(2^{1/\alpha}+\varepsilon_0)^{d+\alpha} R^{d+\alpha}} \mathbb{E}_x\big[\tau^{\kappa/2}_{B(0,\varepsilon_0 R)\setminus\overline{B(0,\varepsilon)}}\big] \int_{B(0,2^{1/d}R)\setminus B(0,R)} dy \\ &= \frac{\mathcal{A}(d,-\alpha)\delta_0 A_{d-1}}{d(2^{1/\alpha}+\varepsilon_0)^{d+\alpha}} \mathbb{E}_x\big[\tau^{\kappa/2}_{B(0,\varepsilon_0 R)\setminus\overline{B(0,\varepsilon)}}\big]. \end{aligned}\tag{3.11}$$

Since $\varepsilon \in (0,\varepsilon_0 R)$ is arbitrary, combining (3.10) with (3.11) and applying the monotone convergence theorem, we conclude that

$$\mathbb{E}_x\big[\tau^{\kappa/2}_{B(0,\varepsilon_0 R)}\big] = \lim_{\varepsilon\to 0} \mathbb{E}_x\big[\tau^{\kappa/2}_{B(0,\varepsilon_0 R)\setminus\overline{B(0,\varepsilon)}}\big] \le \frac{d(2^{1/\alpha}+\varepsilon_0)^{d+\alpha}}{\mathcal{A}(d,-\alpha)\delta_0 A_{d-1}} \psi(|x|).$$

□

Denote by ζ^κ the lifetime of X^κ.

Lemma 3.6. *There exists $C = C(d,\alpha,\beta_1,\beta_2,\Lambda) > 0$ such that for all $R \in (0,1]$ and $x \in \mathbb{R}^d_0$ with $|x| < \varepsilon_0 R$,*

$$\mathbb{P}_x\big(\tau^\kappa_{B(0,\varepsilon_0 R)} < \zeta^\kappa\big) \le \mathbb{P}_x\big(\tau^{\kappa/2}_{B(0,\varepsilon_0 R)} < \zeta^{\kappa/2}\big) = \mathbb{P}_x\Big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)}} \in \mathbb{R}^d_0\Big) \le \frac{C\psi(|x|)}{\psi(R)}. \tag{3.12}$$

Proof. The first inequality in (3.12) is evident. We now present the proof of the second inequality. For any $z \in B(0,R) \setminus B(0,\varepsilon_0 R)$, by (3.1), $\phi_R(z) \ge C_0^{-1}\psi(\varepsilon_0 R)$. Hence, by Markov's inequality, (1.6) and (3.10), it holds that

$$\mathbb{P}_x\Big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)}} \in B(0,R)\Big) \le \frac{C_0}{\psi(\varepsilon_0 R)} \mathbb{E}_x\Big[\phi_R\big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)}}\big)\Big] \le \frac{C_0 \Lambda \psi(|x|)}{\varepsilon_0^{\beta_2}\psi(R)}.$$

On the other hand, by (2.10), we have

$$\mathbb{P}_x\Big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)}} \in B(0,R)^c\Big) = \mathbb{E}_x\left[\int_0^{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)}} \int_{B(0,R)^c} \frac{\mathcal{A}(d,-\alpha)}{|X^{\kappa/2}_s - w|^{d+\alpha}}\, dwds\right]. \tag{3.13}$$

For any $z \in B(0,\varepsilon_0 R)$, by (1.6), since $\psi(1) = 1$, $R \le 1$ and $\beta_1 > \alpha$, it holds that

$$\int_{B(0,R)^c} \frac{\mathcal{A}(d,-\alpha)}{|z-w|^{d+\alpha}} dw \le \int_{B(z,R/2)^c} \frac{\mathcal{A}(d,-\alpha)}{|z-w|^{d+\alpha}} dw = \frac{2^\alpha \mathcal{A}(d,-\alpha) A_{d-1}}{R^\alpha \psi(1)} \le \frac{c_1}{R^{\alpha-\beta_1}\psi(R)} \le \frac{c_1}{\psi(R)}.$$

Combining this with (3.13) and using Proposition 3.5, we obtain

$$\mathbb{P}_x\Big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,\varepsilon_0 R)}} \in B(0,R)^c\Big) \le \frac{c_1}{\psi(R)}\mathbb{E}_x\big[\tau^{\kappa/2}_{B(0,\varepsilon_0 R)}\big] \le \frac{c_2\psi(|x|)}{\psi(R)}.$$

The proof is complete. □

Lemma 3.7. *There exists $C = C(d,\alpha,\beta_1,\beta_2,\Lambda) > 0$ such that for all $t \in (0,1]$ and $x \in \mathbb{R}^d_0$,*

$$\mathbb{P}_x(\zeta^{\kappa/2} > t) \le C\bigg(1 \wedge \frac{\psi(|x|)}{t}\bigg). \tag{3.14}$$

Proof. If $|x| \ge \varepsilon_0\psi^{-1}(t)$, then $\psi(|x|)/t \ge c_1$ by (1.6), and hence (3.14) holds. If $|x| < \varepsilon_0\psi^{-1}(t)$, then using Lemma 3.6 and Markov's inequality in the second line below, and Proposition 3.5 in the third, we obtain

$$\begin{aligned}\mathbb{P}_x(\zeta^{\kappa/2} > t) &\le \mathbb{P}_x\big(\tau^{\kappa/2}_{B(0,\varepsilon_0\psi^{-1}(t))} < \zeta^{\kappa/2}\big) + \mathbb{P}_x\big(\tau^{\kappa/2}_{B(0,\varepsilon_0\psi^{-1}(t))} > t\big)\\ &\le c_2 t^{-1}\psi(|x|) + t^{-1}\mathbb{E}_x[\tau^{\kappa/2}_{B(0,\varepsilon_0\psi^{-1}(t))}] \le c_3 t^{-1}\psi(|x|).\end{aligned}$$

□

Lemma 3.8. *There exists $C = C(d,\alpha,\beta_1,\beta_2,\Lambda) > 0$ such that for all $r \in (0,\varepsilon_0]$, $t \in (0,1]$ and $x \in \mathbb{R}^d_0$ with $|x| < r$,*

$$\mathbb{P}_x\big(\tau^{\kappa/2}_{B(0,3r)} < t \wedge \zeta^{\kappa/2}\big) \le \frac{C\psi(|x|)}{r^\alpha}\bigg(1 \vee \frac{t}{\psi(r)}\bigg).$$

Proof. Using the strong Markov property, we see that

$$\begin{aligned}&\mathbb{P}_x\big(\tau^{\kappa/2}_{B(0,3r)} < t \wedge \zeta^{\kappa/2}\big) \le \mathbb{P}_x\big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,r)}} \in B(0,2r)^c\big) + \mathbb{P}_x\big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,r)}} \in B(0,2r),\, \tau^{\kappa/2}_{B(0,3r)} < t \wedge \zeta^{\kappa/2}\big)\\ &\le \mathbb{P}_x\big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,r)}} \in B(0,2r)^c\big) + \mathbb{P}_x\big(X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,r)}} \in B(0,2r)\big) \sup_{z\in B(0,2r)} \mathbb{P}_z\big(\tau^{\kappa/2}_{B(z,r)} < t \wedge \zeta^{\kappa/2}\big)\\ &=: I_1 + I_2.\end{aligned}$$

For any $z \in B(0,r)$, it holds that

$$\int_{B(0,2r)^c} \frac{\mathcal{A}(d,-\alpha)}{|z-y|^{d+\alpha}}dy \le \int_{B(z,r)^c} \frac{\mathcal{A}(d,-\alpha)}{|z-y|^{d+\alpha}}dy = \frac{\mathcal{A}(d,-\alpha)A_{d-1}}{dr^\alpha}. \tag{3.15}$$

Using (2.10) in the equality below, (3.15) in the first inequality and Proposition 3.5 in the second inequality, we obtain

$$I_1 = \mathbb{E}_x\bigg[\int_0^{\tau^{\kappa/2}_{B(0,r)}} \int_{B(0,2r)^c} \frac{\mathcal{A}(d,-\alpha)}{|X^{\kappa/2}_s - y|^{d+\alpha}}dyds\bigg] \le \frac{c_1}{r^\alpha}\mathbb{E}_x\big[\tau^{\kappa/2}_{B(0,r)}\big] \le \frac{c_2\psi(|x|)}{r^\alpha}.$$

Further, by Lemma 3.6 and Proposition 2.1, we have

$$I_2 \le \mathbb{P}_x\big(\tau^{\kappa/2}_{B(0,r)} < \zeta^{\kappa/2}\big) \sup_{z\in B(0,2r)} \mathbb{P}_z\big(\tau^Y_{B(z,r)} < t\big) \le \frac{c_3\psi(|x|)t}{\psi(r)r^\alpha}.$$

The proof is complete. □

4 Small time estimates

In this section, we continue to assume that $\kappa \in \mathcal{K}_\alpha^0(\psi,\Lambda)$ and that $\varepsilon_0=\varepsilon_0(d,\alpha,\beta_1,\beta_2,\Lambda)\in(0,1/8]$ is the constant in (3.3). The goal of this section is to establish the following two-sided small time heat kernel estimates.

Theorem 4.1. *Suppose that $\kappa \in \mathcal{K}_\alpha^0(\psi,\Lambda)$. Let $T\ge 1$. There exist constants $\lambda_1=\lambda_1(\beta_2,\Lambda), \lambda_2=\lambda_2(\beta_2,\Lambda)>0$ and $C=C(d,\alpha,\beta_1,\beta_2,\Lambda,T)\ge 1$ such that for all $t\in(0,T]$ and $x,y\in\mathbb{R}^d_0$,*

$$p^\kappa(t,x,y)\le C\left(1\wedge\frac{\psi(|x|)}{t}\right)\left(1\wedge\frac{\psi(|y|)}{t}\right) \times\left[e^{-\lambda_1 t/\psi(|x|\vee|y|)}t^{-d/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}+\frac{t^2}{\psi^{-1}(t)^{d+2\alpha}}\left(1\wedge\frac{\psi^{-1}(t)}{|x-y|}\right)^{d+2\alpha}\right] \tag{4.1}$$

and

$$p^\kappa(t,x,y)\ge C^{-1}\left(1\wedge\frac{\psi(|x|)}{t}\right)\left(1\wedge\frac{\psi(|y|)}{t}\right) \times\left[e^{-\lambda_2 t/\psi(|x|\vee|y|)}t^{-d/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}+\frac{t^2}{\psi^{-1}(t)^{d+2\alpha}}\left(1\wedge\frac{\psi^{-1}(t)}{|x-y|}\right)^{d+2\alpha}\right]. \tag{4.2}$$

The proofs for (4.1) and (4.2) will be given at the end of Subsections 4.1 and 4.2, respectively. We first record a consequence of (4.1), see Corollary 4.3 below.

Lemma 4.2. *For any $T\ge1$, there exists $C=C(d,\alpha,\beta_1,\Lambda,T)>0$ such that for all $t\in(0,T]$ and $r>0$,*

$$\frac{t^2}{\psi^{-1}(t)^{d+2\alpha}}\left(1\wedge\frac{\psi^{-1}(t)}{r}\right)^{d+2\alpha}\le Ct^{-d/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{r}\right)^{d+\alpha}.$$

Proof. Let $t\in(0,T]$ and $r>0$. If $r>(t/T)^{1/\beta_1}$, then $r^\alpha>(t/T)^{\alpha/\beta_1}\ge t/T$. Hence,

$$\frac{t^2}{\psi^{-1}(t)^{d+2\alpha}}\left(1\wedge\frac{\psi^{-1}(t)}{r}\right)^{d+2\alpha}\le\frac{t^2}{r^{d+2\alpha}}\le\frac{Tt}{r^{d+\alpha}}\le c_1t^{-d/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{r}\right)^{d+\alpha}.$$

If $(t/T)^{1/\alpha}\le r\le(t/T)^{1/\beta_1}$, then by (4.10), we get

$$\begin{aligned}\frac{t^2}{\psi^{-1}(t)^{d+2\alpha}}\left(1\wedge\frac{\psi^{-1}(t)}{r}\right)^{d+2\alpha}&\le\frac{t^2}{\psi^{-1}(t/T)^{d+2\alpha}}\le\frac{t^2}{(t/(\Lambda T))^{(d+2\alpha)/\beta_1}}\\&\le\frac{\Lambda^{(d+2\alpha)/\beta_1}t^2}{r^{d+2\alpha}}\le\frac{\Lambda^{(d+2\alpha)/\beta_1}Tt}{r^{d+\alpha}}\le c_2t^{-d/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{r}\right)^{d+\alpha}.\end{aligned}$$

If $r<(t/T)^{1/\alpha}$, then by (4.10), since $t/(\Lambda T)\le1$ and $\beta_1>\alpha$, we see that

$$\begin{aligned}\frac{t^2}{\psi^{-1}(t)^{d+2\alpha}}\left(1\wedge\frac{\psi^{-1}(t)}{r}\right)^{d+2\alpha}&\le\frac{t^2}{(t/(\Lambda T))^{(d+2\alpha)/\beta_1}}\\&\le\frac{t^2}{(t/(\Lambda T))^{(d+2\alpha)/\alpha}}\le c_3t^{-d/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{r}\right)^{d+\alpha}.\end{aligned}$$

The proof is complete. □

Corollary 4.3. *There exists $C = C(d, \alpha, \beta_1, \beta_2, \Lambda) > 0$ such that for all $t > 0$ and $x, y \in \mathbb{R}^d_0$,*

$$p^\kappa(t, x, y) \le C\left(1 \wedge \frac{\psi(|x|)}{t \wedge 1}\right)\left(1 \wedge \frac{\psi(|y|)}{t \wedge 1}\right)\widetilde{q}(t, x, y). \tag{4.3}$$

Proof. Let $t > 0$ and $x, y \in \mathbb{R}^d_0$. If $t \le 2$, then using (4.1) and Lemma 4.2, we get (4.3). Suppose that $t > 2$. Using the semigroup property of p^κ in the first line below, the inequality $p^\kappa(s, v, w) \le q(s, v, w)$ for all $s > 0$ and $v, w \in \mathbb{R}^d_0$, (1.1) and (4.3) with $t = 1$ in the second, and (2.2) in the third, we obtain

$$\begin{aligned} p^\kappa(t, x, y) &= \int_{\mathbb{R}^d_0}\int_{\mathbb{R}^d_0} p^\kappa(1, x, v)p^\kappa(t-2, v, w)p^\kappa(1, w, y)dvdw \\ &\le c_1(1 \wedge \psi(|x|))(1 \wedge \psi(|y|))\int_{\mathbb{R}^d_0}\int_{\mathbb{R}^d_0} \widetilde{q}(1, x, v)\widetilde{q}(t-2, v, w)\widetilde{q}(1, w, y)dvdw \\ &\le c_2(1 \wedge \psi(|x|))(1 \wedge \psi(|y|))\widetilde{q}(t, x, y). \end{aligned}$$

The proof is complete. □

4.1 Small time upper heat kernel estimates

Lemma 4.4. *For any $T \ge 1$, there exists $C = C(d, \alpha, \beta_1, \beta_2, \Lambda, T) > 0$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d_0$,*

$$p^\kappa(t, x, y) \le p^{\kappa/2}(t, x, y) \le \frac{C\psi(|x|)}{t^{d/\alpha+1}}. \tag{4.4}$$

Proof. Clearly, $p^\kappa(t, x, y) \le p^{\kappa/2}(t, x, y)$. We now prove the second inequality in (4.4). If $|x| \ge \varepsilon_0\psi^{-1}(t/T)$, then by (1.6), $\psi(|x|) \ge c_1 t$. Hence, since $p^{\kappa/2}(t, x, y) \le q(t, x, y)$, (4.4) follows from (1.1). Suppose that $|x| < \varepsilon_0\psi^{-1}(t/T)$. Using the semigroup property in the equality below, (1.1) and Lemma 3.7 in the second inequality, we get

$$\begin{aligned} p^{\kappa/2}(t, x, y) &= \int_{\mathbb{R}^d_0} p^{\kappa/2}(t/(2T), x, z)\, p^{\kappa/2}((2T-1)t/(2T), z, y)\, dz \\ &\le \sup_{z \in \mathbb{R}^d_0} q((2T-1)t/(2T), z, y)\, \mathbb{P}_x\big(X^{\kappa/2}_{t/(2T)} \in \mathbb{R}^d_0\big) \\ &\le 2Tc_2(t/2)^{-d/\alpha}\psi(|x|)/t, \end{aligned}$$

proving that the second inequality in (4.4) holds. The proof is complete. □

For any Borel set $E \subset \mathbb{R}^d$ with positive Lebesgue measure and $f \in L^1(E)$, we use the usual notation $⨍_E f := \int_E f / \int_E 1$.

Lemma 4.5. *For any $T \ge 1$, there exists $C = C(d, \alpha, \beta_1, \beta_2, \Lambda, T) > 0$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d_0$ with $|x| < \varepsilon_0\psi^{-1}(t/T)/8$ and $|y| > 8|x|$,*

$$p^\kappa(t, x, y) \le p^{\kappa/2}(t, x, y) \le \frac{C\psi(|x|)}{|x-y|^{d+\alpha}}\left(1 \vee \frac{t}{\psi(|y|)}\right).$$

Proof. Let $t \in (0,T]$ and $x, y \in \mathbb{R}^d_0$ be such that $|x| < \varepsilon_0 \psi^{-1}(t/T)/8$ and $|y| > 8|x|$. Set

$$r_0 := \frac{|y| \wedge (\varepsilon_0 \psi^{-1}(t/T))}{8} \quad \text{and} \quad U := B(0, r_0).$$

By the lower semi-continuity of $p^{\kappa/2}$ and the strong Markov property of $X^{\kappa/2}$, we have

$$\begin{aligned}
&p^{\kappa/2}(t,x,y) \le \liminf_{\delta \to 0} \fint_{B(y,\delta)} p^{\kappa/2}(t,x,v)dv \\
&\le \limsup_{\delta\to 0} \mathbb{E}_x \left[\fint_{B(y,\delta)} p^{\kappa/2}(t-\tau_U^{\kappa/2}, X^{\kappa/2}_{\tau_U}, v)dv : \tau_U^{\kappa/2} < t \wedge \zeta^\kappa,\ X^{\kappa/2}_{\tau_U^{\kappa/2}} \in B(y, |x-y|/2) \right] \\
&\quad + \limsup_{\delta\to 0} \mathbb{E}_x \left[\fint_{B(y,\delta)} p^{\kappa/2}(t-\tau_U^{\kappa/2}, X^{\kappa/2}_{\tau_U^{\kappa/2}}, v)dv : \tau_U^{\kappa/2} < t \wedge \zeta^\kappa,\ X^{\kappa/2}_{\tau_U^{\kappa/2}} \in B(y, |x-y|/2)^c \right] \\
&=: I_1 + I_2.
\end{aligned}$$

For all $z \in U$ and $w \in B(y, |x-y|/2)$, since $|x| < r_0 \le |y|/8$ so that $|x| < r_0 \le |x-y|/7$,

$$|z-w| \ge |w| - |z| \ge |x-y| - |x| - \frac{|x-y|}{2} - r_0 \ge \frac{3|x-y|}{14}. \tag{4.5}$$

Using (2.10) in the equality below, (4.5) and the symmetry of p in the first inequality and Proposition 3.5 in the last, we obtain

$$\begin{aligned}
I_1 &= \limsup_{\delta\to 0} \int_0^t \int_U \int_{B(y,|x-y|/2)} \fint_{B(y,\delta)} p^{\kappa/2,U}(s,x,z) \frac{c_{d,\alpha}}{|z-w|^{d+\alpha}} p^{\kappa/2}(t-s,w,v)\, dvdwdzds \\
&\le \limsup_{\delta\to 0} \frac{(14/3)^{d+\alpha} c_{d,\alpha}}{|x-y|^{d+\alpha}} \int_0^t \int_U p^{\kappa/2,U}(s,x,z)dz \fint_{B(y,\delta)} \int_{B(y,|x-y|/2)} p^{\kappa/2}(t-s,v,w)dwdv\, ds \\
&\le \frac{(14/3)^{d+\alpha} c_{d,\alpha}}{|x-y|^{d+\alpha}} \int_0^t \mathbb{P}_x(\tau_U^{\kappa/2} > s)ds \liminf_{\delta\to 0} \fint_{B(y,\delta)} dv \\
&\le \frac{(14/3)^{d+\alpha} c_{d,\alpha} \mathbb{E}_x[\tau_U^{\kappa/2}]}{|x-y|^{d+\alpha}} \le \frac{c_1 \psi(|x|)}{|x-y|^{d+\alpha}}.
\end{aligned}$$

For I_2, we observe that for any $\delta \in (0, |x-y|/4)$, by (1.1),

$$\begin{aligned}
&\sup_{s\in(0,t],\, z\in B(y,|x-y|/2)^c,\, v\in B(y,\delta)} p^{\kappa/2}(s,z,v) \\
&\le \sup_{s\in(0,t],\, z\in B(y,|x-y|/2)^c,\, v\in B(y,\delta)} q(s,z,v) \le c_2 \sup_{z\in B(y,|x-y|/2)^c,\, v\in B(y,\delta)} \frac{t}{|z-v|^{d+\alpha}} \le \frac{c_2 t}{(|x-y|/4)^{d+\alpha}}.
\end{aligned}$$

Using this, Lemma 3.6 and (1.6), we get that

$$\begin{aligned}
I_2 &\le \frac{4^{d+\alpha} c_2 t}{|x-y|^{d+\alpha}} \mathbb{P}_x\Big(X^{\kappa/2}_{\tau_U^{\kappa/2}} \in \mathbb{R}^d_0\Big) \limsup_{\delta\to 0} \fint_{B(y,\delta)} dv \\
&\le \frac{c_3 t \psi(|x|)}{|x-y|^{d+\alpha} \psi(r_0/\varepsilon_0)} \le \frac{c_4 \psi(|x|)}{|x-y|^{d+\alpha}} \Big(1 \vee \frac{t}{\psi(|y|)}\Big).
\end{aligned}$$

The proof is complete. □

Lemma 4.6. *For any $T \ge 1$, there exists $C = C(d, \alpha, \beta_1, \beta_2, \Lambda, T) > 0$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d_0$ with $|x| < \varepsilon_0 \psi^{-1}(t/T)$ and $|x| \le |y|$,*

$$p^\kappa(t, x, y) \le p^{\kappa/2}(t, x, y) \le \frac{C\psi(|x|)}{t}\left(1 \vee \frac{t}{\psi(|y|)}\right) t^{-d/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}. \tag{4.6}$$

Proof. It suffices to prove the second inequality in (4.6). When $|x - y| \le 9t^{1/\alpha}$, (4.6) follows from Lemma 4.4. Suppose that $|x - y| > 9t^{1/\alpha}$. If $|y| > 8|x|$, then (4.6) follows from Lemma 4.5. If $|y| \le 8|x|$, then we get $|x| \le |y| \le 8|x| < \psi^{-1}(t/T)$ so that by (1.6),

$$\frac{\psi(|x|)}{t}\left(1 \vee \frac{t}{\psi(|y|)}\right) = \frac{\psi(|x|)}{\psi(|y|)} \ge c_1.$$

Hence, using (1.1), we get that

$$p^{\kappa/2}(t, x, y) \le q(t, x, y) \le \frac{c_2 t}{|x-y|^{d+\alpha}} \le \frac{c_1^{-1} c_2 \psi(|x|)}{t}\left(1 \vee \frac{t}{\psi(|y|)}\right)\frac{t}{|x-y|^{d+\alpha}}.$$

The proof is complete. □

Lemma 4.7. *Let $T \ge 1$. There exist $\lambda_1 = \lambda_1(\beta_2, \Lambda) > 0$ and $C = C(d, \alpha, \beta_1, \beta_2, \Lambda, T) > 0$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d_0$ with $|x| < \varepsilon_0 \psi^{-1}(t/T)/8$ and $|x| \le |y| < \varepsilon_0 \psi^{-1}(t/T)$,*

$$p^\kappa(t, x, y) \le \frac{C\psi(|x|)\psi(|y|)}{t^2}\left[e^{-\lambda_1 t/\psi(|y|)} t^{-d/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} + \frac{t^2}{\psi^{-1}(t)^{d+2\alpha}}\right].$$

Proof. Define $M_s := \sup_{0 \le u \le s} |X^\kappa_u|$ for $s > 0$. Let $n_0 \in \mathbb{N}$ be such that $2^{n_0-1}|y| < \varepsilon_0 \psi^{-1}(t/T) \le 2^{n_0}|y|$. By the lower-semicontinuity of p^κ, we have

$$\begin{aligned}
&p^\kappa(t, x, y) \le \liminf_{\delta \to 0} ⨍_{B(y,\delta)} p^\kappa(t, x, v)dv \\
&\le \limsup_{\delta \to 0} \frac{c_1}{\delta^d}\mathbb{P}_x\left(X^\kappa_t \in B(y, \delta) : M_t \le 8|y|\right) + \limsup_{\delta \to 0} \frac{c_1}{\delta^d}\mathbb{P}_x\left(X^\kappa_t \in B(y, \delta) : M_t > 2^{n_0+3}|y|\right) \\
&\quad + \sum_{m=1}^{n_0} \limsup_{\delta \to 0} \frac{c_1}{\delta^d}\mathbb{P}_x\left(X^\kappa_t \in B(y, \delta) : 2^{m+2}|y| < M_t \le 2^{m+3}|y|\right) \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

From (2.5), we get that for any $\delta > 0$,

$$\begin{aligned}
\mathbb{P}_x\left(X^\kappa_t \in B(y, \delta) : M_t \le 8|y|\right) &\le \mathbb{P}_x(X^{\kappa/2}_t \in B(y, \delta)) \exp\left(-\frac{t}{2}\inf_{|z| \le 8|y|} \kappa(z)\right) \\
&= \int_{B(y,\delta)} p^{\kappa/2}(t, x, v)dv \exp\left(-\frac{t}{2}\inf_{|z| \le 8|y|} \kappa(z)\right).
\end{aligned} \tag{4.7}$$

By (1.7) and (1.6), we have

$$\frac{t}{2}\inf_{|z| \le 8|y|} \kappa(z) \ge \frac{t}{2\Lambda\psi(8|y|)} \ge \frac{c_2 t}{\psi(|y|)},$$

for some $c_2 = c_2(\beta_2, \Lambda) > 0$. Using this and Lemma 4.6 in the first inequality below and $\sup_{u\ge 1} u^2 e^{-2^{-1}c_2 u} < \infty$ in the third, we get from (4.7) that

$$\begin{aligned} I_1 &\le \frac{c_3\psi(|x|)e^{-c_2t/\psi(|y|)}}{t} \limsup_{\delta\to 0} \fint_{B(y,\delta)} \left(1 \vee \frac{t}{\psi(|v|)}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-v|^{d+\alpha}}\right) dv \\ &= \frac{c_3\psi(|x|)e^{-c_2t/\psi(|y|)}}{\psi(|y|)} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \\ &\le \frac{c_4\psi(|x|)\psi(|y|)e^{-2^{-1}c_2t/\psi(|y|)}}{t^2} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \end{aligned}$$

For I_2, since $2^{n_0}|y| \ge \varepsilon_0\psi^{-1}(t/T)$, using the strong Markov property, we see that for any $\delta > 0$,

$$\begin{aligned} &\mathbb{P}_x\left(X_t^\kappa \in B(y,\delta) : M_t > 2^{n_0+3}|y|\right) \le \mathbb{P}_x\left(X_t^\kappa \in B(y,\delta) : \tau^\kappa_{B(0,8\varepsilon_0\psi^{-1}(t/T))} < t \wedge \zeta^\kappa\right) \\ &= \mathbb{E}_x\left[\mathbb{P}_{X^\kappa_{\tau^\kappa_{B(0,8\varepsilon_0\psi^{-1}(t/T))}}}\left(X^\kappa_{t-\tau^\kappa_{B(0,8\varepsilon_0\psi^{-1}(t/T))}} \in B(y,\delta)\right) : \tau^\kappa_{B(0,8\varepsilon_0\psi^{-1}(t/T))} < t\wedge\zeta^\kappa\right] \\ &\le \mathbb{P}_x\left(\tau^\kappa_{B(0,8\varepsilon_0\psi^{-1}(t/T))} < t \wedge \zeta^\kappa\right) \sup_{0<s\le t,\, z\in B(0,8\varepsilon_0\psi^{-1}(t/T))^c} \mathbb{P}_z\left(X^\kappa_s \in B(y,\delta)\right). \end{aligned}$$

Hence, using (1.6) and Lemmas 3.8 and 4.6, since $|y| < \varepsilon_0\psi^{-1}(t/T)$, we obtain

$$\begin{aligned} I_2 &\le \frac{c_5\psi(|x|)}{\psi^{-1}(t/T)^\alpha} \limsup_{\delta\to 0} \sup_{0<s\le t,\, z\in B(0,8\varepsilon_0\psi^{-1}(t/T))^c} \fint_{B(y,\delta)} p^\kappa(s,z,v)dv \\ &\le \frac{c_6\psi(|x|)}{\psi^{-1}(t/T)^\alpha} \sup_{0<s\le t,\, z\in B(0,8\varepsilon_0\psi^{-1}(t/T))^c} \frac{\psi(|y|)}{s}\left(1 \vee \frac{s}{\psi(|z|)}\right) s^{-d/\alpha}\left(1 \wedge \frac{s^{1/\alpha}}{|y-z|}\right)^{d+\alpha} \\ &\le \frac{c_7\psi(|x|)}{\psi^{-1}(t/T)^\alpha} \sup_{0<s\le t,\, z\in B(0,8\varepsilon_0\psi^{-1}(t/T))^c} \frac{\psi(|y|)}{s}\frac{s}{|y-z|^{d+\alpha}} \\ &\le \frac{c_7\psi(|x|)\psi(|y|)}{\psi^{-1}(t/T)^\alpha(7\varepsilon_0\psi^{-1}(t/T))^{d+\alpha}} \le \frac{c_8\psi(|x|)\psi(|y|)}{\psi^{-1}(t)^{d+2\alpha}}. \end{aligned}$$

It remains to estimate I_3. Let $1 \le m \le n_0$. Using the formula (2.5) in the first inequality below, the strong Markov property, (1.7) and (1.6) in the second, and Lemma 3.8 and $\psi(2^{m+2}|y|) \le \psi(8\varepsilon_0\psi^{-1}(t/T)) \le t/T$ in the fourth, we obtain

$$\begin{aligned} &\mathbb{P}_x\left(X^\kappa_t \in B(y,\delta) : 2^{m+2}|y| < M_t \le 2^{m+3}|y|\right) \\ &\le \mathbb{P}_x\left(X^{\kappa/2}_t \in B(y,\delta) : 2^{m+2}|y| < M_t \le 2^{m+3}|y|\right) \exp\left(-\frac{t}{2}\inf_{|z|\le 2^{m+3}|y|}\kappa(z)\right) \\ &\le \mathbb{E}_x\left[\mathbb{P}_{X^{\kappa/2}_{\tau^{\kappa/2}_{B(0,2^{m+2}|y|)}}}\left(X^{\kappa/2}_{t-\tau^{\kappa/2}_{B(0,2^{m+2}|y|)}} \in B(y,\delta)\right) : \tau^{\kappa/2}_{B(0,2^{m+2}|y|)} < t\wedge\zeta^{\kappa/2}\right] \exp\left(-\frac{c_9 t}{\psi(2^{m+2}|y|)}\right) \\ &\le \mathbb{P}_x\left(\tau^{\kappa/2}_{B(0,2^{m+2}|y|)} < t\wedge\zeta^{\kappa/2}\right) \sup_{0<s\le t,\, z\in B(0,2^{m+2}|y|)^c} \mathbb{P}_z\left(X^{\kappa/2}_s \in B(y,\delta)\right) \exp\left(-\frac{c_9 t}{\psi(2^{m+2}|y|)}\right) \\ &\le \frac{c_{10}\psi(|x|)t}{(2^{m+2}|y|)^\alpha\psi(2^{m+2}|y|)} \sup_{0<s\le t,\, z\in B(0,2^{m+2}|y|)^c} \int_{B(y,\delta)} p^{\kappa/2}(s,z,v)dv \exp\left(-\frac{c_9 t}{\psi(2^{m+2}|y|)}\right). \qquad (4.8) \end{aligned}$$

By Lemma 4.6 and the inequality $\psi(2^{m+2}|y|) \le t/T$, we have

$$
\begin{aligned}
&\limsup_{\delta\to 0} \frac{1}{\delta^d} \sup_{0<s\le t,\, z\in B(0,2^{m+2}|y|)^c} \int_{B(y,\delta)} p^{\kappa/2}(s,z,v)dv \\
&\le \sup_{0<s\le t,\, z\in B(0,2^{m+2}|y|)^c} \frac{c_{11}\psi(|y|)}{s}\left(1 \vee \frac{s}{\psi(|z|)}\right)\frac{s}{|y-z|^{d+\alpha}} \le \frac{c_{11}\psi(|y|)t}{(2^{m+1}|y|)^{d+\alpha}\psi(2^{m+2}|y|)}. \qquad (4.9)
\end{aligned}
$$

Combining (4.8) with (4.9), and using $\sup_{a\ge 1} a^{3+(d+2\alpha)/\beta_1}e^{-c_9 a} < \infty$ and (1.6), we deduce that

$$
\begin{aligned}
&\limsup_{\delta\to 0} \frac{1}{\delta^d}\mathbb{P}_x\left(X_t^\kappa \in B(y,\delta) : 2^{m+2}|y| < M_t \le 2^{m+3}|y|\right) \\
&\le \frac{2^{d+\alpha}c_{12}\psi(|x|)\psi(|y|)t^2}{(2^{m+2}|y|)^{d+2\alpha}\psi(2^{m+2}|y|)^2}\exp\left(-\frac{c_9 t}{\psi(2^{m+2}|y|)}\right) \\
&\le \frac{c_{13}\psi(|x|)\psi(|y|)t^2}{(2^{m+2}|y|)^{d+2\alpha}\psi(2^{m+2}|y|)^2}\left(\frac{\psi(2^{m+2}|y|)}{t}\right)^{3+(d+2\alpha)/\beta_1} \\
&\le \frac{c_{14}\psi(|x|)\psi(|y|)\psi(2^{m+2}|y|)}{t\psi^{-1}(t)^{d+2\alpha}}.
\end{aligned}
$$

Using this, (1.6) and $2^{n_0+2}|y| < 8\varepsilon_0\psi^{-1}(t/T) \le \psi^{-1}(t/T)$, we conclude that

$$
I_2 \le \frac{c_{14}\Lambda\psi(|x|)\psi(|y|)\psi(2^{n_0+2}|y|)}{t\psi^{-1}(t)^{d+2\alpha}}\sum_{m=1}^{n_0} 2^{-(n_0-m)\beta_1} \le \frac{c_{15}\psi(|x|)\psi(|y|)}{\psi^{-1}(t)^{d+2\alpha}}.
$$

The proof is complete. □

Note that for any $t \in (0,T]$, by (1.6),

$$
\psi^{-1}(t/T) \ge \psi^{-1}(1)(t/(\Lambda T))^{1/\beta_1} = (t/(\Lambda T))^{1/\beta_1} \ge (t/(\Lambda T))^{1/\alpha}. \qquad (4.10)
$$

Proof of Theorem 4.1 (Upper estimates). Let $t \in (0,T]$ and $x,y \in \mathbb{R}_0^d$. Without loss of generality, we assume that $|x| \le |y|$. We deal with three cases separately.

Case 1: $|x| \ge \varepsilon_0\psi^{-1}(t/T)/8$. By (1.1), we get

$$
p^\kappa(t,x,y) \le q(t,x,y) \le c_1 t^{-d/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.
$$

Further, by (1.6), we have $\psi(|y|)/t \ge \psi(|x|)/t \ge \psi(\varepsilon_0\psi^{-1}(t/T)/8)/t \ge c_2$. Thus, (4.1) holds.

Case 2: $|x| < \varepsilon_0\psi^{-1}(t/T)/8$ and $|y| \ge \varepsilon_0\psi^{-1}(t/T)$. In this case, by (1.6), we have $\psi(|x|)/t \le c_3$ and $\psi(|y|)/t \ge c_4$. Moreover, by (4.10),

$$
|x-y| \ge 7\varepsilon_0\psi^{-1}(t/T)/8 \ge 7\varepsilon_0(t/(\Lambda T))^{1/\alpha}/8.
$$

Hence, the right-hand side of (4.1) is bounded below by

$$
\frac{c_5\psi(|x|)}{t^{1+d/\alpha}}\left(\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} = \frac{c_5\psi(|x|)}{|x-y|^{d+\alpha}}.
$$

Therefore, from Lemma 4.5, since $\psi(|y|)/t \ge c_4$, we conclude that (4.1) holds in this case.

Case 3: $|x| < \varepsilon_0 \psi^{-1}(t/T)/8$ and $|y| < \varepsilon_0 \psi^{-1}(t/T)$. In this case, since $\psi(|x|) \le \psi(|y|) \le t$ and $|x-y| \le |x| + |y| < \psi^{-1}(t)$, the right-hand side of (4.1) is equal to

$$\frac{c_6 \psi(|x|)\psi(|y|)}{t^2}\left[e^{-\lambda_1 t/\psi(|y|)} t^{-d/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} + \frac{t^2}{\psi^{-1}(t)^{d+2\alpha}}\right].$$

Thus, (4.1) follows from Lemma 4.7. The proof is complete. □

4.2 Small time lower heat kernel estimates

Lemma 4.8. *For any $T \ge 1$, there exists $C = C(d,\alpha,\beta_1,\beta_2,\Lambda,T) > 0$ such that for all $t \in (0,T]$ and $x, y \in \mathbb{R}^d_0$ with $|x| \le \psi^{-1}(t/T)$ and $|y| \ge 2\psi^{-1}(t/T)$,*

$$p^\kappa(t,x,y) \ge \frac{C\psi(|x|)}{|x-y|^{d+\alpha}}.$$

Proof. Fix $t \in (0,T]$ and $x, y \in \mathbb{R}^d_0$ such that $|x| \le \psi^{-1}(t/T)$ and $|y| \ge 2\psi^{-1}(t/T)$. Let $U := B(0, 3|y|) \setminus B(0, |x|/2)$ and $r_t := (t/(\Lambda T))^{1/\alpha}$. By the strong Markov property and the joint continuity of $p^{\kappa,U}$, we have

$$\begin{aligned} p^{\kappa,U}(t,x,y) &= \mathbb{E}_x\left[p^{\kappa,U}(t - \tau^\kappa_{B(x,|x|/2)}, X^\kappa_{\tau^\kappa_{B(x,|x|/2)}}, y) : \tau^\kappa_{B(x,|x|/2)} < t\right] \\ &\ge \mathbb{E}_x\left[p^{\kappa,U}(t - \tau^\kappa_{B(x,|x|/2)}, X^\kappa_{\tau^\kappa_{B(x,|x|/2)}}, y) : \tau^\kappa_{B(x,|x|/2)} \le t/2,\ X^\kappa_{\tau^\kappa_{B(x,|x|/2)}} \in B(y,r_t)\right] \\ &\ge \mathbb{P}_x\left(\tau^\kappa_{B(x,|x|/2)} \le t/2,\ X^\kappa_{\tau^\kappa_{B(x,|x|/2)}} \in B(y,r_t)\right) \inf_{s\in[t/2,t],\, z\in B(y,r_t)} p^{\kappa,U}(s,z,y). \qquad (4.11) \end{aligned}$$

For all $s \in [t/2,t]$ and $z \in B(y,r_t)$, by (4.10), we have $|y| \wedge |z| > 2\psi^{-1}(t/T) - r_t \ge \psi^{-1}(t/T)$ and $|y| \vee |z| < |y| + r_t \le 3|y|/2$. Hence, using Proposition 2.5 (with $R = 3|y|$, $r = \psi^{-1}(t/T)/2$ and $a = 2^{\beta_2}\Lambda T$) and (1.1), we get

$$\begin{aligned} \inf_{s\in[t/2,t],\, z\in B(y,r_t)} p^{\kappa,U}(s,z,y) &\ge \inf_{s\in[t/2,t],\, z\in B(y,r_t)} p^{\kappa, B(0,3|y|)\setminus B(0,\psi^{-1}(t/T)/2)}(s,z,y) \\ &\ge c_1 \inf_{s\in[t/2,t]} \inf_{z\in B(y,r_t)} q(s,y,z) \ge c_2 \inf_{s\in[t/2,t]} s^{-d/\alpha} = c_2 t^{-d/\alpha}. \qquad (4.12) \end{aligned}$$

Besides, we note that for all $w \in B(x,|x|/2)$ and $z \in B(y,r_t)$, since $|y| \ge 2|x|$ and $|y| \ge 2r_t$,

$$|w-z| \le 2|x| + |y| + r_t \le 5|y|/2 \le 5(|y| - |x|) \le 5|y-x|. \qquad (4.13)$$

Further, by Lemma 2.7 and (1.6),

$$\mathbb{E}_x\left[\tau^\kappa_{B(x,|x|/2)} \wedge (t/2)\right] \ge \left(\psi(|x|/2) \wedge (t/2)\right) \mathbb{P}_x\left(\tau^\kappa_{B(x,|x|/2)} \ge \psi(|x|/2)\right) \ge c_3 \psi(|x|). \qquad (4.14)$$

Using (2.10) in the equality below, (4.13) in the first inequality and (4.14) in the second inequality, we obtain

$$\mathbb{P}_x\left(\tau^\kappa_{B(x,|x|/2)} \le t/2,\ X^\kappa_{\tau^\kappa_{B(x,|x|/2)}} \in B(y,r_t)\right) = \mathbb{E}_x\left[\int_0^{\tau^\kappa_{B(x,|x|/2)} \wedge (t/2)} \int_{B(y,r_t)} \frac{\mathcal{A}(d,-\alpha)}{|X^\kappa_s - z|^{d+\alpha}} dz ds\right]$$

$$\geq \frac{\mathcal{A}(d,-\alpha)\mathbb{E}_x[\tau^\kappa_{B(x,|x|/2)} \wedge (t/2)]}{5^{d+\alpha}|x-y|^{d+\alpha}} \int_{B(y,r_t)} dz \geq \frac{c_4 t^{d/\alpha}\psi(|x|)}{|x-y|^{d+\alpha}}.$$

Combining this with (4.11) and (4.12), and using the inequality $p^\kappa(t,x,y) \geq p^{\kappa,U}(t,x,y)$, we arrive at the result. □

Lemma 4.9. *For any $T \geq 1$, there exists $C = C(d,\alpha,\beta_1,\beta_2,\Lambda,T) > 0$ such that for all $t \in (0,T]$ and $x,y \in \mathbb{R}^d_0$ with $|x| \leq \psi^{-1}(t/(2T))$ and $|x| \leq |y| \leq 2\psi^{-1}(t/T)$,*

$$p^\kappa(t,x,y) \geq \frac{C\psi(|x|)\psi(|y|)}{\psi^{-1}(t/T)^{d+2\alpha}}.$$

Proof. Let $z_0 \in \mathbb{R}^d$ be such that $|z_0| = 3\psi^{-1}(t/T)$. By Lemma 4.8, for any $z \in B(z_0, \psi^{-1}(t/T))$,

$$p^\kappa(t/2,x,z) \geq \frac{c_1\psi(|x|)}{|x-z|^{d+\alpha}} \geq \frac{c_1\psi(|x|)}{(5\psi^{-1}(t/T))^{d+\alpha}}. \tag{4.15}$$

Similarly, by Lemma 4.8 and the symmetry of $p^\kappa(t,\cdot,\cdot)$, if $|y| \leq \psi^{-1}(t/(2T))$, then $p^\kappa(t/2,z,y) \geq c_2\psi(|y|)/\psi^{-1}(t/T)^{d+\alpha}$ for any $z \in B(z_0,\psi^{-1}(t/T))$. If $\psi^{-1}(t/(2T)) < |y| \leq 2\psi^{-1}(t/T)$, then by using (1.6), Proposition 2.5 (with $R = 8\psi^{-1}(t/T)$, $r = \psi^{-1}(t/(2T))/2$ and $a = 2^{\beta_2}\Lambda T$), (1.1) and (4.10), we get that for any $z \in B(z_0,\psi^{-1}(t/T))$,

$$\begin{aligned} p^\kappa(t/2,z,y) &\geq p^{\kappa,B(0,8\psi^{-1}(t/T))\setminus B(0,\psi^{-1}(t/(2T))/2)}(t/2,z,y) \\ &\geq c_3 q(t/2,z,y) \geq \frac{c_4 t/2}{(6\psi^{-1}(t/T))^{d+\alpha}} \geq \frac{c_5\psi(|y|)}{\psi^{-1}(t/T)^{d+\alpha}}. \end{aligned}$$

Hence, in both cases, we have

$$p^\kappa(t/2,z,y) \geq \frac{c_6\psi(|y|)}{\psi^{-1}(t/T)^{d+\alpha}} \quad \text{for all } z \in B(z_0,\psi^{-1}(t/T)). \tag{4.16}$$

Using the semigroup property, (4.15) and (4.16), we conclude that

$$p^\kappa(t,x,y) \geq \int_{B(z_0,\psi^{-1}(t/T))} p^\kappa(t/2,x,z)p^\kappa(t/2,z,y)dz \geq \frac{c_5\psi(|x|)\psi(|y|)}{\psi^{-1}(t/T)^{d+2\alpha}}.$$

□

Lemma 4.10. *Let $T \geq 1$. There exist $\lambda_2 = \lambda_2(\beta_2,\Lambda) > 0$ and $C = C(d,\alpha,\beta_1,\beta_2,\Lambda,T) > 0$ such that for all $t \in (0,T]$ and $x,y \in \mathbb{R}^d_0$ with $|y| \geq |x| \vee (t/(2\Lambda T))^{1/\alpha}$ and $|x-y| \leq (t/(2\Lambda T))^{1/\alpha}/4$,*

$$p^\kappa(t,x,y) \geq Ct^{-d/\alpha}e^{-\lambda_2 t/\psi(|y|)}.$$

Proof. From (2.6), using (1.7) and (1.6), we get that

$$p^\kappa(t,x,y) \geq p^{\kappa,B(y,|y|/2)}(t,x,y) \geq q^{B(y,|y|/2)}(t,x,y)\exp\Big(-t \sup_{z\in B(y,|y|/2)} \kappa(z)\Big)$$

$$\geq q^{B(y,|y|/2)}(t,x,y)\exp\Big(-t\sup_{|z|\geq|y|/2}\kappa(z)\Big)\geq q^{B(y,|y|/2)}(t,x,y)e^{-\lambda_2 t/\psi(|y|)}, \tag{4.17}$$

for some constant $\lambda_2=\lambda_2(\beta_2,\Lambda)>0$. On the other hand, since $|y|\geq 4|x-y|$ and $|y|\geq (t/(2\Lambda T))^{1/\alpha}$, applying Proposition 2.2 with $k=2(2\Lambda T)^{1/\alpha}$, we obtain

$$\begin{aligned} q^{B(y,|y|/2)}(t,x,y) &\geq c_1\bigg(1\wedge\frac{\delta_{B(y,|y|/2)}(x)^\alpha}{t}\bigg)^{1/2}\bigg(1\wedge\frac{\delta_{B(y,|y|/2)}(y)^\alpha}{t}\bigg)^{1/2}t^{-d/\alpha}\bigg(1\wedge\frac{t^{1/\alpha}}{|x-y|}\bigg)^{d+\alpha}\\ &= c_1\bigg(1\wedge\frac{(|y|/2-|x-y|)^\alpha}{t}\bigg)^{1/2}\bigg(1\wedge\frac{(|y|/2)^\alpha}{t}\bigg)^{1/2}t^{-d/\alpha}\geq c_2 t^{-d/\alpha}. \end{aligned}$$

Combining this with (4.17), we arrive at the result. □

Lemma 4.11. *For any $T\geq 1$, there exists $C=C(d,\alpha,\beta_1,\beta_2,\Lambda,T)>0$ such that for all $t\in(0,T]$ and $x,y\in\mathbb{R}^d_0$ with $|x|\leq|y|\leq 2\psi^{-1}(t/T)$,*

$$p^\kappa(t,x,y)\geq\frac{C\psi(|x|)\psi(|y|)}{t^{2+d/\alpha}}e^{-\lambda_2 t/\psi(|y|)}\bigg(1\wedge\frac{t^{1/\alpha}}{|x-y|}\bigg)^{d+\alpha}, \tag{4.18}$$

where $\lambda_2=\lambda_2(\beta_2,\Lambda)>0$ is the constant in Lemma 4.10.

Proof. Let $b_t:=(t/(2\Lambda T))^{1/\alpha}$ and let $\lambda_2>0$ be the constant in Lemma 4.10. Note that $b_t\leq\psi^{-1}(t/(2T))$ by (4.10). We consider the following three cases separately.

Case 1: $|y|\leq b_t$. By (1.6), it holds that

$$\frac{t}{\psi(|y|)}\geq\frac{t}{\Lambda|y|^{\beta_1}}\geq c_1 t^{-(\beta_1-\alpha)/\alpha}.$$

Using this and the inequality $\sup_{u\in(0,1]}u^{(d+2\alpha)/\beta_2-2-d/\alpha}e^{-c_1\lambda_2 u^{-(\beta_1-\alpha)/\alpha}}=c_2<\infty$, we see that

$$\frac{\psi(|x|)\psi(|y|)}{t^{2+d/\alpha}}e^{-\lambda_2 t/\psi(|y|)}\leq\frac{\psi(|x|)\psi(|y|)}{t^{2+d/\alpha}}e^{-c_1\lambda_2 t^{-(\beta_1-\alpha)/\alpha}}\leq\frac{c_2\psi(|x|)\psi(|y|)}{t^{(d+2\alpha)/\beta_2}}. \tag{4.19}$$

On the other hand, since $|x|\leq|y|\leq b_t\leq\psi^{-1}(t/(2T))$, from Lemma 4.9 and (1.6), we obtain

$$p^\kappa(t,x,y)\geq\frac{c_3\psi(|x|)\psi(|y|)}{\psi^{-1}(t/T)^{d+2\alpha}}\geq\frac{c_4\psi(|x|)\psi(|y|)}{(t/T)^{(d+2\alpha)/\beta_2}}.$$

Combining this with (4.19), we conclude that (4.18) holds in this case.

Case 2: $|y|>b_t$ and $|x-y|\leq b_t/4$. Since $\psi(|x|)\psi(|y|)/t^2\leq(2^{\beta_2}\Lambda)^2$ by (1.6), the result follows from Lemma 4.10.

Case 3: $|y|>b_t$ and $|x-y|>b_t/4$. Let

$$z_0:=\bigg(\frac{|y|+2^{-3-1/\alpha}b_t}{|y|}\bigg)y.$$

For any $z \in B(z_0, 2^{-3-1/\alpha} b_t)$, we have $|z| \geq |y| \geq |x|$,

$$|y - z| \leq |y - z_0| + |z_0 - z| < 2^{-2-1/\alpha} b_t = (t/(4\Lambda T))^{1/\alpha}/4 \tag{4.20}$$

and $|x - z| \leq |x - y| + |y - z| < 2|x - y|$. Hence, applying Proposition 2.5 (with $a = 2^{\beta_2+1}\Lambda$) and using (1.1), we get that for all $z \in B(z_0, 2^{-3-1/\alpha} b_t)$,

$$\begin{aligned} p^\kappa(\psi(|x|)/2, x, z) &\geq p^{\kappa, B(0, 2(|z_0|+b_t)) \setminus B(0,|x|/2)}(\psi(|x|)/2, x, z) \\ &\geq c_5 q(\psi(|x|)/2, x, z) \geq \frac{c_6 \psi(|x|)/2}{(2|x-y|)^{d+\alpha}}. \end{aligned} \tag{4.21}$$

Besides, for any $z \in B(z_0, 2^{-3-1/\alpha} b_t)$, since $t - \psi(|x|)/2 \in [t/2, t]$, $|z| \geq |y| > b_t$ and (4.20) holds, by Lemma 4.10, we obtain

$$p^\kappa(t - \psi(|x|)/2, z, y) \geq c_7 (t - \psi(|x|)/2)^{-d/\alpha} e^{-\lambda_2 (t - \psi(|x|/2))/\psi(|z|)} \geq c_7 t^{-d/\alpha} e^{-\lambda_2 t/\psi(|y|)}. \tag{4.22}$$

Using the semigroup property, (4.21) and (4.22), we arrive at

$$\begin{aligned} p^\kappa(t, x, y) &\geq \int_{B(z_0, 2^{-3-1/\alpha} b_t)} p^\kappa(\psi(|x|)/2, x, z) p^\kappa(t - \psi(|x|)/2, z, y) dz \\ &\geq \frac{c_8 \psi(|x|) t^{-d/\alpha} e^{-\lambda_2 t/\psi(|y|)}}{|x-y|^{d+\alpha}} \int_{B(z_0, 2^{-3-1/\alpha} b_t)} dz \\ &= \frac{c_9 \psi(|x|) e^{-\lambda_2 t/\psi(|y|)}}{|x-y|^{d+\alpha}} \geq \frac{c_9 \psi(|x|)\psi(|y|) e^{-\lambda_2 t/\psi(|y|)}}{t|x-y|^{d+\alpha}}. \end{aligned}$$

The proof is complete. □

Proof of Theorem 4.1 (Lower estimates). Let $t \in (0, T]$ and $x, y \in \mathbb{R}^d_0$. Without loss of generality, by symmetry, we assume that $|x| \leq |y|$. We consider three cases separately.

Case 1: $|x| \geq \psi^{-1}(t/(2T))$. In this case, we have $t \leq 2T\psi(|x|) \leq 2^{\beta_2+1}\Lambda T \psi(|x|/2)$. Using Proposition 2.5 (with $a = 2^{\beta_2+1}\Lambda T$) and (1.1), we obtain

$$p^\kappa(t, x, y) \geq p^{\kappa, B(0, 3|y|) \setminus B(0, |x|/2)}(t, x, y) \geq c_1 q(t, x, y) \geq c_2 t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.$$

Combining this with Lemma 4.2, we conclude that (4.2) holds.

Case 2: $|x| < \psi^{-1}(t/(2T))$ and $|y| \geq 2\psi^{-1}(t/T)$. By (4.10), $|x - y| \geq \psi^{-1}(t/T) \geq c_3 t^{1/\alpha}$ in this case. Hence, by Lemma 4.8, we have

$$p^\kappa(t, x, y) \geq \frac{c_3 \psi(|x|)}{|x-y|^{d+\alpha}} \geq \frac{c_4 \psi(|x|)}{t} t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.$$

Using Lemma 4.2 again, we obtain (4.2).

Case 3: $|x| < \psi^{-1}(t/(2T))$ and $|y| < 2\psi^{-1}(t/T)$. Note that $|x - y| \leq |x| + |y| < 3\psi^{-1}(t)$. Thus, from Lemmas 4.9 and 4.11, (4.2) follows.

The proof is complete. □

5 Key proposition for large time estimates

Starting from this section, we assume that $\kappa \in \mathcal{K}_\alpha(\psi, \Lambda)$. Recall that $\widetilde{q}(t,x,y)$ is defined by (1.2). By [9, (9)], the following 3P inequality holds for $\widetilde{q}(t,x,y)$.

Proposition 5.1. *There exists $C = C(d,\alpha) > 0$ such that for all $t > s > 0$ and $x, y, z \in \mathbb{R}^d$,*

$$\frac{\widetilde{q}(t-s,x,z)\widetilde{q}(s,z,y)}{\widetilde{q}(t,x,y)} \le C(\widetilde{q}(t-s,x,z) + \widetilde{q}(s,y,z)).$$

The next proposition is the main result of this section. The proof will be provided at the end of this section.

Proposition 5.2. *There exists $C = C(d,\alpha,\beta_1,\Lambda) > 0$ such that for all $R \ge 1$, $t \ge R^\alpha$ and $x, y \in B(0,2R)^c$,*

$$\int_0^t \int_{B(0,R)^c} \frac{q^{B(0,R)^c}(s,x,z)q^{B(0,R)^c}(t-s,z,y)}{q^{B(0,R)^c}(t,x,y)} \kappa(z) dz ds \le \frac{C}{R^{\beta_1 - \alpha}}. \tag{5.1}$$

In the remainder of this section, we fix $R \ge 1$. Note that by (1.9) and (1.6),

$$\kappa(z) \le \frac{\Lambda}{\psi(|z|)} \le \frac{\Lambda^2}{|z|^{\beta_1}} \quad \text{for all } z \in B(0,R)^c. \tag{5.2}$$

Further, we see that

$$\delta_{B(0,R)^c}(z) \ge |z|/2 \ge R \quad \text{for all } z \in B(0,2R)^c. \tag{5.3}$$

We consider the cases $d > \alpha$, $d = 1 < \alpha$ and $d = \alpha = 1$ separately.

5.1 The case of $d > \alpha$

When $d > \alpha$, we have for all $x, y \in \mathbb{R}^d$ with $x \ne y$,

$$\int_0^\infty \widetilde{q}(s,x,y) ds = \int_0^{|x-y|^\alpha} \frac{s}{|x-y|^{d+\alpha}} ds + \int_{|x-y|^\alpha}^\infty s^{-d/\alpha} ds = \frac{d+\alpha}{2(d-\alpha)|x-y|^{d-\alpha}}. \tag{5.4}$$

Lemma 5.3. *Suppose that $d > \alpha$. There exists $C = C(d,\alpha,\beta_1) > 0$ independent of R such that*

$$\int_{B(0,R)^c} \frac{dz}{|w-z|^{d-\alpha}|z|^{\beta_1}} \le \frac{C}{R^{\beta_1-\alpha}} \quad \text{for all } w \in B(0,2R)^c.$$

Proof. Let $w \in B(0,2R)^c$. We decompose the integral as follows:

$$\begin{aligned}
&\int_{B(0,R)^c} \frac{dz}{|w-z|^{d-\alpha}|z|^{\beta_1}} \\
&= \left(\int_{B(w,|w|/2)\setminus B(0,R)} + \int_{B(w,3|w|)\setminus(B(w,|w|/2)\cup B(0,R))} + \int_{B(w,3|w|)^c} \right) \frac{dz}{|w-z|^{d-\alpha}|z|^{\beta_1}} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For any $z \in B(w, |w|/2)$, we have $|z| \geq |w|/2$. Using this, we get that

$$I_1 \leq \frac{1}{(|w|/2)^{\beta_1}} \int_{B(w,|w|/2)} \frac{dz}{|w-z|^{d-\alpha}} = \frac{A_{d-1}(|w|/2)^{\alpha}}{\alpha(|w|/2)^{\beta_1}} = \frac{c_1}{|w|^{\beta_1-\alpha}} \leq \frac{c_1}{(2R)^{\beta_1-\alpha}}.$$

Set $\widetilde{\beta}_1 := \beta_1 \wedge ((d+\alpha)/2)$. Then $\alpha < \widetilde{\beta}_1 < d$. For I_2, we have

$$\begin{aligned} I_2 &\leq \frac{1}{(|w|/2)^{d-\alpha}} \int_{B(w,3|w|)\setminus(B(w,|w|/2)\cup B(0,R))} \frac{dz}{|z|^{\beta_1}} \\ &\leq \frac{1}{R^{\beta_1-\widetilde{\beta}_1}(|w|/2)^{d-\alpha}} \int_{B(0,4|w|)\setminus B(0,R)} \frac{dz}{|z|^{\widetilde{\beta}_1}} \\ &\leq \frac{c_2(4|w|)^{d-\widetilde{\beta}_1}}{R^{\beta_1-\widetilde{\beta}_1}|w|^{d-\alpha}} \leq \frac{c_3}{R^{\beta_1-\alpha}}. \end{aligned}$$

For any $z \in B(w, 3|w|)^c$, we have $|z| \geq 2|w|$ and $|w-z| \geq |z| - |w| \geq |z|$. Hence,

$$I_3 \leq \int_{B(0,2|w|)^c} \frac{dz}{|z|^{d-\alpha+\beta_1}} = \frac{c_4}{(2|w|)^{\beta_1-\alpha}} \leq \frac{c_4}{(4R)^{\beta_1-\alpha}}.$$

The proof is complete. □

Lemma 5.4. *When $d > \alpha$, (5.1) holds true.*

Proof. Let $t \geq R^{\alpha}$ and $x, y \in B(0, 2R)^c$. By Proposition 2.4(i), we have

$$q^{B(0,r)^c}(t,x,y) \geq c_1 \widetilde{q}(t,x,y). \tag{5.5}$$

On the other hand, using (1.1) and Proposition 5.1 in the first inequality below, (5.2) and (5.4) in the third, and Lemma 5.3 in the last, we see that

$$\begin{aligned} &\frac{1}{\widetilde{q}(t,x,y)} \int_0^t \int_{B(0,R)^c} q(s,x,z)q(t-s,z,y)\kappa(z)dzds \\ &\leq c_2 \int_0^t \int_{B(0,R)^c} (\widetilde{q}(s,x,z) + \widetilde{q}(t-s,y,z))\kappa(z)dzds \\ &\leq 2c_2 \sup_{w \in B(0,2R)^c} \int_0^{\infty} \int_{B(0,R)^c} \widetilde{q}(s,w,z)\kappa(z)dzds \\ &\leq c_3 \sup_{w \in B(0,2R)^c} \int_{B(0,R)^c} \frac{dz}{|w-z|^{d-\alpha}|z|^{\beta_1}} \\ &\leq c_4 R^{-(\beta_1-\alpha)}. \end{aligned} \tag{5.6}$$

Combining (5.5) with (5.6), we arrive at (5.1). □

5.2 The case of $d = 1 < \alpha$

Throughout this subsection, we assume that $d = 1 < \alpha$. Recall that $R \geq 1$ is fixed. Define

$$h_R(s,z) := 1 \wedge \frac{|z|^{\alpha-1} R^{(2-\alpha)/2}}{s^{(\alpha-1)/\alpha}(s \wedge R^\alpha)^{(2-\alpha)/(2\alpha)}} \quad \text{for } s > 0 \text{ and } z \in B(0,R)^c.$$

By Proposition 2.4(ii) and (5.3), there exists $C = C(d,\alpha) > 0$ such that

$$q^{B(0,R)^c}(t,x,y) \geq C h_R(t,x) h_R(t,y) \widetilde{q}(t,x,y) \quad \text{for all } t \geq R^\alpha \text{ and } x,y \in B(0,2R)^c. \tag{5.7}$$

For any $z \in B(0,R)^c$ and $s \leq R^\alpha$, it holds that $h_R(s,z) = 1$. Hence,

$$h_R(s,z) = 1 \wedge \frac{|z|^{\alpha-1}}{s^{(\alpha-1)/\alpha}}, \quad s > 0, \ z \in B(0,R)^c.$$

Note that, for each fixed $z \in B(0,R)^c$, the map $s \mapsto h_R(s,z)$ is non-increasing and

$$s^{(\alpha-1)/\alpha} h_R(s,z) \leq u^{(\alpha-1)/\alpha} h_R(u,z) \quad \text{for all } u \geq s > 0. \tag{5.8}$$

Lemma 5.5. *Suppose that $d = 1 < \alpha$. There exists $C = C(\alpha,\beta_1,\Lambda) > 0$ independent of R such that for all $t \geq R^\alpha$ and $v \in B(0,2R)^c$,*

$$\int_0^t \int_{B(0,R)^c} h_R(s,v) h_R(s,z) h_R(t,z) \kappa(z) dz ds \leq \frac{C t^{1/\alpha} h_R(t,v)}{R^{\beta_1-\alpha}}.$$

Proof. Let $t \geq R^\alpha$ and $v \in R(0,2R)^c$. Using (5.2) and the fact that $h_R \leq 1$ in the first inequality below, and (5.8) in the second, we obtain

$$\begin{aligned}
&\int_0^t \int_{B(0,R)^c} h_R(s,v) h_R(s,z) h_R(t,z) \kappa(z) dz ds \leq \frac{\Lambda^2}{t^{(\alpha-1)/\alpha}} \int_0^t \int_{B(0,R)^c} \frac{h_R(s,v)|z|^{\alpha-1}}{|z|^{\beta_1}} dz ds \\
&\leq \Lambda^2 h_R(t,v) \int_0^t \int_{B(0,R)^c} \frac{1}{s^{(\alpha-1)/\alpha}|z|^{\beta_1-\alpha+1}} dz ds = c_1 R^{-\beta_1+\alpha} t^{1/\alpha} h_R(t,v).
\end{aligned}$$

□

Lemma 5.6. *Suppose that $d = 1 < \alpha$. There exists $C = C(\alpha,\beta_1,\Lambda) > 0$ independent of R such that for all $t \geq R^\alpha$ and $v \in B(0,2R)^c$,*

$$\int_0^t \int_{B(0,R)^c} h_R(s,v) h_R(s,z) h_R(t,z) \widetilde{q}(s,v,z) \kappa(z) dz ds \leq \frac{C h_R(t,v)}{R^{\beta_1-\alpha}}.$$

Proof. Let $t \geq R^\alpha$ and $v \in R(0,2R)^c$. Using (5.2), (5.8), and the inequalities $h_R \leq 1$ and $\widetilde{q}(s,\cdot,\cdot) \leq s^{-1/\alpha}$, we get

$$\int_0^t \int_{B(0,R)^c} h_R(s,v) h_R(s,z) h_R(t,z) \widetilde{q}(s,v,z) \kappa(z) dz ds$$

$$
\begin{aligned}
&\le \frac{\Lambda^2}{t^{(\alpha-1)/\alpha}} \int_0^t \int_{B(0,R)^c} \frac{h_R(s,v)\widetilde{q}(s,v,z)}{|z|^{\beta_1-\alpha+1}} dzds \\
&\le \Lambda^2 h_R(t,v) \int_0^t \int_{B(0,R)^c} \frac{h_R(s,z)\widetilde{q}(s,v,z)}{s^{(\alpha-1)/\alpha}|z|^{\beta_1-\alpha+1}} dzds \\
&\le \Lambda^2 h_R(t,v) \int_0^{R^\alpha} \int_{B(0,R)^c} \frac{\widetilde{q}(s,v,z)}{s^{(\alpha-1)/\alpha}|z|^{\beta_1-\alpha+1}} dzds + \Lambda^2 h_R(t,v) \int_{R^\alpha}^t \int_{B(0,R)^c} \frac{h_R(s,z)}{s|z|^{\beta_1-\alpha+1}} dzds \\
&=: \Lambda^2 h_R(t,v)(I_1 + I_2).
\end{aligned}
$$

By using (2.1), we have

$$
I_1 \le \frac{1}{R^{\beta_1-\alpha+1}} \int_0^{R^\alpha} \int_{B(0,R)^c} \frac{\widetilde{q}(s,v,z)}{s^{(\alpha-1)/\alpha}} dzds \le \frac{c_1}{R^{\beta_1-\alpha+1}} \int_0^{R^\alpha} \frac{ds}{s^{(\alpha-1)/\alpha}} = \frac{\alpha c_1}{R^{\beta_1-\alpha}}.
$$

On the other hand, we observe that

$$
\begin{aligned}
I_2 &\le \int_{R^\alpha}^t \int_{B(0,s^{1/\alpha})\setminus B(0,R)} \frac{|z|^{\alpha-1}}{s^{1+(\alpha-1)/\alpha}|z|^{\beta_1-\alpha+1}} dzds + \int_{R^\alpha}^t \int_{B(0,s^{1/\alpha})^c} \frac{1}{s|z|^{\beta_1-\alpha+1}} dzds \\
&=: I_{2,1} + I_{2,2}.
\end{aligned}
$$

Set $\eta_1 := ((\alpha-1) \wedge (\beta_1-\alpha))/2$. For $I_{2,1}$, we have

$$
I_{2,1} \le \int_{R^\alpha}^t \int_{B(0,s^{1/\alpha})\setminus B(0,R)} \frac{(s^{1/\alpha})^{\alpha-1-\eta_1}}{s^{1+(\alpha-1)/\alpha}|z|^{\beta_1-\alpha+1-\eta_1}} dzds \le \frac{c_2}{R^{\beta_1-\alpha-\eta_1}} \int_{R^\alpha}^t \frac{1}{s^{1+\eta_1/\alpha}} ds \le \frac{c_3}{R^{\beta_1-\alpha}}.
$$

For $I_{2,2}$, we see that

$$
I_{2,2} = c_4 \int_{R^\alpha}^t \frac{ds}{s^{1+(\beta_1-\alpha)/\alpha}} \le \frac{c_5}{R^{\beta_1-\alpha}}.
$$

The proof is complete. □

Lemma 5.7. *When $d = 1 < \alpha$, (5.1) holds true.*

Proof. Let $t \ge R^\alpha$ and $x, y \in B(0,2R)^c$. Using Proposition 2.4(ii) and Proposition 5.1 in the first inequality below, (5.8) and the inequality $\widetilde{q}(t-s,w,z) \le (t/2)^{-1/\alpha}$ for all $s \in (0,t/2]$ and $w,z \in \mathbb{R}^d$ in the second, and Lemmas 5.5 and 5.6 in the last, we obtain

$$
\begin{aligned}
&\frac{1}{\widetilde{q}(t,x,y)} \int_0^t \int_{B(0,R)^c} q^{B(0,R)^c}(s,x,z) q^{B(0,R)^c}(t-s,z,y)\kappa(z) dzds \\
&\le c_2 \int_0^t \int_{B(0,R)^c} h_R(s,x)h_R(s,z)h_R(t-s,y)h_R(t-s,z)(\widetilde{q}(s,x,z) + \widetilde{q}(t-s,y,z))\kappa(z) dzds \\
&\le c_3 h_R(t,y) \int_0^{t/2} \int_{B(0,R)^c} h_R(s,x)h_R(s,z)h_R(t,z)\widetilde{q}(s,x,z)\kappa(z) dzds \\
&\quad + c_3 t^{-1/\alpha} h_R(t,y) \int_0^{t/2} \int_{B(0,R)^c} h_R(s,x)h_R(s,z)h_R(t,z)\kappa(z) dzds
\end{aligned}
$$

$$
\begin{aligned}
&+ c_3 h_R(t,x)\int_0^{t/2}\int_{B(0,R)^c} h_R(s,y)h_R(s,z)h_R(t,z)\widetilde{q}(s,y,z)\kappa(z)dzds \\
&+ c_3 t^{-1/\alpha} h_R(t,x)\int_0^{t/2}\int_{B(0,R)^c} h_R(s,y)h_R(s,z)h_R(t,z)\kappa(z)dzds \\
\le\, & c_4 R^{-\beta_1+\alpha} h_R(t,x)h_R(t,y).
\end{aligned}
$$

Combining this with (5.7), we conclude that (5.1) holds when $d = 1 < \alpha$. □

5.3 The case of $d = 1 = \alpha$

In this subsection, we assume that $d = 1 = \alpha$. Recall that $\operatorname{Log} r$ is defined in (1.3). Note that

$$\operatorname{Log} sr \le \operatorname{Log} s + \operatorname{Log} r \quad \text{for all } r, s > 0, \tag{5.9}$$

$$\operatorname{Log} r \ge 1 \quad \text{for all } r \ge 1 \tag{5.10}$$

and

$$\operatorname{Log} r \le \frac{r \operatorname{Log} a}{a} \le r \quad \text{for all } r \ge a \ge 1. \tag{5.11}$$

Moreover, for any $\varepsilon > 0$, there exists $c(\varepsilon) \ge 1$ such that

$$\frac{\operatorname{Log} u}{\operatorname{Log} s} \le c(\varepsilon)\left(\frac{u}{s}\right)^{\varepsilon} \quad \text{for all } u \ge s > 0. \tag{5.12}$$

Define for $s > 0$ and $z \in B(0,R)^c$,

$$f_R(s,z) := 1 \wedge \frac{R^{1/2} \operatorname{Log}(|z|/R)}{(s \wedge R)^{1/2} \operatorname{Log}(s/R)}.$$

By Proposition 2.4(iii), using (5.3), we see that there exists $C = C(d,\alpha) > 0$ such that

$$q^{B(0,R)^c}(t,x,y) \ge C f_R(t,x) f_R(t,y)\widetilde{q}(t,x,y) \quad \text{for all } t \ge R \text{ and } x, y \in B(0,2R)^c. \tag{5.13}$$

For any $s \le R$ and $z \in B(0,R)^c$, $f_R(s,z) = 1$. Hence

$$f_R(s,z) = 1 \wedge \frac{\operatorname{Log}(|z|/R)}{\operatorname{Log}(s/R)}.$$

Consequently, for each fixed $z \in B(0,R)^c$, we have

$$f_R(s,z)\operatorname{Log}(s/R) \le f_R(u,z)\operatorname{Log}(u/R) \quad \text{for all } u \ge s > 0. \tag{5.14}$$

Lemma 5.8. *Suppose that $d = 1 = \alpha$. There exists $C = C(\beta_1, \Lambda) > 0$ independent of R such that for all $t \ge R$ and $v \in B(0,2R)^c$,*

$$\int_0^t \int_{B(0,R)^c} f_R(s,v) f_R(s,z) f_R(t,z)\kappa(z)dzds \le \frac{Ct f_R(t,v)}{R^{\beta_1 - 1}}.$$

Proof. Let $t \geq R$ and $v \in R(0,2R)^c$. Using (5.2) and $f_R(s,z) \leq 1$ in the first inequality below, (5.14) in the second, and (5.12) twice (with $\varepsilon = 1/2$ and $\varepsilon = (\beta_1 - 1)/2$) in the third, we obtain

$$\begin{aligned}
&\int_0^t \int_{B(0,R)^c} f_R(s,v) f_R(s,z) f_R(t,z) \kappa(z) dz ds \\
&\leq \frac{\Lambda^2}{\operatorname{Log}(t/R)} \int_0^t \int_{B(0,R)^c} \frac{f_R(s,v) \operatorname{Log}(|z|/R)}{|z|^{\beta_1}} dz ds \\
&\leq \Lambda^2 f_R(t,v) \int_0^t \int_{B(0,R)^c} \frac{\operatorname{Log}(|z|/R)}{|z|^{\beta_1} \operatorname{Log}(s/R)} dz ds \\
&\leq \frac{c_1 t^{1/2} f_R(t,v) \operatorname{Log} 1}{R^{(\beta_1-1)/2} \operatorname{Log}(t/R)} \int_0^t \frac{ds}{s^{1/2}} \int_{B(0,R)^c} \frac{dz}{|z|^{(\beta_1+1)/2}} \\
&= \frac{c_2 t f_R(t,v)}{R^{\beta_1 - 1} \operatorname{Log}(t/R)} \leq \frac{c_2 t f_R(t,v)}{R^{\beta_1-1}}.
\end{aligned}$$

□

Lemma 5.9. *Suppose that $d = 1 = \alpha$. There exists $C = C(\beta_1, \Lambda) > 0$ independent of R such that for all $t \geq R$ and $v \in B(0,2R)^c$,*

$$\int_0^t \int_{B(0,R)^c} f_R(s,v) f_R(s,z) f_R(t,z) \widetilde{q}(s,v,z) \kappa(z) dz ds \leq \frac{C f_R(t,v)}{R^{\beta_1-1}}.$$

Proof. Let $t \geq R$ and $v \in R(0,2R)^c$. Using (5.2), (5.14), and the inequalities $f_R \leq 1$ and $\widetilde{q}(s,\cdot,\cdot) \leq s^{-1}$, we get

$$\begin{aligned}
&\int_0^t \int_{B(0,R)^c} f_R(s,v) f_R(s,z) f_R(t,z) \widetilde{q}(s,v,z) \kappa(z) dz ds \\
&\leq \frac{\Lambda^2}{\operatorname{Log}(t/R)} \int_0^t \int_{B(0,R)^c} \frac{f_R(s,v) f_R(s,z) \widetilde{q}(s,v,z) \operatorname{Log}(|z|/R)}{|z|^{\beta_1}} dz ds \\
&\leq \Lambda^2 f_R(t,v) \int_0^t \int_{B(0,R)^c} \frac{f_R(s,z) \widetilde{q}(s,v,z) \operatorname{Log}(|z|/R)}{|z|^{\beta_1} \operatorname{Log}(s/R)} dz ds \\
&\leq \Lambda^2 f_R(t,v) \int_0^R \int_{B(0,R)^c} \frac{\widetilde{q}(s,v,z) \operatorname{Log}(|z|/R)}{|z|^{\beta_1} \operatorname{Log}(s/R)} dz ds \\
&\quad + \Lambda^2 f_R(t,v) \int_R^t \int_{B(0,R)^c} \frac{(\operatorname{Log}(|z|/R))^2}{s|z|^{\beta_1} (\operatorname{Log}(s/R))^2} dz ds \\
&=: \Lambda^2 f_R(t,v)(I_1 + I_2).
\end{aligned}$$

Using (5.12) twice (with $\varepsilon = 1/2$ and $\varepsilon = \beta_1$) in the first inequality below and (2.1) in the second, we obtain

$$I_1 \leq \frac{c_1 R^{1/2} \operatorname{Log} 1}{R^{\beta_1} \operatorname{Log} 1} \int_0^R \int_{B(0,R)^c} \frac{\widetilde{q}(s,v,z)}{s^{1/2}} dz ds \leq \frac{c_2}{R^{\beta_1 - 1/2}} \int_0^R \frac{ds}{s^{1/2}} = \frac{2c_2}{R^{\beta_1-1}}.$$

For I_2, using (5.12) (with $\varepsilon = (\beta_1 - 1)/4$), we see that

$$\begin{aligned} I_2 &\le \frac{c_3 (\mathrm{Log}\, 1)^2}{R^{(\beta_1-1)/2}} \int_R^t \int_{B(0,R)^c} \frac{1}{s(\mathrm{Log}\,(s/R))^2\, |z|^{(\beta_1+1)/2}} dz ds \\ &= \frac{c_4}{R^{\beta_1-1}} \int_R^t \frac{1}{s(\mathrm{Log}\,(s/R))^2} ds \\ &\le \frac{c_4}{R^{\beta_1-1}} \int_R^\infty \frac{e}{(s+(e-1)R)(\mathrm{Log}\,(s/R))^2} ds = \frac{e c_4}{R^{\beta_1-1}\,\mathrm{Log}\, 1}. \end{aligned}$$

The proof is complete. □

Lemma 5.10. *When $d = 1 = \alpha$, (5.1) holds true.*

Proof. Let $t \ge R$ and $x, y \in B(0, 2R)^c$. Using Proposition 2.4(iii) and Proposition 5.1 in the first inequality below, (5.14) and the inequality $\widetilde{q}(t-s, w, z) \le (t/2)^{-1}$ for all $s \in (0, t/2]$ and $w, z \in \mathbb{R}^d$ in the second, and Lemmas 5.8 and 5.9 in the last, we obtain

$$\begin{aligned} &\frac{1}{\widetilde{q}(t,x,y)} \int_0^t \int_{B(0,R)^c} q^{B(0,R)^c}(s,x,z) q^{B(0,R)^c}(t-s,z,y) \kappa(z) dz ds \\ &\le c_2 \int_0^t \int_{B(0,R)^c} f_R(s,x) f_R(s,z) f_R(t-s,y) f_R(t-s,z) (\widetilde{q}(s,x,z) + \widetilde{q}(t-s,y,z)) \kappa(z) dz ds \\ &\le c_3 f_R(t,y) \int_0^{t/2} \int_{B(0,R)^c} f_R(s,x) f_R(s,z) f_R(t,z) \widetilde{q}(s,x,z) \kappa(z) dz ds \\ &\quad + c_3 t^{-1} f_R(t,y) \int_0^{t/2} \int_{B(0,R)^c} f_R(s,x) f_R(s,z) f_R(t,z) \kappa(z) dz ds \\ &\quad + c_3 f_R(t,x) \int_0^{t/2} \int_{B(0,R)^c} f_R(s,y) f_R(s,z) f_R(t,z) \widetilde{q}(s,y,z) \kappa(z) dz ds \\ &\quad + c_3 t^{-1} f_R(t,x) \int_0^{t/2} \int_{B(0,R)^c} f_R(s,y) f_R(s,z) f_R(t,z) \kappa(z) dz ds \\ &\le \frac{c_4 f_R(t,x) f_R(t,y)}{R^{\beta_1-1}}. \end{aligned}$$

Combining this with (5.13), we conclude that (5.1) holds. □

Now, the proof of Proposition 5.2 is straightforward.

Proof of Proposition 5.2. The result follows from Lemmas 5.4, 5.7 and 5.10. □

6 Large time estimates

Throughout this section, we continue to assume that $\kappa \in \mathcal{K}_\alpha(\psi, \Lambda)$. The goal of this section is to establish the following large time estimates for $p^\kappa(t, x, y)$.

Theorem 6.1. *Suppose that $\kappa \in \mathcal{K}_\alpha(\psi, \Lambda)$. Then there exist comparison constants depending only on $d, \alpha, \beta_1, \beta_2$ and Λ such that the following estimates hold for all $t \ge 2$ and $x, y \in \mathbb{R}^d_0$:*

$$\frac{p^\kappa(t,x,y)}{\widetilde{q}(t,x,y)} \asymp \begin{cases} (1 \wedge \psi(|x|))(1 \wedge \psi(|y|)) & \text{if } d > \alpha, \\ \left(1 \wedge \frac{\psi(|x|) \wedge |x|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right)\left(1 \wedge \frac{\psi(|y|) \wedge |y|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right) & \text{if } d = 1 < \alpha, \\ \left(1 \wedge \frac{\psi(|x|) \wedge \mathrm{Log}\,|x|}{\mathrm{Log}\, t}\right)\left(1 \wedge \frac{\psi(|y|) \wedge \mathrm{Log}\,|y|}{\mathrm{Log}\, t}\right) & \text{if } d = 1 = \alpha. \end{cases}$$

6.1 Large time lower heat kernel estimates

Recall that for any open subset U of $\mathbb{R}^d_0$ with $\overline{U} \subset \mathbb{R}^d_0$,

$$p^{\kappa,U}(t,x,y) = q^U(t,x,y) + \sum_{k=1}^\infty p_k^{\kappa,U}(t,x,y)$$

where $p_k^{\kappa,U}(t,x,y)$, $k \ge 1$, are defined by (2.7), and $p^\kappa(t,x,y)$ is defined as the increasing limit of $p^{\kappa,B(0,1/n)^c}(t,x,y)$ as $n \to \infty$.

Lemma 6.2. *There exists $R_1 = R_1(d,\alpha,\beta_1,\Lambda) \ge 2$ such that for all $t \ge R_1^\alpha$ and $x, y \in B(0,2R_1)^c$,*

$$p^\kappa(t,x,y) \ge \frac{1}{2} q^{B(0,R_1)^c}(t,x,y).$$

Proof. By Proposition 5.2, there exists $R_1 = R_1(d,\alpha,\beta_1,\Lambda) \ge 2$ such that for all $t \ge R_1^\alpha$ and $x, y \in B(0,2R_1)^c$,

$$\int_0^t \int_{B(0,R_1)^c} \frac{q^{B(0,R_1)^c}(s,x,z) q^{B(0,R_1)^c}(t-s,z,y)}{q^{B(0,R_1)^c}(t,x,y)} \kappa(z) dz ds \le \frac{1}{3}. \tag{6.1}$$

Let $t \ge R_1^\alpha$ and $x, y \in B(0,2R_1)^c$. By using induction and (6.1), we see that

$$|p_k^{\kappa,B(0,R_1)^c}(t,x,y)| \le 3^{-k} q^{B(0,R_1)^c}(t,x,y) \quad \text{for all } k \ge 1.$$

Therefore, we conclude that

$$p^\kappa(t,x,y) \ge p^{\kappa,B(0,R_1)^c}(t,x,y) \ge q^{B(0,R_1)^c}(t,x,y) - q^{B(0,R_1)^c}(t,x,y) \sum_{k=1}^\infty 3^{-k} = \frac{1}{2} q^{B(0,R_1)^c}(t,x,y).$$

□

Proof of Theorem 6.1 (Lower estimates). Let $t \ge 2$ and $x, y \in \mathbb{R}^d_0$. Without loss of generality, we assume that $|x| \le |y|$. Let $R_1 \ge 2$ be the constant in Lemma 6.2 and $z_0 \in \mathbb{R}^d$ be such that $|z_0| = 3R_1$. We deal with four cases separately.

Case 1: $1 \le t \le 4R_1^\alpha$ and $|y| < 8R_1$. Note that $|x-y| \le |x| + |y| < 16R_1$ and $t \asymp \psi^{-1}(t) \asymp 1$. In particular, $\widetilde{q}(t,x,y) \asymp 1$. Applying Theorem 4.1 with $T = 4R_1^\alpha$, we obtain

$$p^\kappa(t,x,y) \ge c_1 \left(1 \wedge \frac{\psi(|x|)}{t}\right)\left(1 \wedge \frac{\psi(|y|)}{t}\right) \frac{t^2}{\psi^{-1}(t)^{d+2\alpha}} \left(1 \wedge \frac{\psi^{-1}(t)}{|x-y|}\right)^{d+2\alpha}$$

$$\geq c_2(1 \wedge \psi(|x|))(1 \wedge \psi(|y|)$$
$$\geq c_3(1 \wedge \psi(|x|))(1 \wedge \psi(|y|)\widetilde{q}(t,x,y).$$

Since $t^{(\alpha-1)/\alpha} \asymp \mathrm{Log}\, t \asymp 1$ in this case, this yields the desired lower bound.

Case 2: $1 \leq t \leq 4R_1^\alpha$ and $|y| \geq 8R_1$. Applying Theorem 4.1 with $T = 4R_1^\alpha$, we obtain

$$\begin{aligned} p^\kappa(t,x,y) &\geq c_4\left(1 \wedge \frac{\psi(|x|)}{t}\right)\left(1 \wedge \frac{\psi(|y|)}{t}\right)e^{-\lambda_2 t/\psi(|y|)}\,\widetilde{q}(t,x,y) \\ &\geq c_4\left(1 \wedge \frac{\psi(|x|)}{4R_1^\alpha}\right)\left(1 \wedge \frac{\psi(|y|)}{4R_1^\alpha}\right)e^{-4\lambda_2 R_1^\alpha/\psi(8R_1)}\,\widetilde{q}(t,x,y) \\ &\geq c_5(1 \wedge \psi(|x|))(1 \wedge \psi(|y|))\,\widetilde{q}(t,x,y). \end{aligned}$$

Using $t^{(\alpha-1)/\alpha} \asymp \mathrm{Log}\, t \asymp 1$ for $t \in [1, 4R_1^\alpha]$, we get the desired lower bound.

Case 3: $t > 4R_1^\alpha$ and $|y| < 8R_1$. For all $z \in B(z_0, R_1)$, we have $|z| > 2R_1$ and

$$|x - z| \vee |y - z| \leq |y| + |z| < 12R_1.$$

Hence, by Theorem 4.1, we get that for any $z \in B(z_0, R_1)$,

$$\begin{aligned} p^\kappa(1,x,z) &\geq c_6(1 \wedge \psi(|x|))(1 \wedge \psi(2R_1))e^{-\lambda_2/\psi(2R_1)}\left(1 \wedge (12R_1)^{-1}\right)^{d+\alpha} \\ &\geq c_7(1 \wedge \psi(|x|)) \geq c_8\psi(|x|) \end{aligned} \tag{6.2}$$

and $p^\kappa(1,z,y) \geq c_8\psi(|y|)$. Besides, for all $z, w \in B(z_0, R_1)$, since $t - 2 \geq t/2 \geq R_1^\alpha$, $|z| \wedge |w| > 2R_1$ and $|z - w| < 2R_1 \leq 2(t-2)^{1/\alpha}$, from Lemma 6.2 and Proposition 2.4, we deduce that

$$\begin{aligned} p^\kappa(t-2,z,w) &\geq \frac{1}{2}q^{B(0,R_1)^c}(t-2,z,w) \\ &\geq c_9 t^{-d/\alpha} \times \begin{cases} 1 & \text{if } d > \alpha, \\ 1 \wedge \left((2R_1)^{\alpha-1}/t^{(\alpha-1)/\alpha}\right)^2 & \text{if } d = 1 < \alpha, \\ 1 \wedge (\mathrm{Log}\,(2R_1)/\mathrm{Log}\, t)^2 & \text{if } d = 1 = \alpha \end{cases} \\ &\geq c_{10} t^{-d/\alpha} \times \begin{cases} 1 & \text{if } d > \alpha, \\ t^{-2(\alpha-1)/\alpha} & \text{if } d = 1 < \alpha, \\ (\mathrm{Log}\, t)^{-2} & \text{if } d = 1 = \alpha. \end{cases} \end{aligned}$$

Therefore, by using the semigroup property, we arrive at

$$\begin{aligned} p^\kappa(t,x,y) &\geq \int_{B(z_0,R_1)}\int_{B(z_0,R_1)} p^\kappa(1,x,z)p^\kappa(t-2,z,w)p^\kappa(1,w,y)dzdw \\ &\geq c_8^2 c_{10}\psi(|x|)\psi(|y|)t^{-d/\alpha}\int_{B(z_0,R_1)} dzdw \times \begin{cases} 1 & \text{if } d > \alpha, \\ t^{-2(\alpha-1)/\alpha} & \text{if } d = 1 < \alpha, \\ (\mathrm{Log}\, t)^{-2} & \text{if } d = 1 = \alpha. \end{cases} \end{aligned}$$

$$\geq c_{11}\psi(|x|)\psi(|y|)\widetilde{q}(t,x,y)\times\begin{cases}1 & \text{if } d>\alpha,\\ t^{-2(\alpha-1)/\alpha} & \text{if } d=1<\alpha,\\ (\operatorname{Log} t)^{-2} & \text{if } d=1=\alpha.\end{cases}$$

Case 4: $t > 4R_1^\alpha$ and $|y| \geq 8R_1$. If $|x| \geq 2R_1$, then the lower bound follows from Lemma 6.2 and Proposition 2.4. Suppose that $|x| < 2R_1$. Note that (6.2) is still valid. Further, for any $z \in B(z_0, R_1)$, since $|z| < 4R_1 < 2t^{1/\alpha}$, by (2.4) and (2.3),

$$\widetilde{q}(t-1,z,y) \geq c_{12}\widetilde{q}(t,z,y) \geq c_{13}\widetilde{q}(t,0,y) \geq c_{14}\widetilde{q}(t,x,y).$$

Thus, by Lemma 6.2 and Proposition 2.4, we get that for any $z \in B(z_0, R_1)$,

$$\begin{aligned} p^\kappa(t-1,z,y) &\geq \frac{1}{2}q^{B(0,R_1)^c}(t-1,z,y)\\ &\geq c_{15}\widetilde{q}(t-1,z,y)\times\begin{cases}1 & \text{if } d>\alpha,\\ \left(1\wedge\frac{(2R_1)^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right)\left(1\wedge\frac{|y|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right) & \text{if } d=1<\alpha,\\ \left(1\wedge\frac{\operatorname{Log}(2R_1)}{\operatorname{Log} t}\right)\left(1\wedge\frac{\operatorname{Log}|y|}{\operatorname{Log} t}\right) & \text{if } d=1=\alpha\end{cases}\\ &\geq c_{16}\widetilde{q}(t,x,y)\times\begin{cases}1 & \text{if } d>\alpha,\\ t^{-(\alpha-1)/\alpha}\left(1\wedge\frac{|y|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right) & \text{if } d=1<\alpha,\\ (\operatorname{Log} t)^{-1}\left(1\wedge\frac{\operatorname{Log}|y|}{\operatorname{Log} t}\right) & \text{if } d=1=\alpha.\end{cases}\end{aligned}$$

Combining this with (6.2) and using the semigroup property, we conclude that

$$\begin{aligned} p^\kappa(t,x,y) &\geq \int_{B(z_0,R_1)} p^\kappa(1,x,z)p^\kappa(t-1,z,y)dz\\ &\geq c_{17}\int_{B(z_0,R_1)} dz\,\psi(|x|)\widetilde{q}(t,x,y)\times\begin{cases}1 & \text{if } d>\alpha,\\ t^{-(\alpha-1)/\alpha}\left(1\wedge\frac{|y|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right) & \text{if } d=1<\alpha,\\ (\operatorname{Log} t)^{-1}\left(1\wedge\frac{\operatorname{Log}|y|}{\operatorname{Log} t}\right) & \text{if } d=1=\alpha.\end{cases}\end{aligned}$$

The proof is complete. □

6.2 Large time upper heat kernel estimates

As preparation for the proof of the upper estimates in Theorem 6.1, we first prove some lemmas.

From (1.6), we see that $\psi(r) \leq \Lambda r^{\beta_1}$ for all $r \in (0,1]$ and $\psi(r) \geq \Lambda^{-1}r^{\beta_1}$ for all $r \in [1,\infty)$. Thus, for each fixed $a \in (0,\beta_1)$ and $R > 0$, there exist comparison constants depending on a and R such that

$$\psi(r)\wedge r^a \asymp \begin{cases}\psi(r) & \text{if } r\leq R,\\ r^a & \text{if } r>R.\end{cases} \tag{6.3}$$

Lemma 6.3. *Suppose that $d = 1 < \alpha$. Then there exists $C = C(\alpha, \beta_1, \beta_2, \Lambda) \ge 1$ such that for all $t \ge 1$ and $x, y \in \mathbb{R}^1_0$,*

$$\frac{p^\kappa(t,x,y)}{\widetilde{q}(t,x,y)} \le C\bigg(1 \wedge \frac{\psi(|x|) \wedge |x|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\bigg). \tag{6.4}$$

Proof. Let $t \ge 1$ and $x, y \in \mathbb{R}^1_0$. By Proposition 2.4(ii), we get that for all $s > 0$ and $v, w \in \mathbb{R}^1_0$,

$$\begin{aligned}\frac{p^\kappa(s,v,w)}{\widetilde{q}(s,v,w)} &= \lim_{n\to\infty} \frac{p^{\kappa, B(0,1/n)^c}(s,v,w)}{\widetilde{q}(s,v,w)} \le \lim_{n\to\infty} \frac{q^{B(0,1/n)^c}(s,v,w)}{\widetilde{q}(s,v,w)} \\ &\le c_1 \lim_{n\to\infty}\bigg(1 \wedge \frac{(|v|-n^{-1})^{\alpha-1}(|v| \wedge n^{-1})^{(2-\alpha)/2}}{s^{(\alpha-1)/\alpha}(s \wedge n^{-\alpha})^{(2-\alpha)/(2\alpha)}}\bigg) = c_1\bigg(1 \wedge \frac{|v|^{\alpha-1}}{s^{(\alpha-1)/\alpha}}\bigg).\end{aligned} \tag{6.5}$$

Hence, by (6.3), we deduce that (6.4) holds if $|x| > 1$.

Suppose that $|x| \le 1$. Using the semigroup property in the first line below and Corollary 4.3 in the second, we obtain

$$\begin{aligned}p^\kappa(t,x,y) &\le \bigg(\int_{B(0,2)} + \int_{B(0,t^{1/\alpha})\setminus B(0,2)} + \int_{B(0,t^{1/\alpha})^c}\bigg) p^\kappa(1/2,x,z)p^\kappa(t-1/2,z,y)dz \\ &\le c_2\psi(|x|)\bigg(\int_{B(0,2)} + \int_{B(0,t^{1/\alpha})\setminus B(0,2)} + \int_{B(0,t^{1/\alpha})^c}\bigg) \widetilde{q}(1/2,x,z)p^\kappa(t-1/2,z,y)dz \\ &=: c_2\psi(|x|)(I_1+I_2+I_3).\end{aligned}$$

By (6.5), (2.4) and (2.3), for any $z \in B(0,t^{1/\alpha})$,

$$p^\kappa(t-1/2,z,y) \le \frac{c_1|z|^{\alpha-1}\widetilde{q}(t-1/2,z,y)}{(t-1/2)^{(\alpha-1)/\alpha}} \le \frac{c_3|z|^{\alpha-1}\widetilde{q}(t,0,y)}{t^{(\alpha-1)/\alpha}} \le \frac{c_4|z|^{\alpha-1}\widetilde{q}(t,x,y)}{t^{(\alpha-1)/\alpha}}. \tag{6.6}$$

Using this, we get

$$I_1 \le \frac{2c_4\widetilde{q}(t,x,y)}{t^{(\alpha-1)/\alpha}}\int_{B(0,2)}|z|^{\alpha-1}dz = \frac{c_5\widetilde{q}(t,x,y)}{t^{(\alpha-1)/\alpha}}.$$

For I_2, we note that for any $z \in B(0,2)^c$,

$$\widetilde{q}(1/2,x,z) \le \frac{1/2}{|x-z|^{\alpha+1}} \le \frac{2^\alpha}{|z|^{\alpha+1}}. \tag{6.7}$$

Using (6.6) and (6.7), we obtain

$$I_2 \le \frac{2^\alpha c_4\widetilde{q}(t,x,y)}{t^{(\alpha-1)/\alpha}}\int_{B(0,t^{1/\alpha})\setminus B(0,2)}\frac{dz}{|z|^2} \le \frac{c_6\widetilde{q}(t,x,y)}{t^{(\alpha-1)/\alpha}}.$$

For I_3, we consider the cases $|y| \le 2t^{1/\alpha}$ and $|y| \le 2t^{1/\alpha}$ separately. If $|y| \le 2t^{1/\alpha}$, then using (6.7), we see that

$$I_3 \le \frac{2^\alpha}{(t^{1/\alpha})^{\alpha+1}}\int_{B(0,t^{1/\alpha})^c}p^\kappa(t-1/2,z,y)dz \le \frac{2^\alpha}{t^{1+1/\alpha}} \le \frac{2^{1+2\alpha}\widetilde{q}(t,0,y)}{t} \le \frac{2^{1+2\alpha}\widetilde{q}(t,0,y)}{t^{(\alpha-1)/\alpha}}.$$

Suppose that $|y| > 2t^{1/\alpha}$. Then for any $z \in B(0, |y|/2) \setminus B(0, t^{1/\alpha})$, by (1.1) and (2.4), we get

$$p^\kappa(t-1/2, z, y) \le c_7 \widetilde{q}(t, z, y) = \frac{c_7 t}{|y-z|^{\alpha+1}} \le \frac{2^{\alpha+1} c_7 t}{|y|^{\alpha+1}}.$$

Using this and (6.7), we get that

$$\begin{aligned}
I_3 &\le 2^\alpha \Big(\int_{B(0,|y|/2)\setminus B(0,t^{1/\alpha})} + \int_{B(0,|y|/2)^c} \Big) \frac{p^\kappa(t-1/2, z, y)}{|z|^{\alpha+1}} dz \\
&\le \frac{c_8 t}{|y|^{\alpha+1}} \int_{B(0,|y|/2)\setminus B(0,t^{1/\alpha})} \frac{dz}{|z|^{\alpha+1}} + \frac{2^\alpha}{(|y|/2)^{\alpha+1}} \int_{B(0,|y|/2)^c} p^\kappa(t-1/2, z, y) dz \\
&\le \frac{c_9}{|y|^{\alpha+1}} = \frac{c_9 \widetilde{q}(t, 0, y)}{t} \le \frac{c_9 \widetilde{q}(t, 0, y)}{t^{(\alpha-1)/\alpha}}.
\end{aligned}$$

Thus, in both cases, using (2.3), we deduce that

$$I_3 \le c_{10} t^{-(\alpha-1)/\alpha} \widetilde{q}(t, x, y).$$

Now combining the estimates for I_1, I_2 and I_3 above and using (6.3), we conclude that (6.4) is also valid in this case. The proof is complete. □

For $n \ge 1$, we denote by $\mathrm{Log}^n r := \mathrm{Log} \circ \cdots \circ \mathrm{Log}\, r$ the n-th iterated function of Log.

Lemma 6.4. *Suppose that $d = 1 = \alpha$. Let $n \in \mathbb{N}$. If there exists $a_0 \ge 1$ such that*

$$\frac{p^\kappa(t, z, y)}{\widetilde{q}(t, 0, y)} \le \frac{a_0 \psi(|z|) \, \mathrm{Log}^n t}{\mathrm{Log}\, t}, \quad \text{for all } t \ge 1 \text{ and } z, y \in \mathbb{R}^1_0 \text{ with } |z| \le 1, \tag{6.8}$$

then there exists $C = C(\Lambda) > 0$ such that

$$\frac{p^\kappa(t, x, y)}{\widetilde{q}(t, 0, y)} \le \frac{C \mathrm{Log}\, a_0 (\mathrm{Log}\, |x| + \mathrm{Log}^{n+1} t)}{\mathrm{Log}\, t}, \quad \text{for all } t \ge 1 \text{ and } x, y \in \mathbb{R}^1_0 \text{ with } |x| \le t. \tag{6.9}$$

Proof. Let $t \ge 1$ and $x, y \in \mathbb{R}^1_0$ with $|x| \le t$. Set

$$\varepsilon := \left(\frac{(\mathrm{Log}\, |x|) \wedge (\mathrm{Log}^n t)}{a_0 \mathrm{Log}^n t} \right)^{1/\beta_1} \in (0, 1].$$

For all $w \in \mathbb{R}^1_0$, using Proposition 2.4(iii) in the first inequality below, (5.9) and $\beta_1 > \alpha = 1$ in the second, $\mathrm{Log}\, r > \log(e-1)$ for all $r > 0$ in the third and (5.12) in the last, we obtain

$$\begin{aligned}
\frac{q^{B(0,\varepsilon)^c}(t, x, w)}{\widetilde{q}(t, x, w)} &\le \frac{c_1 ((|x| - \varepsilon)_+ \wedge \varepsilon)^{1/2} \, \mathrm{Log}\, (|x|/\varepsilon)}{\varepsilon^{1/2} \mathrm{Log}\, (t/\varepsilon)} \\
&\le \frac{c_1 (\mathrm{Log}\, |x| + \mathrm{Log}\, (\mathrm{Log}^n t / ((\mathrm{Log}\, |x|) \wedge (\mathrm{Log}^n t)) + \mathrm{Log}\, a_0)}{\mathrm{Log}\, t} \\
&\le \frac{c_1 (\mathrm{Log}\, a_0)(\mathrm{Log}\, |x| + \mathrm{Log}\, (\mathrm{Log}^n t / (\log(e-1))))}{(\log(e-1)) \, \mathrm{Log}\, t}
\end{aligned}$$

$$\leq \frac{c_2(\operatorname{Log} a_0)(\operatorname{Log}|x| + \operatorname{Log}^{n+1} t)}{\operatorname{Log} t}. \tag{6.10}$$

We deal with the cases $|y| \leq 2t$ and $|y| > 2t$ separately.

Case 1: $|y| \leq 2t$. By (6.8), (1.6) (with $\psi(1) = 1$) and (5.12), we get that for all $w \in \mathbb{R}^1_0$,

$$\sup_{s\in(t/2,t],\, z\in B(0,\varepsilon)\setminus\{0\}} p^\kappa(s,z,w) \leq \sup_{s\in(t/2,t]} \frac{a_0\psi(\varepsilon)\operatorname{Log}^n s}{s\operatorname{Log} s} \leq \frac{c_3 a_0 \varepsilon^{\beta_1}\operatorname{Log}^n t}{t\operatorname{Log} t} \leq \frac{c_3\operatorname{Log}|x|}{t\operatorname{Log} t}.$$

Combining this with (6.10) and applying Lemma 8.1, since $\operatorname{Log} a_0 \geq \operatorname{Log} 1 = 1$ and $t \geq 1$, we deduce that

$$\begin{aligned} p^\kappa(t,x,y) &\leq \frac{c_2(\operatorname{Log} a_0)(\operatorname{Log}|x| + \operatorname{Log}^{n+1} t)}{\operatorname{Log} t} + \frac{2c_3\operatorname{Log}|x|}{t\operatorname{Log} t} \\ &\leq \frac{(c_2+2c_3)(\operatorname{Log} a_0)(\operatorname{Log}|x| + \operatorname{Log}^{n+1} t)}{\operatorname{Log} t}. \end{aligned}$$

Case 2: $|y| > 2t$. By the strong Markov property, we see that for a.e. $w \in B(0,2t)^c$,

$$p^\kappa(t,x,w) \leq p^{\kappa,B(0,\varepsilon)^c}(t,x,w) + \mathbb{E}_x[p^\kappa(t-\tau^\kappa_{B(0,\varepsilon)^c}, X^\kappa_{\tau^\kappa_{B(0,\varepsilon)^c}}, w) : \tau^\kappa_{B(0,\varepsilon)^c} < t] =: I_1 + I_2.$$

For all $w \in B(0,2t)^c$, we have $|w-x| \geq |w|/2$ so that $\widetilde{q}(t,x,w) \leq t|x-w|^{-2} \leq 4t|w|^{-2}$. Hence, by (6.10), we obtain

$$I_1 \leq q^{B(0,\varepsilon)^c}(t,x,w) \leq \frac{4c_2(\operatorname{Log} a_0)t(\operatorname{Log}|x| + \operatorname{Log}^{n+1} t)}{|w|^2\operatorname{Log} t}. \tag{6.11}$$

For I_2, we note that

$$I_2 \leq \sup_{s\in(0,t],\, z\in B(0,\varepsilon)\setminus\{0\}} p^\kappa(s,z,w). \tag{6.12}$$

For all $s \in (0,1]$ and $z \in B(0,\varepsilon)\setminus\{0\}$, since $|w| > 2t \geq 2$, we get from (1.1) that

$$p^\kappa(s,z,w) \leq q(s,z,w) \leq c_4\widetilde{q}(s,z,w) \leq \frac{c_4 s}{|w-z|^2} \leq \frac{4c_4}{|w|^2} \leq \frac{4c_4 t}{|w|^2\operatorname{Log} t}, \tag{6.13}$$

where we used (5.11) in the last inequality. Besides, for all $s \in (1,t]$ and $z \in B(0,\varepsilon)\setminus\{0\}$, using (6.8) and (1.6), we see that

$$p^\kappa(s,z,w) \leq \frac{a_0\widetilde{q}(s,0,w)\psi(|z|)\operatorname{Log}^n t}{\operatorname{Log} t} \leq \frac{a_0\Lambda\varepsilon^{\beta_1} t\operatorname{Log}^n t}{|w|^2\operatorname{Log} t} \leq \frac{\Lambda t\operatorname{Log}|x|}{|w|^2\operatorname{Log} t}. \tag{6.14}$$

By (6.12), (6.13) and (6.14), we deduce that

$$I_2 \leq \frac{4\Lambda c_4 t\operatorname{Log}|x|}{|w|^2\operatorname{Log} t}.$$

Combining this with (6.11), we obtain

$$p^\kappa(t,x,w) \le \frac{c_5(\mathrm{Log}\, a_0)t(\mathrm{Log}\,|x| + \mathrm{Log}^{n+1} t)}{|w|^2 \mathrm{Log}\, t} \quad \text{for a.e. } w \in B(0,2t)^c,$$

where $c_5 := 4c_2 + 4\Lambda c_4$. By the lower semi-continuity of p^κ, it follows that

$$\begin{aligned} p^\kappa(t,x,y) &\le \liminf_{\delta\to 0} \fint_{B(y,\delta)} p^\kappa(t,x,w)dw \\ &\le \frac{c_5(\mathrm{Log}\, a_0)t(\mathrm{Log}\,|x| + \mathrm{Log}^{n+1} t)}{|y|^2 \mathrm{Log}\, t} = \frac{c_5(\mathrm{Log}\, a_0)\widetilde{q}(t,0,y)(\mathrm{Log}\,|x| + \mathrm{Log}^{n+1} t)}{\mathrm{Log}\, t}. \end{aligned}$$

Combining Case 1 and Case 2, we conclude that (6.9) holds with $C_1 := c_2 + 2c_3 + c_5$. □

Lemma 6.5. *Suppose that $d = 1 = \alpha$. Let $n \in \mathbb{N}$. If there exists $b_0 \ge 1$ such that*

$$\frac{p^\kappa(t,x,y)}{\widetilde{q}(t,0,y)} \le \frac{b_0(\mathrm{Log}\,|x| + \mathrm{Log}^n t)}{\mathrm{Log}\, t}, \qquad \textit{for all } t \ge 1 \textit{ and } x,y \in \mathbb{R}^1_0 \textit{ with } |x| \le t, \tag{6.15}$$

then there exists $C = C(\beta_1,\beta_2,\Lambda) \ge 1$ such that

$$\frac{p^\kappa(t,z,y)}{\widetilde{q}(t,0,y)} \le \frac{C_2 b_0 \psi(|z|)\,\mathrm{Log}^n t}{\mathrm{Log}\, t}, \qquad \textit{for all } t \ge 1 \textit{ and } z,y \in \mathbb{R}^1_0 \textit{ with } |z| \le 1. \tag{6.16}$$

Proof. Let $t \ge 1$ and $z, y \in \mathbb{R}^1_0$ with $|z| \le 1$. If $t \le 10$, then by Corollary 4.3 and (2.3), we get

$$p^\kappa(t,z,y) \le c_1\psi(|z|)\widetilde{q}(t,z,y) \le c_2\psi(|z|)\widetilde{q}(t,0,y) \le \frac{(c_2 \mathrm{Log}\, 10)b_0\psi(|z|)\widetilde{q}(t,0,y)\mathrm{Log}^n t}{\mathrm{Log}\, t}.$$

Hence, by taking C_2 larger than $c_2\mathrm{Log}\, 10$, (6.16) holds in this case.

Suppose that $t > 10$. By (5.10) and (5.11), we see that $1 \le \mathrm{Log}^n t \le \mathrm{Log}^{n-1}(t/4) \le t/4$. By the semigroup property,

$$\begin{aligned} p^\kappa(t,z,y) &= \left(\int_{B(0,\mathrm{Log}^n t)} + \int_{B(0,t/2)\setminus B(0,\mathrm{Log}^n t)} + \int_{B(0,t/2)^c} \right) p^\kappa(\mathrm{Log}^n t, z, w)\, p^\kappa(t - \mathrm{Log}^n t, w, y)dw \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Since $t/2 \le t - \mathrm{Log}^n t < t$, we have $\widetilde{q}(t - \mathrm{Log}^n t, 0, y) \le c_3\widetilde{q}(t,0,y)$ and by (5.12), $\mathrm{Log}(t - \mathrm{Log}^n t) \ge c_4 \mathrm{Log}\, t$. Thus, we get from (6.15) that for all $w \in B(0,t/2)$,

$$p^\kappa(t - \mathrm{Log}^n t, w, y) \le \frac{c_3 b_0 \widetilde{q}(t,0,y)(\mathrm{Log}\,|w| + \mathrm{Log}^n t)}{c_4 \mathrm{Log}\, t}. \tag{6.17}$$

By Corollary 4.3, we have for all $w \in B(0, \mathrm{Log}^n t)$,

$$p^\kappa(\mathrm{Log}^n t, z, w) \le \frac{c_5\psi(|z|)}{\mathrm{Log}^n t}.$$

Using this and (6.17), we obtain

$$I_1 \le \frac{c_6 b_0 \psi(|z|)\widetilde{q}(t,0,y)}{\operatorname{Log} t \cdot \operatorname{Log}^n t} \int_{B(0,\operatorname{Log}^n t)} (\operatorname{Log}|w| + \operatorname{Log}^n t) dw$$
$$\le \frac{c_6 b_0 \psi(|z|)\widetilde{q}(t,0,y)(\operatorname{Log}^{n+1} t + \operatorname{Log}^n t)}{\operatorname{Log} t} \le \frac{2c_6 b_0 \psi(|z|)\widetilde{q}(t,0,y)\operatorname{Log}^n t}{\operatorname{Log} t},$$

where we used (5.11) in the last inequality. For I_2, using Corollary 4.3 and (2.3), since $|z| \le 1 \le \operatorname{Log}^n t$, we see that for all $w \in B(0,t/2) \setminus B(0,\operatorname{Log}^n t)$,

$$p^\kappa(\operatorname{Log}^n t, z, w) \le c_7 \psi(|z|)\widetilde{q}(\operatorname{Log}^n t, 0, w) \le \frac{c_7 \psi(|z|)\operatorname{Log}^n t}{|w|^2}.$$

Thus, using (6.17) and (5.12) (with $\varepsilon = 1/2$), we obtain

$$I_2 \le \frac{c_8 b_0 \psi(|z|)\widetilde{q}(t,0,y)}{\operatorname{Log} t} \int_{B(0,t/2)\setminus B(0,\operatorname{Log}^n t)} \frac{(\operatorname{Log}|w| + \operatorname{Log}^n t)\operatorname{Log}^n t}{|w|^2} dw$$
$$\le \frac{c_9 b_0 \psi(|z|)\widetilde{q}(t,0,y)(\operatorname{Log}^{n+1} t + \operatorname{Log}^n t)}{\operatorname{Log} t \cdot (\operatorname{Log}^n t)^{1/2}} \int_{B(0,t/2)\setminus B(0,\operatorname{Log}^n t)} \frac{\operatorname{Log}^n t}{|w|^{3/2}} dw$$
$$\le \frac{c_{10} b_0 \psi(|z|)\widetilde{q}(t,0,y)\operatorname{Log}^n t}{\operatorname{Log} t}.$$

For I_3, we note that for all $w \in B(0,t/2)^c$, by Corollary 4.3 and (2.3),

$$p^\kappa(\operatorname{Log}^n t, z, w) \le c_{11} \psi(|z|)\widetilde{q}(\operatorname{Log}^n t, 0, w) \le \frac{c_{11} \psi(|z|)\operatorname{Log}^n t}{|w|^2}$$

and by (1.1), since $t - \operatorname{Log}^n t \in [t/2, t]$,

$$p^\kappa(t - \operatorname{Log}^n t, w, y) \le c_{12}\widetilde{q}(t - \operatorname{Log}^n t, w, y) \le c_{13}\widetilde{q}(t,w,y) = c_{13}\widetilde{q}(t,y,w).$$

Hence, by (2.1), if $|y| \le t$, then

$$I_3 \le c_{14}\psi(|z|)\operatorname{Log}^n t \int_{B(0,t/2)^c} \frac{\widetilde{q}(t,y,w)}{|w|^2} dw$$
$$\le \frac{c_{14}\psi(|z|)\operatorname{Log}^n t}{(t/2)^2} \int_{B(0,t/2)^c} \widetilde{q}(t,y,w) dw \le \frac{c_{15}\psi(|z|)\operatorname{Log}^n t}{t^2}$$

and if $|y| > t$, then

$$I_3 \le c_{14}\psi(|z|)\operatorname{Log}^n t \int_{B(0,t/2)^c} \frac{\widetilde{q}(t,y,w)}{|w|^2} dw$$
$$\le \frac{c_{14}\psi(|z|) t \operatorname{Log}^n t}{(|y|/2)^2} \int_{B(0,|y|/2)\setminus B(0,t/2)} \frac{dw}{|w|^2} + \frac{c_{14}\psi(|z|)\operatorname{Log}^n t}{(|y|/2)^2} \int_{B(0,|y|/2)^c} \widetilde{q}(t,y,w) dw$$
$$\le \frac{c_{16}\psi(|z|)\operatorname{Log}^n t}{|y|^2}.$$

Therefore, in both cases, using (5.11), we get

$$I_3 \le \frac{(c_{15} \vee c_{16})\psi(|z|)\widetilde{q}(t,0,y)\mathrm{Log}^n\, t}{t} \le \frac{(c_{15} \vee c_{16})b_0\psi(|z|)\widetilde{q}(t,0,y)\mathrm{Log}^n\, t}{\mathrm{Log}\, t}.$$

The proof is complete. □

Lemma 6.6. *Suppose that $d = 1 = \alpha$. Then there exists $C = C(\beta_1, \beta_2, \Lambda) \ge 1$ such that for all $t \ge 1$ and $x, y \in \mathbb{R}^1_0$,*

$$\frac{p^\kappa(t,x,y)}{\widetilde{q}(t,x,y)} \le C\left(1 \wedge \frac{\psi(|x|) \wedge \mathrm{Log}\,|x|}{\mathrm{Log}\, t}\right). \tag{6.18}$$

Proof. By Corollary 4.3 and (2.3), there exists $c_1 = c_1(\beta_1, \beta_2, \Lambda) \ge 1$ such that for all $s \ge 1$ and $z, w \in \mathbb{R}^1_0$ with $|z| \le 1$,

$$p^\kappa(s,z,w) \le c_1\psi(|z|)\widetilde{q}(s,0,w).$$

Hence, (6.8) holds with $a_0 = c_1$ and $n = 1$. Applying Lemma 6.4, we get that for all $s \ge 1$ and $v, w \in \mathbb{R}^1_0$ with $|v| \le s$,

$$\frac{p^\kappa(s,v,w)}{\widetilde{q}(s,0,w)} \le \frac{C_1(\mathrm{Log}\, c_1)(\mathrm{Log}\,|v| + \mathrm{Log}^2 s)}{\mathrm{Log}\, s}.$$

Then using Lemma 6.5, we see that for all $s \ge 1$ and $z, w \in \mathbb{R}^1_0$ with $|z| \le 1$,

$$\frac{p^\kappa(s,z,w)}{\widetilde{q}(s,0,w)} \le \frac{C_1C_2(\mathrm{Log}\, c_1)\psi(|z|)\mathrm{Log}^2 s}{\mathrm{Log}\, s}$$

Iterating this procedure, we deduce that the following inequalities hold for all $n \ge 1$ and $s \ge 1$:

$$\frac{p^\kappa(s,z,w)}{\widetilde{q}(s,0,w)} \le \frac{a_n\psi(|z|)\mathrm{Log}^{n+1} s}{\mathrm{Log}\, s} \quad \text{for all } z, w \in \mathbb{R}^1_0 \text{ with } |z| \le 1 \tag{6.19}$$

and

$$\frac{p^\kappa(s,v,w)}{\widetilde{q}(s,0,w)} \le \frac{b_n(\mathrm{Log}\,|v| + \mathrm{Log}^{n+1} s)}{\mathrm{Log}\, s} \quad \text{for all } v, w \in \mathbb{R}^1_0 \text{ with } |v| \le s, \tag{6.20}$$

where the sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ are defined by $b_1 := C_1(\log c_1)$,

$$a_n := C_2 b_n \quad \text{and} \quad b_{n+1} := C_1(\mathrm{Log}\, a_n) = C_1(\mathrm{Log}\,(C_2 b_n)), \quad n \ge 1.$$

By (5.12) (with $\varepsilon = 1/2$), we have for all $r \ge 1$,

$$C_1(\mathrm{Log}\,(C_2 r)) \le c_2 C_1 (C_2 r)^{1/2} = c_3 r^{1/2},$$

where $c_3 := c_2 C_1 C_2^{1/2} \ge 1$. Hence, for all $n \ge 1$, if $b_n \ge 4c_3^2$, then $b_{n+1} \le c_3 b_n^{1/2} \le b_n/2$. It follows that $\limsup_{n\to\infty} b_n \le 4c_3^2$. Taking $\limsup_{n\to\infty}$ in (6.19) and (6.20), since $\lim_{n\to\infty} \mathrm{Log}^{n+1} s = 1$ for all $s \ge 1$ and $\mathrm{Log}\, r \ge \log(e-1)$ for all $r > 0$, we conclude that for all $s \ge 1$,

$$\frac{p^\kappa(s,z,w)}{\widetilde{q}(s,0,w)} \le \frac{4C_2c_3^2\psi(|z|)}{\mathrm{Log}\, s} \quad \text{for all } z, w \in \mathbb{R}^1_0 \text{ with } |z| \le 1 \tag{6.21}$$

and

$$\frac{p^\kappa(s,v,w)}{\widetilde{q}(s,0,w)} \le \frac{4c_3^2(1+(\log(e-1))^{-1})(\mathrm{Log}\,|v|)}{\mathrm{Log}\,s} \quad \text{for all } v,w\in\mathbb{R}^1_0 \text{ with } |v|\le s. \tag{6.22}$$

Now, we prove (6.18). Let $t\ge 1$ and $x,y\in\mathbb{R}^1_0$. If $|x|\le 1$, then using (6.21) and (2.3), we get that

$$p^\kappa(t,x,y) \le \frac{4C_2c_3^2\psi(|x|)\widetilde{q}(t,0,y)}{\mathrm{Log}\,t} \le \frac{c_4\psi(|x|)\widetilde{q}(t,x,y)}{\mathrm{Log}\,t}.$$

By (1.6) and (5.12), we have $\psi(|x|)\le \Lambda|x|^{\beta_1}\le c_5\mathrm{Log}\,|x|\le c_5\mathrm{Log}\,t$. Thus, (6.18) holds when $|x|\le 1$. If $1<|x|\le t$, then using (6.22) and (2.3), we get that

$$p^\kappa(t,x,y) \le \frac{c_6(\mathrm{Log}\,|x|)\widetilde{q}(t,0,y)}{\mathrm{Log}\,t} \le \frac{c_7(\mathrm{Log}\,|x|)\widetilde{q}(t,x,y)}{\mathrm{Log}\,t}.$$

In this case, by (5.12) and (1.6), we have

$$\mathrm{Log}\,|x| \le c_8|x|^{\beta_1} \le c_8\Lambda\psi(|x|) \quad \text{and} \quad \mathrm{Log}\,|x|\le \mathrm{Log}\,t.$$

Thus, (6.18) holds. If $|x|\ge t$, then by (1.6) and (5.12) (with $\varepsilon=\beta_1$),

$$\psi(|x|)\wedge \mathrm{Log}\,|x| \ge \psi(t)\wedge\mathrm{Log}\,t \ge \Lambda t^{\beta_1}\wedge \mathrm{Log}\,t \ge c_9\mathrm{Log}\,t.$$

Hence, since $p^\kappa(t,x,y)\le q(t,x,y)\le c_{10}\widetilde{q}(t,x,y)$ by (1.1), we conclude that (6.18) holds. □

Proof of Theorem 6.1 (Upper estimates). When $d>\alpha$, the upper bound follows from Corollary 4.3. When $d=1<\alpha$, using the semigroup property and symmetry of p^κ in the first line below, Lemma 6.3 in the second and (2.2) in the third, we get that for all $t\ge 2$ and $x,y\in\mathbb{R}^1_0$,

$$\begin{aligned}
p^\kappa(t,x,y) &= \int_{\mathbb{R}^1_0} p^\kappa(t/2,x,z)p^\kappa(t/2,y,z)dz\\
&\le c_1\left(1\wedge\frac{\psi(|x|)\wedge|x|^{\alpha-1}}{(t/2)^{(\alpha-1)/\alpha}}\right)\left(1\wedge\frac{\psi(|y|)\wedge|y|^{\alpha-1}}{(t/2)^{(\alpha-1)/\alpha}}\right)\int_{\mathbb{R}^1_0}\widetilde{q}(t/2,x,z)\widetilde{q}(t/2,y,z)dz\\
&\le c_2\left(1\wedge\frac{\psi(|x|)\wedge|x|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right)\left(1\wedge\frac{\psi(|y|)\wedge|y|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right)\widetilde{q}(t,x,y).
\end{aligned}$$

When $d=\alpha=1$, using the semigroup property and symmetry of p^κ in the first line below, Lemma 6.6 in the second, and (5.12) and (2.2) in the third, we deduce that for all $t\ge 2$ and $x,y\in\mathbb{R}^1_0$,

$$\begin{aligned}
p^\kappa(t,x,y) &= \int_{\mathbb{R}^1_0} p^\kappa(t/2,x,z)p^\kappa(t/2,y,z)dz\\
&\le c_3\left(1\wedge\frac{\psi(|x|)\wedge\mathrm{Log}\,|x|}{\mathrm{Log}\,(t/2)}\right)\left(1\wedge\frac{\psi(|y|)\wedge\mathrm{Log}\,|y|}{\mathrm{Log}\,(t/2)}\right)\int_{\mathbb{R}^1_0}\widetilde{q}(t/2,x,z)\widetilde{q}(t/2,y,z)dz\\
&\le c_4\left(1\wedge\frac{\psi(|x|)\wedge\mathrm{Log}\,|x|}{\mathrm{Log}\,t}\right)\left(1\wedge\frac{\psi(|y|)\wedge\mathrm{Log}\,|y|}{\mathrm{Log}\,t}\right)\widetilde{q}(t,x,y).
\end{aligned}$$

The proof of the theorem is complete. □

7 Green function estimates

Define

$$G^\kappa(x,y) := \int_0^\infty p^\kappa(t,x,y)dt, \quad x,y \in \mathbb{R}^d_0.$$

The goal of this section is to prove the following two-sided Green function estimates.

Theorem 7.1. *Suppose that $\kappa \in \mathcal{K}_\alpha(\psi,\Lambda)$. Then there exist comparison constants depending only on d,α,β_1,β_2 and Λ such that the following estimates hold for all $x,y \in \mathbb{R}^d_0$:*
(i) If $d > \alpha$, then

$$G^\kappa(x,y) \asymp \left(1 \wedge \frac{\psi(|x|)\wedge 1}{(|x-y|\wedge 1)^\alpha}\right)\left(1 \wedge \frac{\psi(|y|)\wedge 1}{(|x-y|\wedge 1)^\alpha}\right)\frac{1}{|x-y|^{d-\alpha}}. \tag{7.1}$$

(ii) If $d = 1 < \alpha$, then

$$\begin{aligned}G^\kappa(x,y) \asymp &\left(1 \wedge \frac{\psi(|x|)\wedge 1}{(|x-y|\wedge 1)^\alpha}\right)\left(1 \wedge \frac{\psi(|y|)\wedge 1}{(|x-y|\wedge 1)^\alpha}\right)\left(1 \wedge \frac{|x|\vee 1}{|x-y|\vee 1}\right)^{\alpha-1}\left(1 \wedge \frac{|y|\vee 1}{|x-y|\vee 1}\right)^{\alpha-1} \\ &\times \big((\psi(|x|)\wedge |x|^\alpha)\vee |x-y|^\alpha\big)^{(\alpha-1)/(2\alpha)}\big((\psi(|y|)\wedge |y|^\alpha)\vee |x-y|^\alpha\big)^{(\alpha-1)/(2\alpha)}.\end{aligned} \tag{7.2}$$

(iii) If $d = 1 = \alpha$, then

$$\begin{aligned}G^\kappa(x,y) \asymp &\left(1 \wedge \frac{\psi(|x|)\wedge 1}{|x-y|\wedge 1}\right)\left(1 \wedge \frac{\psi(|y|)\wedge 1}{|x-y|\wedge 1}\right)\left(1 \wedge \frac{\mathrm{Log}\,|x|}{\mathrm{Log}\,|x-y|}\right)^{1/2}\left(1 \wedge \frac{\mathrm{Log}\,|y|}{\mathrm{Log}\,|x-y|}\right)^{1/2} \\ &\times \left[\mathrm{Log}\left(\frac{\psi(|x|)\wedge |x|}{|x-y|\wedge 1}\right)\mathrm{Log}\left(\frac{\psi(|y|)\wedge |y|}{|x-y|\wedge 1}\right)\right]^{1/2}.\end{aligned} \tag{7.3}$$

The following lemma will be used in the proof of Theorem 7.1.

Lemma 7.2. *Suppose that $\kappa \in \mathcal{K}_\alpha(\psi,\Lambda)$. Then there exist comparison constants depending only on d,α,β_1,β_2 and Λ such that the following estimates hold for all $x,y \in \mathbb{R}^d_0$ with $|x-y|\vee\psi(|x|\wedge|y|) \le 1$:*
(i) If $|x-y|^\alpha > \psi(|x|\wedge|y|)$, then

$$G^\kappa(x,y) \asymp \frac{\psi(|x|)\psi(|y|)}{|x-y|^{d+\alpha}}.$$

(ii) If $|x-y|^\alpha \le \psi(|x|\wedge|y|)$, then

$$G^\kappa(x,y) \asymp \int_{|x-y|^\alpha}^{2\psi(|x|\wedge|y|)} t^{-d/\alpha}dt \asymp \begin{cases} |x-y|^{-d+\alpha} & \text{if } d > \alpha, \\ \psi(|x|\wedge|y|)^{(\alpha-1)/\alpha} & \text{if } d = 1 < \alpha, \\ \mathrm{Log}\big(\psi(|x|\wedge|y|)/|x-y|\big) & \text{if } d = 1 = \alpha. \end{cases} \tag{7.4}$$

Proof. Let $x,y \in \mathbb{R}^d_0$ be such that $|x-y|\vee\psi(|x|\wedge|y|) \le 1$. Without loss of generality, we assume that $|x| \le |y|$.

(i) Suppose that $|x-y|^\alpha > \psi(|x|)$. By (1.6), since $\beta_1 > \alpha$, we have

$$
\begin{aligned}
&\psi(|y|) \le \psi(|x|+|x-y|) \le \psi(2|x|) \vee \psi(2|x-y|) \\
&\le 2^{\beta_2}\Lambda(\psi(|x|) \vee \psi(|x-y|)) \le 2^{\beta_2}\Lambda(\psi(|x|) \vee (\Lambda|x-y|^{\beta_1})) \le 2^{\beta_2}\Lambda^2|x-y|^\alpha. \qquad (7.5)
\end{aligned}
$$

Applying Theorem 4.1 with $T = 2^{\beta_2+1}\Lambda^2$, we get

$$
\begin{aligned}
G^\kappa(x,y) &\ge c_1 \int_{\psi(|y|)}^{2\psi(|y|)} \left(1 \wedge \frac{\psi(|x|)}{t}\right)\left(1 \wedge \frac{\psi(|y|)}{t}\right) e^{-\lambda_2 t/\psi(|y|)} t^{-d/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} dt \\
&\ge c_2 \psi(|x|)\psi(|y|) \int_{\psi(|y|)}^{2\psi(|y|)} t^{-2-d/\alpha}\left(\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} dt \\
&= \frac{c_2\psi(|x|)\psi(|y|)}{|x-y|^{d+\alpha}} \int_{\psi(|y|)}^{2\psi(|y|)} \frac{dt}{t} = \frac{(\log 2)c_2\psi(|x|)\psi(|y|)}{|x-y|^{d+\alpha}}.
\end{aligned}
$$

On the other hand, by using Theorems 4.1 and 6.1, we have

$$
\begin{aligned}
G^\kappa(x,y) \le\ & c_3 \int_0^{\psi(|x|)} \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{t^2}{|x-y|^{d+2\alpha}}\right) dt \\
&+ c_3\psi(|x|) \int_{\psi(|x|)}^{\psi(|y|)} \left(\frac{1}{|x-y|^{d+\alpha}} + \frac{t}{|x-y|^{d+2\alpha}}\right) dt \\
&+ c_3\psi(|x|)\psi(|y|) \int_{\psi(|y|)}^{2^{\beta_2}\Lambda^2|x-y|^\alpha} \left(\frac{e^{-\lambda_1 t/\psi(|y|)}}{t|x-y|^{d+\alpha}} + \frac{1}{|x-y|^{d+2\alpha}}\right) dt \\
&+ c_3\psi(|x|)\psi(|y|) \int_{2^{\beta_2}\Lambda^2|x-y|^\alpha}^{2^{\beta_2}\Lambda^2} \left(\frac{e^{-\lambda_1 t/\psi(|y|)}}{t^{2+d/\alpha}} + \frac{1}{\psi^{-1}(t)^{d+2\alpha}}\right) dt \\
&+ c_3\psi(|x|)\psi(|y|) \int_{2^{\beta_2}\Lambda^2}^{\infty} \frac{dt}{t^{d/\alpha}} \\
=:\ & c_3(I_1 + I_2 + I_3 + I_4 + I_5).
\end{aligned}
$$

Since $\psi(|x|) < |x-y|^\alpha$ and $|x| \le |y|$, we have

$$
I_1 \le \frac{\psi(|x|)^2}{2|x-y|^{d+\alpha}} + \frac{\psi(|x|)^3}{3|x-y|^{d+2\alpha}} \le \frac{5\psi(|x|)^2}{6|x-y|^{d+\alpha}} \le \frac{5\psi(|x|)\psi(|y|)}{6|x-y|^{d+\alpha}}.
$$

For I_2, using (7.5), we obtain

$$
I_2 \le \frac{\psi(|x|)\psi(|y|)}{|x-y|^{d+\alpha}} + \frac{\psi(|x|)\psi(|y|)^2}{2|x-y|^{d+2\alpha}} \le \frac{(2^{\beta_2}\Lambda^2+1)\psi(|x|)\psi(|y|)}{|x-y|^{d+\alpha}}.
$$

For I_3, using the inequality $e^{-r} \le 1/r$ for $r > 0$, we see that

$$
I_3 \le \frac{\psi(|x|)\psi(|y|)^2}{\lambda_1|x-y|^{d+\alpha}} \int_{\psi(|y|)}^{\infty} \frac{dt}{t^2} + \frac{\psi(|x|)\psi(|y|)}{|x-y|^{d+\alpha}} = \frac{(\lambda_1^{-1}+1)\psi(|x|)\psi(|y|)}{|x-y|^{d+\alpha}}.
$$

For I_4, using (4.10), we get

$$I_4 \le \psi(|x|)\psi(|y|) \int_{2^{\beta_2}\Lambda^2|x-y|^\alpha}^{2^{\beta_2}\Lambda^2} \left(\frac{1}{t^{2+d/\alpha}} + \frac{c_4}{t^{2+d/\alpha}} \right) dt \le \frac{c_5\psi(|x|)\psi(|y|)}{|x-y|^{d+\alpha}}.$$

For I_5, since $|x-y| \le 1$, we have

$$I_5 \le c_6\psi(|x|)\psi(|y|) \le \frac{c_6\psi(|x|)\psi(|y|)}{|x-y|^{d+\alpha}}.$$

The proof of (i) is complete.

(ii) The second comparison in (7.4) is straightforward. We now prove the first comparison. Applying Theorem 4.1 with $T=2$, we get

$$\begin{aligned} G^\kappa(x,y) &\ge c_1 \int_{|x-y|^\alpha}^{2\psi(|x|)} \left(1 \wedge \frac{\psi(|x|)}{t}\right)\left(1 \wedge \frac{\psi(|y|)}{t}\right) e^{-\lambda_2 t/\psi(|y|)} t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} dt \\ &\ge 2^{-2}c_1 e^{-2\lambda_2} \int_{|x-y|^\alpha}^{2\psi(|x|)} t^{-d/\alpha} dt. \end{aligned}$$

On the other hand, using Corollary 4.3 and Theorem 6.1, we see that

$$\begin{aligned} G^\kappa(x,y) &\le c_2 \int_0^{|x-y|^\alpha} \frac{t}{|x-y|^{d+\alpha}} dt + c_2 \int_{|x-y|^\alpha}^{2\psi(|x|)} \frac{dt}{t^{d/\alpha}} + c_2\psi(|x|) \int_{2\psi(|x|)}^2 \frac{dt}{t^{1+d/\alpha}} \\ &\quad + c_2 \mathbf{1}_{\{d=1<\alpha\}} \psi(|x|)\psi(|y|) \int_2^\infty \frac{dt}{t^{1+(\alpha-1)/\alpha}} + c_2 \mathbf{1}_{\{d=1=\alpha\}} \psi(|x|)\psi(|y|) \int_2^\infty \frac{dt}{t(\operatorname{Log} t)^2} \\ &=: c_2(I_1 + I_2 + I_3 + \mathbf{1}_{\{d=1<\alpha\}} I_4 + \mathbf{1}_{\{d=1=\alpha\}} I_5). \end{aligned}$$

Observe that

$$I_2 \ge \int_{|x-y|^\alpha}^{2|x-y|^\alpha} \frac{dt}{t^{d/\alpha}} \ge \frac{1}{2^{d/\alpha}|x-y|^{d-\alpha}}$$

and

$$I_2 \ge \int_{\psi(|x|)}^{2\psi(|x|)} \frac{dt}{t^{d/\alpha}} \ge \frac{1}{2^{d/\alpha}\psi(|x|)^{d/\alpha-1}}. \tag{7.6}$$

Hence,

$$I_1 = 2^{-1}|x-y|^{\alpha-d} \le 2^{d/\alpha-1} I_2 \qquad \text{and} \qquad I_3 \le (\alpha/d)\psi(|x|)^{1-d/\alpha} \le 2^{d/\alpha}(\alpha/d) I_2.$$

By using (1.6) and the fact that $\psi(r) \le \Lambda r^{\beta_1} \le \Lambda r^\alpha$ for all $r \in (0,1]$, we have

$$\psi(|x|) \le \psi(|y|) \le 2^{\beta_2}\Lambda(\psi(|x|) \vee \psi(|x-y|)) \le 2^{\beta_2}\Lambda(\psi(|x|) \vee (\Lambda|x-y|^\alpha)) \le 2^{\beta_2}\Lambda^2\psi(|x|). \tag{7.7}$$

When $d=1<\alpha$, using (7.7) and (7.6), since $\psi(|x|) \le 1$, we get that

$$I_4 = c_3\psi(|x|)\psi(|y|) \le 2^{\beta_2}\Lambda^2 c_3 \psi(|x|)^2 \le 2^{\beta_2}\Lambda^2 c_3 \psi(|x|)^{1-1/\alpha} \le c_4 I_2.$$

Similarly, when $d = 1 = \alpha$, using (7.7), (7.6) and $\psi(|x|) \le 1$, we obtain

$$I_5 = c_5\psi(|x|)\psi(|y|) \le 2^{\beta_2}\Lambda^2 c_5 \psi(|x|)^2 \le 2^{\beta_2}\Lambda^2 c_5 \le c_6 I_2.$$

The proof is complete. □

Proof of Theorem 7.1. Let $x, y \in \mathbb{R}^d_0$. Without loss of generality, we assume that $|x| \le |y|$.

(i) Suppose that $d > \alpha$. By (1.1) and (5.4), we have

$$G^\kappa(x,y) \le \int_0^\infty q(t,x,y)dt \le c_1 \int_0^\infty \widetilde{q}(t,x,y)dt = \frac{c_2}{|x-y|^{d-\alpha}}. \tag{7.8}$$

We deal with four cases separately.

Case 1: $|x-y| \vee \psi(|x|) \le 1$. By (7.5), we have $\psi(|y|) \le 2^{\beta_2}\Lambda^2|x-y|^\alpha \le 2^{\beta_2}\Lambda^2$ in this case. Moreover, by Lemma 7.2, it holds that

$$G^\kappa(x,y) \asymp \left(1 \wedge \frac{\psi(|x|)}{|x-y|^\alpha}\right)\left(1 \wedge \frac{\psi(|y|)}{|x-y|^\alpha}\right)\frac{1}{|x-y|^{d-\alpha}}.$$

Hence, (7.1) holds true.

Case 2: $|x-y| \le 1 < \psi(|x|)$. Applying Theorem 4.1 (with $T = 2$), we get

$$\begin{aligned} G^\kappa(x,y) &\ge c_1 \int_{|x-y|^\alpha}^{2|x-y|^\alpha} \left(1 \wedge \frac{\psi(|x|)}{t}\right)\left(1 \wedge \frac{\psi(|y|)}{t}\right) e^{-\lambda_2 t/\psi(|y|)} t^{-d/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} dt \\ &\ge 2^{-2}c_1 e^{-2\lambda_2} \int_{|x-y|^\alpha}^{2|x-y|^\alpha} t^{-d/\alpha}dt = \frac{c_2}{|x-y|^{d-\alpha}}. \end{aligned}$$

Combining this with (7.8), we get (7.1).

Case 3: $|x-y| > 1$ and $|x| \ge 1/2$. By Theorem 6.1,

$$G^\kappa(x,y) \ge c_3(1 \wedge \psi(1/2))^2 \int_{2|x-y|^\alpha}^{3|x-y|^\alpha} \frac{t}{|x-y|^{d+\alpha}}dt = \frac{c_4}{|x-y|^{d-\alpha}}.$$

Combining this with (7.8), we get (7.1).

Case 4: $|x-y| > 1$ and $|x| < 1/2$. In this case, we have $|y| \ge |y-x| - |x| > 1/2$. Applying Theorem 6.1, we get that

$$G^\kappa(x,y) \ge c_5(1 \wedge \psi(|x|))(1 \wedge \psi(1/2)) \int_{2|x-y|^\alpha}^{3|x-y|^\alpha} \frac{t}{|x-y|^{d+\alpha}}dt \ge \frac{c_6\psi(|x|)}{|x-y|^{d-\alpha}}.$$

On the other hand, by Corollary 4.3, Theorem 6.1 and (5.4), since $|x-y| > 1$, we have

$$\begin{aligned} G^\kappa(x,y) &\le c_7 \int_0^2 \frac{\psi(|x|)}{|x-y|^{d+\alpha}}dt + c_7\psi(|x|)\int_2^\infty \widetilde{q}(t,x,y)dt \\ &\le \frac{2c_7\psi(|x|)}{|x-y|^{d+\alpha}} + \frac{c_8\psi(|x|)}{|x-y|^{d-\alpha}} \le \frac{(2c_7+c_7)\psi(|x|)}{|x-y|^{d-\alpha}}. \end{aligned}$$

The proof of (i) is complete.

(ii) Suppose that $d = 1 < \alpha$. We deal with five cases separately.

Case 1: $\psi(|x|) < |x-y|^\alpha \le 1$. Using (6.3) and (7.5), we see that the right-hand side of (7.2) is comparable to

$$\left(\frac{\psi(|x|)}{|x-y|^\alpha}\right)\left(\frac{\psi(|y|)}{|x-y|^\alpha}\right)(|x-y|^\alpha)^{(\alpha-1)/(2\alpha)}(|x-y|^\alpha)^{(\alpha-1)/(2\alpha)} = \frac{\psi(|x|)\psi(|y|)}{|x-y|^{1+\alpha}}.$$

By Lemma 7.2(i), (7.2) holds true.

Case 2: $|x-y|^\alpha \le \psi(|x|) \le 1$. By using (6.3) and (7.7), we see that the right-hand side of (7.2) is comparable to $\psi(|x|)^{(\alpha-1)/(2\alpha)}\psi(|y|)^{(\alpha-1)/(2\alpha)} \asymp \psi(|x|)^{(\alpha-1)/\alpha}$. By Lemma 7.2(ii), we get (7.2).

Case 3: $|x-y| \le 1 \le |x|$. Note that $|x| \le |y| \le |x| + |x-y| \le 2|x|$. Hence, using (6.3), we see that the right-hand side of (7.2) is comparable to $(|x|^\alpha)^{(\alpha-1)/(2\alpha)}(|y|^\alpha)^{(\alpha-1)/(2\alpha)} \asymp |x|^{\alpha-1}$ in this case. For the lower bound, using Theorem 6.1 and (6.3), we obtain

$$G^\kappa(x,y) \ge c_1 \int_{2|x|^\alpha}^{3|x|^\alpha} \left(1 \wedge \frac{|x|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right)^2 \frac{dt}{t^{1/\alpha}} \ge c_2 \int_{2|x|^\alpha}^{3|x|^\alpha} \frac{dt}{t^{1/\alpha}} = c_3|x|^{\alpha-1}.$$

For the upper bound, using Corollary 4.3, Theorem 6.1 and the inequality $|x| \le |y| \le 2|x|$, we get that

$$\begin{aligned} G^\kappa(x,y) \le\ & c_4 \int_0^{|x-y|^\alpha} \frac{t}{|x-y|^{1+\alpha}} dt + c_4 \int_{|x-y|^\alpha}^{2|x|^\alpha} \frac{dt}{t^{1/\alpha}} + c_4|x|^{\alpha-1} \int_{2|x|^\alpha}^{2|y|^\alpha} \frac{dt}{t} \\ & + c_4|x|^{\alpha-1}|y|^{\alpha-1} \int_{2|y|^\alpha}^\infty \frac{dt}{t^{1+(\alpha-1)/\alpha}} \\ \le\ & c_5\left(|x-y|^{\alpha-1} + |x|^{\alpha-1} + |x|^{\alpha-1}\log(|y|^\alpha/|x|^\alpha) + |x|^{\alpha-1}\right) \\ \le\ & (3+\log 2^\alpha)c_5|x|^{\alpha-1}. \end{aligned}$$

Case 4: $|x-y| > 1$ and $1/2 \le |x| < |x-y|$. Using (6.3), we see that the right-hand side of (7.2) is comparable to

$$\left(\frac{|x|}{|x-y|}\right)^{\alpha-1}\left(\frac{|y|}{|x-y|}\right)^{\alpha-1}\left(|x-y|^\alpha\right)^{(\alpha-1)/(2\alpha)}\left(|x-y|^\alpha\right)^{(\alpha-1)/(2\alpha)} = \frac{|x|^{\alpha-1}|y|^{\alpha-1}}{|x-y|^{\alpha-1}}.$$

By Theorem 6.1, we have

$$\begin{aligned} G^\kappa(x,y) &\ge c_6 \int_{2|x-y|^\alpha}^{3|x-y|^\alpha} \left(1 \wedge \frac{|x|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right)\left(1 \wedge \frac{|y|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right) t^{-1/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{1+\alpha} dt \\ &\ge c_7|x|^{\alpha-1}|y|^{\alpha-1} \int_{2|x-y|^\alpha}^{3|x-y|^\alpha} \frac{1}{t^{1-2/\alpha}|x-y|^{1+\alpha}} dt = \frac{c_8|x|^{\alpha-1}|y|^{\alpha-1}}{|x-y|^{\alpha-1}}. \end{aligned}$$

For the upper bound, by Corollary 4.3 and Theorem 6.1, since $|x| \le |y| \le |x| + |x-y| < 2|x-y|$, we have

$$G^\kappa(x,y) \le c_9 \int_0^{8|x|^\alpha} \frac{t}{|x-y|^{1+\alpha}} dt + c_9|x|^{\alpha-1} \int_{8|x|^\alpha}^{8|y|^\alpha} \frac{t^{1/\alpha}}{|x-y|^{1+\alpha}} dt$$

$$+ c_9|x|^{\alpha-1}|y|^{\alpha-1}\int_{8|y|^\alpha}^{8|x-y|^\alpha} \frac{t^{-1+2/\alpha}}{|x-y|^{1+\alpha}}dt + c_9|x|^{\alpha-1}|y|^{\alpha-1}\int_{8|x-y|^\alpha}^{\infty} \frac{dt}{t^{1+(\alpha-1)/\alpha}}$$
$$\le c_{10}\left(\frac{|x|^{2\alpha}}{|x-y|^{\alpha+1}} + \frac{|x|^{\alpha-1}|y|^{\alpha+1}}{|x-y|^{\alpha+1}} + \frac{|x|^{\alpha-1}|y|^{\alpha-1}}{|x-y|^{\alpha-1}} + \frac{|x|^{\alpha-1}|y|^{\alpha-1}}{|x-y|^{\alpha-1}}\right)$$
$$\le \frac{c_{11}|x|^{\alpha-1}|y|^{\alpha-1}}{|x-y|^{\alpha-1}}.$$

Case 5: $|x-y| > 1$ and $|x| < 1/2$. In this case, we have $|y| \le |x| + |x-y| \le (3/2)|x-y|$ and $|y| \ge |x-y| - |x| \ge (1/2)|x-y|$. In particular, $|y| > 1/2$ so that $\psi(|y|) \wedge |y|^\alpha \asymp |y|^\alpha$ and $\psi(|y|) \wedge |y|^{\alpha-1} \asymp |y|^{\alpha-1}$ by (6.3). Therefore, the right-hand side of (7.2) is comparable to

$$\psi(|x|)|x-y|^{1-\alpha}(|x-y|^\alpha)^{(\alpha-1)/(2\alpha)}(|x-y|^\alpha)^{(\alpha-1)/(2\alpha)} = \psi(|x|).$$

Applying Theorem 6.1, and using (6.3) and $|y| \asymp |x-y|$, we get that

$$G^\kappa(x,y) \ge c_{12}\int_{2|x-y|^\alpha}^{3|x-y|^\alpha}\left(1 \wedge \frac{\psi(|x|)}{t^{(\alpha-1)/\alpha}}\right)\left(1 \wedge \frac{|y|^{\alpha-1}}{t^{(\alpha-1)/\alpha}}\right)t^{-1/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{1+\alpha}dt$$
$$\ge c_{13}\psi(|x|)\int_{2|x-y|^\alpha}^{3|x-y|^\alpha} t^{-1}dt = (\log(3/2))c_{13}\psi(|x|).$$

On the other hand, using Corollary 4.3 and Theorem 6.1, since $|y| \asymp |x-y| > 1$, we obtain

$$G^\kappa(x,y) \le c_{14}\psi(|x|)\int_0^{2|x-y|^\alpha} \frac{1}{|x-y|^{1+\alpha}}dt + c_{14}\psi(|x|)|y|^{\alpha-1}\int_{2|x-y|^\alpha}^{\infty}\frac{1}{t^{1+(\alpha-1)/\alpha}}dt$$
$$\le c_{15}\left(\psi(|x|)|x-y|^{-1} + \psi(|x|)|y|^{\alpha-1}|x-y|^{-\alpha+1}\right) \le c_{16}\psi(|x|).$$

The proof of (ii) is complete.

(iii) Suppose that $d = 1 = \alpha$. We deal with five cases separately.

Case 1: $\psi(|x|) < |x-y| \le 1$. Note that $|x| \le \psi^{-1}(1) = 1$ and $|y| \le |x| + |x-y| \le 2$ in this case. Using (6.3), (7.5) and the fact that $\mathrm{Log}\, r \asymp 1$ for $r \in (0,2]$, we see that the right-hand side of (7.3) is comparable to $\psi(|x|)\psi(|y|)|x-y|^{-2}$ in this case. Hence, by Lemma 7.2(i), (7.3) holds.

Case 2: $|x-y| \le \psi(|x|) \le 1$. We have $|x| \le 1$ and $|y| \le |x| + |x-y| \le 2$. By (6.3), (7.7), (5.12) and the fact that $\mathrm{Log}\, r \asymp 1$ for $r \in (0,2]$, the right-hand side of (7.3) is comparable to

$$\left[\mathrm{Log}\left(\frac{\psi(|x|)}{|x-y|}\right)\mathrm{Log}\left(\frac{\psi(|y|)}{|x-y|}\right)\right]^{1/2} \asymp \mathrm{Log}\left(\frac{\psi(|x|)}{|x-y|}\right)$$

in this case. Hence, (7.3) follows from Lemma 7.2(ii).

Case 3: $|x-y| \vee 1 \le |x|$. We have $|x| \le |y| \le 2|x|$. Using this, (6.3) and (5.12), we see that the right-hand side of (7.3) is comparable to

$$\left[\mathrm{Log}\left(\frac{|x|}{|x-y| \wedge 1}\right)\mathrm{Log}\left(\frac{|y|}{|x-y| \wedge 1}\right)\right]^{1/2} \asymp \mathrm{Log}\left(\frac{|x|}{|x-y| \wedge 1}\right)$$

in this case. For the lower bound, by using Theorems 4.1 and 6.1, $\psi(r)\wedge \operatorname{Log} r \asymp \operatorname{Log} r$ for $r\ge 1$ and (5.9), since $|y|\ge |x|\ge 1$, we have

$$
\begin{aligned}
&G^\kappa(x,y)\\
&\ge c_1 \mathbf{1}_{\{|x-y|\le 1\}} \int_{|x-y|}^2 \left(1\wedge \frac{|x|}{t}\right)\left(1\wedge \frac{|y|}{t}\right) e^{-\lambda_2 t/|y|} t^{-1} \left(1\wedge \frac{t}{|x-y|}\right)^2 dt + c_1 (\operatorname{Log}|x|)^2 \int_{2|x|}^\infty \frac{dt}{t(\operatorname{Log} t)^2}\\
&\ge c_2 \mathbf{1}_{\{|x-y|\le 1\}} \int_{|x-y|}^2 t^{-1} dt + c_2 (\operatorname{Log}|x|)^2 \int_{2|x|}^\infty \frac{dt}{t(\log t)^2}\\
&= c_2 \mathbf{1}_{\{|x-y|\le 1\}} \log\left(\frac{2}{|x-y|}\right) + \frac{c_2(\operatorname{Log}|x|)^2}{\log(2|x|)}\\
&\ge c_3 \mathbf{1}_{\{|x-y|\le 1\}} \operatorname{Log}\left(\frac{1}{|x-y|}\right) + c_3 \operatorname{Log}|x| \ge c_3 \operatorname{Log}\left(\frac{|x|}{|x-y|\wedge 1}\right).
\end{aligned}
$$

For the upper bound, using Corollary 4.3, Theorem 6.1, $|x|\asymp|y|$ and (5.12), we see that

$$
\begin{aligned}
&G^\kappa(x,y)\\
&\le c_4 \int_0^{|x-y|} \frac{t}{|x-y|^2} dt + c_4 \int_{|x-y|}^{2|x|} \frac{dt}{t} + c_4 \operatorname{Log}|x| \int_{2|x|}^{2|y|} \frac{dt}{t} + c_4 (\operatorname{Log}|x|)(\operatorname{Log}|y|) \int_{2|y|}^\infty \frac{dt}{t(\operatorname{Log} t)^2}\\
&\le c_5 \left(1 + \operatorname{Log}\left(\frac{|x|}{|x-y|}\right) + (\log 2)\operatorname{Log}|x| + \operatorname{Log}|x|\right) \le c_6 \operatorname{Log}\left(\frac{|x|}{|x-y|\wedge 1}\right).
\end{aligned}
$$

Case 4: $|x-y|>1$ and $1/2\le |x|<|x-y|$. We see that the right-hand side of (7.3) is comparable to

$$
\left(\frac{\operatorname{Log}|x|}{\operatorname{Log}|x-y|}\right)^{1/2}\left(\frac{\operatorname{Log}|y|}{\operatorname{Log}|x-y|}\right)^{1/2} [(\operatorname{Log}|x|)(\operatorname{Log}|y|)]^{1/2} = \frac{(\operatorname{Log}|x|)(\operatorname{Log}|y|)}{\operatorname{Log}|x-y|}.
$$

Since $|y|\le |x|+|x-y|<2|x-y|$, by Theorem 6.1, we have

$$
G^\kappa(x,y) \ge c_7 (\operatorname{Log}|x|)(\operatorname{Log}|y|) \int_{2|x-y|}^\infty \frac{1}{t(\operatorname{Log} t)^2} dt \ge \frac{c_8 (\operatorname{Log}|x|)(\operatorname{Log}|y|)}{\operatorname{Log}|x-y|}.
$$

For the upper bound, by using Corollary 4.3 and Theorem 6.1, we obtain

$$
\begin{aligned}
G^\kappa(x,y) &\le c_9 \int_0^{4|x|} \frac{t}{|x-y|^2} dt + \frac{c_9(\operatorname{Log}|x|)}{|x-y|^2} \int_{4|x|}^{4|y|} \frac{t}{\operatorname{Log} t} dt\\
&\quad + \frac{c_9 (\operatorname{Log}|x|)(\operatorname{Log}|y|)}{|x-y|^2} \int_{4|y|}^{4|x-y|} \frac{t}{(\operatorname{Log} t)^2} dt + c_9 (\operatorname{Log}|x|)(\operatorname{Log}|y|) \int_{4|x-y|}^\infty \frac{dt}{t(\operatorname{Log} t)^2}\\
&\le c_{10} \left(\frac{|x|^2}{|x-y|^2} + \frac{(\operatorname{Log}|x|)|y|^2}{|x-y|^2 \operatorname{Log}|y|} + \frac{(\operatorname{Log}|x|)(\operatorname{Log}|y|)}{(\operatorname{Log}|x-y|)^2} + \frac{(\operatorname{Log}|x|)|y|^2}{|x-y|^2 \operatorname{Log}|y|} + \frac{(\operatorname{Log}|x|)(\operatorname{Log}|y|)}{\operatorname{Log}|x-y|} \right)\\
&\le \frac{c_{11} (\operatorname{Log}|x|)(\operatorname{Log}|y|)}{\operatorname{Log}|x-y|}.
\end{aligned}
$$

Case 5: $|x-y|>1$ and $|x|<1/2$. In this case, we have $(1/2)|x-y|\le |y|\le (3/2)|x-y|$. Hence, using (6.3), (5.12) and the fact that $\mathrm{Log}\, r \asymp 1$ for $r\in(0,1]$, we see that the right-hand side of (7.3) is comparable to

$$\psi(|x|)\left(\frac{\mathrm{Log}\,|x|}{\mathrm{Log}\,|x-y|}\right)^{1/2}[(\mathrm{Log}\,\psi(|x|))(\mathrm{Log}\,|y|)]^{1/2}\asymp \frac{\psi(|x|)(\mathrm{Log}\,|y|)^{1/2}}{(\mathrm{Log}\,|x-y|)^{1/2}}\asymp \psi(|x|).$$

By Theorem 6.1, since $\psi(|x|)\wedge \mathrm{Log}\,|x|\asymp\psi(|x|)$ and $|y|\asymp|x-y|>1$, we get that

$$\begin{aligned}G^\kappa(x,y)&\ge c_{12}\int_{2|x-y|}^{3|x-y|}\left(1\wedge\frac{\psi(|x|)}{\mathrm{Log}\,t}\right)\left(1\wedge\frac{\mathrm{Log}\,|y|}{\mathrm{Log}\,t}\right)t^{-1}\left(1\wedge\frac{t}{|x-y|}\right)^2dt\\&\ge c_{13}\psi(|x|)\int_{2|x-y|}^{3|x-y|}t^{-1}dt=(\log(3/2))c_{13}\psi(|x|).\end{aligned}$$

Besides, using Corollary 4.3, Theorem 6.1 and (5.12), since $|x-y|>1$, we obtain

$$\begin{aligned}G^\kappa(x,y)&\le c_{14}\psi(|x|)\int_0^1\frac{1}{|x-y|^2}dt+c_{14}\psi(|x|)\int_1^{2|x-y|}\frac{t}{|x-y|^2(\mathrm{Log}\,t)}dt\\&\quad+c_{14}\psi(|x|)(\mathrm{Log}\,|y|)\int_{2|x-y|}^\infty\frac{1}{t(\mathrm{Log}\,t)^2}dt\\&\le c_{15}\psi(|x|)\left(|x-y|^{-2}+(\mathrm{Log}\,|x-y|)^{-1}+(\mathrm{Log}\,|y|)/(\mathrm{Log}\,|x-y|)\right)\\&\le c_{16}\psi(|x|).\end{aligned}$$

The proof is complete. □

8 Appendix

In this appendix, we prove the following general result used in getting large time heat kernel upper bound in the case $d=1=\alpha$. We believe it is of independent interest.

Lemma 8.1. *Let $t>0$ and $\varepsilon>0$. If there exists a non-negative continuous function $F_{t,\varepsilon}$ on $\mathbb{R}^d_0$ such that*

$$\sup_{s\in(t/2,t],\, z\in B(0,\varepsilon)\setminus\{0\}}p^\kappa(s,z,y)\le F_{t,\varepsilon}(y)\quad\textit{for all } y\in\mathbb{R}^d_0,\tag{8.1}$$

then we have

$$p^\kappa(t,x,y)\le p^{\kappa,B(0,\varepsilon)^c}(t,x,y)+F_{t,\varepsilon}(x)+F_{t,\varepsilon}(y)\quad\textit{for all } x,y\in\mathbb{R}^d_0.$$

Proof. Let $x,y\in\mathbb{R}^d_0$ and $\delta>0$. Using the strong Markov property, we see that for all $w,u\in\mathbb{R}^d_0$,

$$\begin{aligned}&\int_{B(u,\delta)}p^\kappa(t/2,w,v)dv=\mathbb{P}_w(X^\kappa_{t/2}\in B(u,\delta))\\&=\mathbb{P}_w(X^{\kappa,B(0,\varepsilon)^c}_{t/2}\in B(u,\delta))+\mathbb{P}_w(X^\kappa_{t/2}\in B(u,\delta),\ \tau^\kappa_{B(0,\varepsilon)^c}<t/2)\end{aligned}$$

$$= \mathbb{P}_w(X^{\kappa,B(0,\varepsilon)^c}_{t/2} \in B(u,\delta)) + \mathbb{E}_w\left[\mathbb{P}_{X^\kappa_{\tau^\kappa_{B(0,\varepsilon)^c}}}(X^\kappa_{t/2-\tau^\kappa_{B(0,\varepsilon)^c}} \in B(u,\delta)) : \tau^\kappa_{B(0,\varepsilon)^c} < t/2\right]$$
$$\le \int_{B(u,\delta)} p^{\kappa,B(0,\varepsilon)^c}(t/2,w,v)dv + \sup_{s\in(0,t/2],\, z\in B(0,\varepsilon)\setminus\{0\}} \int_{B(u,\delta)} p^\kappa(s,z,v)dv.$$

By the Lebesgue differentiation theorem, for a.e. $(w,u) \in \mathbb{R}^d_0 \times \mathbb{R}^d_0$,

$$p^\kappa(t/2,w,u) \le p^{\kappa,B(0,\varepsilon)^c}(t/2,w,u) + \sup_{s\in(0,t/2],\, z\in B(0,\varepsilon)\setminus\{0\}} p^\kappa(s,z,u). \tag{8.2}$$

Using the semigroup property of X^κ in the equality below, (8.2) in the first inequality, the symmetry of p^κ and (8.2) in the second, the symmetry and the semigroup properties of p^κ and $p^{\kappa,B(0,\varepsilon)^c}$ in the third, and (8.1) in the fourth, we obtain

$$\begin{aligned}
&⨍_{B(y,\delta)} p^\kappa(t,x,v)dv = \int_{\mathbb{R}^d_0} p^\kappa(t/2,x,u) ⨍_{B(y,\delta)} p^\kappa(t/2,u,v)dvdu \\
&\le \int_{\mathbb{R}^d_0} p^{\kappa,B(0,\varepsilon)^c}(t/2,x,u) ⨍_{B(y,\delta)} p^\kappa(t/2,u,v)dvdu \\
&\quad + \sup_{s\in(0,t/2],\, z\in B(0,\varepsilon)\setminus\{0\}} \int_{\mathbb{R}^d_0} p^\kappa(s,z,u) ⨍_{B(y,\delta)} p^\kappa(t/2,u,v)dvdu \\
&\le \int_{\mathbb{R}^d_0} p^{\kappa,B(0,\varepsilon)^c}(t/2,x,u) ⨍_{B(y,\delta)} p^{\kappa,B(0,\varepsilon)^c}(t/2,v,u)dvdu \\
&\quad + \sup_{s\in(0,t/2],\, z\in B(0,\varepsilon)\setminus\{0\}} \int_{\mathbb{R}^d_0} p^{\kappa,B(0,\varepsilon)^c}(t/2,x,u) ⨍_{B(y,\delta)} p^\kappa(s,z,u)dvdu \\
&\quad + \sup_{s\in(0,t/2],\, z\in B(0,\varepsilon)\setminus\{0\}} \int_{\mathbb{R}^d_0} p^\kappa(s,z,u) ⨍_{B(y,\delta)} p^\kappa(t/2,u,v)dvdu \\
&\le ⨍_{B(y,\delta)} p^{\kappa,B(0,\varepsilon)^c}(t,x,v)dv \\
&\quad + \sup_{s\in(0,t/2],\, z\in B(0,\varepsilon)\setminus\{0\}} ⨍_{B(y,\delta)} p^\kappa(t/2+s,z,x)dv \\
&\quad + \sup_{s\in(0,t/2],\, z\in B(0,\varepsilon)\setminus\{0\}} ⨍_{B(y,\delta)} p^\kappa(t/2+s,z,v)dv \\
&\le ⨍_{B(y,\delta)} p^{\kappa,B(0,\varepsilon)^c}(t,x,v)dv + F_{t,\varepsilon}(x) ⨍_{B(y,\delta)} dv + ⨍_{B(y,\delta)} F_{t,\varepsilon}(v)dv.
\end{aligned}$$

Since $p^\kappa(t,x,\cdot)$ is lower semi-continuous on $\mathbb{R}^d_0$, we have $p^\kappa(t,x,y) \le \liminf_{\delta\to 0} ⨍_{B(y,\delta)} p^\kappa(t,x,v)dv$. Therefore, by the continuities of $p^{\kappa,B(0,\varepsilon)^c}(t,x,\cdot)$ and $F_{t,\varepsilon}$, we get the desired result. □

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