The Seneta-Heyde scaling for supercritical super-Brownian motion *

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Abstract

We consider the additive martingale $W_t(\lambda)$ and the derivative martingale $\partial W_t(\lambda)$ for one-dimensional supercritical super-Brownian motions with general branching mechanism. In the critical case $\lambda = \lambda_0$, we prove that $\sqrt{t}W_t(\lambda_0)$ converges in probability to a positive limit, which is a constant multiple of the almost sure limit $\partial W_\infty(\lambda_0)$ of the derivative martingale $\partial W_t(\lambda_0)$. We also prove that, on the survival event, $\limsup_{t\to\infty} \sqrt{t}W_t(\lambda_0) = \infty$ almost surely.

Résumé: Nous considérons la martingale additive $W_t(\lambda)$ et la martingale dérivée $\partial W_t(\lambda)$ pour les super-mouvements browniens surcritiques unidimensionnels avec mécanisme général de branchement. Dans le cas critique où $\lambda = \lambda_0$, nous prouvons que $\sqrt{t}W_t(\lambda_0)$ converge en probabilité vers une limite positive, qui est un multiple constant de la limite presque sûre $\partial W_\infty(\lambda_0)$ de la martingale dérivée $\partial W_t(\lambda_0)$. Nous prouvons également que, dans l'événement de survie, $\limsup_{t\to\infty} \sqrt{t}W_t(\lambda_0) = \infty$ presque sûrement.

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1 Introduction

Let $\{Z_n, n \geq 0\}$ be a supercritical Galton-Waston process with $Z_0 = 1$ and mean $m = \mathbb{E}Z_1 \in (1, \infty)$. It is well known that $\{m^{-n}Z_n; n \geq 0\}$ is a non-negative martingale and thus converges almost surely to a limit W. The Kesten-Stigum theorem says that W is non-degenerate if and only if $\mathbb{E}[Z_1 \log Z_1] < \infty$. Seneta [25] and Heyde [16] proved that if $\mathbb{E}[Z_1 \log Z_1] = \infty$, then there exists a non-random sequence $\{c_n\}_{n\geq 0}$ such that Z_n/c_n converges almost surely to a non-degenerate random variable as $n \to \infty$. This result is known as the Seneta-Heyde theorem and the sequence $\{c_n\}$ is therefore called a Seneta-Heyde norming.

A branching random walk is defined as follows. At generation 0, there is a particle at the origin of the real line $\mathbb{R}$. At generation $n = 1$, this particle dies and splits into a finite number of offspring. The law of the number of offspring and the positions of the offspring relative to their parent are given by a point process $\mathcal{Z}$. Each of these offspring evolves independently as its parent. Let $\mathcal{Z}_n$ denote the point process formed by the position of the particles in the n-th generation. Biggins and Kyprianou [3, 4] considered the non-negative martingale $W_n(\theta) := m(\theta)^{-n} \int \exp(-\theta x)\mathcal{Z}_n(\mathrm{d}x)$,

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which is referred to as the additive martingale, where $m(\theta) = \mathbb{E}\int \exp(-\theta x)\mathcal{Z}_1(\mathrm{d}x)$. They proved that, if $m(0) > 1$ and $m(\theta) < \infty$ for some $\theta > 0$, then the limit of $W_n(\theta)$, denoted by $W(\theta)$, is non-degenerate if and only if $\log m(\theta) - \theta m'(\theta)/m(\theta) > 0$ (supercritical) and $\mathbb{E}\left[W_1(\theta)\log_+ W_1(\theta)\right] < \infty$, where $\log_+ x := \max\{\log x, 0\}$. They also showed that, when $\log m(\theta) - \theta m'(\theta)/m(\theta) > 0$ holds but $\mathbb{E}\left[W_1(\theta)\log_+ W_1(\theta)\right] = \infty$, there exist a Seneta-Heyde norming $\{c_n\}_{n\geq 0}$ and a non-degenerate random variable Δ such that $W_n(\theta)/c_n$ converges to Δ in probability as $n \to \infty$.

For the critical case of $\log m(\theta) - \theta m'(\theta)/m(\theta) = 0$, without loss of generality, we assume that $m(\theta) = \theta = 1$. According to [3, 4], the additive martingale $W_n := W_n(1) = \int \exp(-x)\mathcal{Z}_n(x)$ converges to 0 almost surely, as $n \to \infty$. The study of the additive martingale W_n in the critical case relies on analyzing another fundamental martingale. Under the assumption that $\mathbb{E}\left[\int x\exp(-x)\mathcal{Z}_1(\mathrm{d}x)\right] = 0$, $D_n := \int x\exp(-x)\mathcal{Z}_n(\mathrm{d}x)$ is a mean 0 martingale which is referred to as the derivative martingale. Convergence of the derivative martingale was studied by Biggins and Kyprianou [5]. In order to state their result, we introduce the following integrability conditions:

$$\sigma^2 := \mathbb{E}\left[\int x^2 e^{-x}\mathcal{Z}_1(\mathrm{d}x)\right] < \infty, \tag{1.1}$$

$$\mathbb{E}\left[\left(\int e^{-x}\mathcal{Z}_1(\mathrm{d}x)\right)\log_+^2\left(\int e^{-x}\mathcal{Z}_1(\mathrm{d}x)\right)\right] < \infty, \tag{1.2}$$

$$\mathbb{E}\left[\left(\int \left((x)_+ e^{-x}\right)\mathcal{Z}_1(\mathrm{d}x)\right)\log_+\left(\int \left((x)_+ e^{-x}\right)\mathcal{Z}_1(\mathrm{d}x)\right)\right] < \infty. \tag{1.3}$$

Biggins and Kyprianou [5] proved that under the assumptions (1.1)-(1.3), D_n converges almost surely to a non-degenerate non-negative limit D_∞ as $n \to \infty$, see also Aïdekon and Shi [1, Theorem B]. Hu and Shi [17, Theorem 1.1] proved that there exists a deterministic sequence $(a_n)_{n\geq 1}$ such that, conditioned on survival, $\frac{W_n}{a_n}$ converges in distribution to some random variable W with $W > 0$ a.s. It was further proved in Aïdékon and Shi [1] that, under the assumptions (1.1)-(1.3),

$$\lim_{n\to\infty} \sqrt{n}W_n = \sqrt{\frac{2}{\pi\sigma^2}}D_\infty \quad \text{in probability.} \tag{1.4}$$

They also proved that $\limsup_{n\to\infty} \sqrt{n}W_n = +\infty$ almost surely conditioned on survival. Under the assumption that the associated random walk is in the domain of attraction of an α-stable law, $\alpha \in (1,2)$, He, Liu and Zhang [15] proved $n^{1/\alpha}W_n$ converges to $CD_\infty(\alpha)$ in probability, where $C > 0$ is a constant and $D_\infty(\alpha)$ is the limit of the derivative martingale under different moment conditions. For the subcritical case $\log m(\theta) - \theta m'(\theta)/m(\theta) < 0$, Hu and Shi [17, Theorem 1.4] gave some convergence results for $\log W_n(\theta)$.

A branching Brownian motion (BBM) can be defined as follows. Initially, there is a single particle at the origin. It lives an exponential amount of time with parameter 1. Each particle moves according to a Brownian motion with drift 1 during its lifetime and then splits into a random number, say L, of new particles. These new particles start the same process from their place of birth behaving independently of the others. The system goes on indefinitely, unless there is no particle at some time. Assume that the BBM is supercritical, i.e., $\mathbb{E}L > 1$, and $2\mathbb{E}[L-1] = 1$. Let Z_t be the point process formed by the position of the particles at time t. The non-negative martingale $W_t(\theta) := e^{-(\theta-1)^2 t/2}\int \exp(-\theta x)Z_t(\mathrm{d}x)$ is called the additive martingale and plays an important role in the study of BBMs. It is known that the limit $W(\theta)$ of $W_t(\theta)$ is non-degenerate if and only if $|\theta| < 1$ (supercritical case) and $\mathbb{E}\left[L \log_+ L\right] < \infty$, see [6, 23]. Another key object for BBMs is the derivative martingale $D_t := \int x \exp(-x) Z_t(\mathrm{d}x)$ in the critical case $\theta = 1$. Yang and Ren [27] proved that D_t converges almost surely to a non-degenerate non-negative limit D_∞ as $t \to \infty$ if and only if $\mathbb{E}\left[L \log_+^2 L\right] < \infty$, and if $\mathbb{E}\left[L \log_+^2 L\right] < \infty$ holds, $D_\infty > 0$ almost surely on the event of survival. Fluctuation of the derivative martingale D_t around its limit D_∞ was given by Maillard and Pain [22]. The analog of (1.4) is also valid for BBMs, see [22, (1.7)].

In this paper we consider supercritical super-Brownian motions in $\mathbb{R}$. A super-Brownian motion arises as the high density limit of branching Brownian motions or branching random walks. Let $\mathcal{B}_b(\mathbb{R})$ (respectively $\mathcal{B}^+(\mathbb{R})$, respectively $\mathcal{B}_b^+(\mathbb{R})$) be the set of all bounded (respectively non-negative, respectively bounded and non-negative) real-valued Borel functions on $\mathbb{R}$. Let $\mathcal{M}(\mathbb{R})$ denote the space of finite Borel measures on $\mathbb{R}$. For any $f \in B_b^+(\mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R})$, we use $\langle f, \mu\rangle$ or $\mu(f)$ to denote the integral of f with respect to μ whenever the integral is well-defined. For simplicity, we sometimes write $\|\mu\| := \langle 1, \mu\rangle$.

We will always assume that $B = \{(B_t)_{t\geq 0}; \Pi_x, x \in \mathbb{R}\}$ is a Brownian motion on $\mathbb{R}$. Let *the branching mechanism* ψ be given by

$$\psi(\lambda) := -\alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right) \nu(\mathrm{d}x), \quad \lambda \geq 0, \tag{1.5}$$

where $\beta \geq 0$, $\alpha = -\psi'(0^+)$ and ν is a σ-finite measure supported on $(0,\infty)$ with $\int_{(0,\infty)}(x \wedge x^2)\nu(\mathrm{d}x) < \infty$. There exists an $\mathcal{M}(\mathbb{R})$-valued Markov process $X = \{(X_t)_{t\geq 0}; \mathbb{P}_\mu, \mu \in \mathcal{M}(\mathbb{R})\}$ such that

$$\mathbb{P}_\mu\left[e^{-X_t(f)}\right] = e^{-\mu(U_t f)}, \quad t \geq 0, f \in \mathcal{B}_b^+(\mathbb{R}),$$

where $(t,x) \mapsto U_t f(x)$ is the unique locally bounded non-negative map on $\mathbb{R}_+ \times \mathbb{R}$ such that

$$U_t f(x) + \Pi_x\left[\int_0^t \psi\left(U_{t-s} f(B_s)\right) \mathrm{d}s\right] = \Pi_x[f(B_t)], \quad t \geq 0, x \in \mathbb{R}.$$

This process X is known as a super-Brownian motion with branching mechanism ψ. For the existence of X we refer our readers to [10, 11, 12] or [21, Section 2.3].

The super-Brownian motion with branching mechanism ψ is called supercritical, critical or subcritical according to $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ or $\psi'(0^+) > 0$. In this paper we concentrate on supercritical super-Brownian motions, i.e., we assume $\psi'(0^+) < 0$. We always assume that $\psi(\infty) = \infty$ which guarantees that the event $\mathcal{E} := \{\lim_{t\to\infty} \|X_t\| = 0\}$ will occur with positive probability. Let λ^* be the largest root of the equation $\psi(\lambda) = 0$. For any $\mu \in \mathcal{M}(\mathbb{R})$, $\mathbb{P}_\mu(\mathcal{E}) = e^{-\lambda^*\|\mu\|}$.

In this paper we shall also assume that

$$\int^\infty \frac{1}{\sqrt{\int_{\lambda^*}^\xi \psi(u)\mathrm{d}u}}\mathrm{d}\xi < \infty. \tag{1.6}$$

Under condition (1.6), it holds that (see, for instance, [20]) $\mathcal{E} = \{\exists t > 0 \text{ such that } \|X_t\| = 0\}$.

Denote by $\mathbf{0}$ the null measure on $\mathbb{R}$. Write $\mathcal{M}^0(\mathbb{R}) := \mathcal{M}(\mathbb{R}) \setminus \{\mathbf{0}\}$. Set $c_\lambda = -\psi'(0^+)/\lambda + \lambda/2$ and define

$$W_t(\lambda) := e^{-\lambda c_\lambda t}\langle e^{-\lambda\cdot}, X_t\rangle, \quad t \geq 0,\ \lambda \in \mathbb{R}.$$

Then according to [20], for any $\mu \in \mathcal{M}^0(\mathbb{R})$, $W(\lambda) := \{W_t(\lambda) : t \geq 0\}$ is a non-negative $\mathbb{P}_\mu$-martingale and thus has an almost sure limit $W_\infty(\lambda)$. $W(\lambda)$ is called the *additive martingale*. By

[20, Theorem 2.4], $W_\infty(\lambda)$ is also an $L^1(\mathbb{P}_\mu)$ limit if and only if $|\lambda| < \lambda_0$ and $\int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) < \infty$, where $\lambda_0 = \sqrt{-2\psi'(0^+)}$.

Another important martingale $\partial W(\lambda)$, called the *derivative martingale*, is defined as follows:

$$\partial W_t(\lambda) := \langle (\lambda t + \cdot)e^{-\lambda(c_\lambda t + \cdot)}, X_t\rangle, \quad t \geq 0.$$

Under condition (1.6), Kyprianou et al. [20, Theorem 2.4] proved that when $|\lambda| \geq \lambda_0$, $\partial W_t(\lambda)$ has a $\mathbb{P}_\mu$ almost surely non-negative limit $\partial W_\infty(\lambda)$ for any $\mu \in \mathcal{M}^0(\mathbb{R})$, and when $|\lambda| > \lambda_0$, $\partial W_\infty(\lambda) = 0$ $\mathbb{P}_\mu$ almost surely. When $|\lambda| = \lambda_0$ (called the critical case), $\partial W_\infty(\lambda)$ is almost surely positive on $\mathcal{E}^c$ if and only if

$$\int_{[1,\infty)} r(\log r)^2 \nu(\mathrm{d}r) < \infty. \tag{1.7}$$

In this paper we concentrate on the critical case $|\lambda| = \lambda_0$. Due to symmetry, without loss of generality, we assume $\lambda = \lambda_0$. The derivative martingale $\partial W_t(\lambda_0)$ plays an important role in the study of the extremal process of super-Brownian motions, see [24].

The additive martingale $W_t(\lambda_0)$ converges to 0 as $t \to \infty$. The goal of this paper is to find the rate at which $W_t(\lambda_0)$ converges to 0. For simplicity, we write

$$W_t := W_t(\lambda_0), \quad \partial W_t := \partial W_t(\lambda_0), \quad \partial W_\infty := \partial W_\infty(\lambda_0).$$

Let $\{(X_t^{\lambda_0})_{t\geq 0}; \mathbb{P}_\mu, \mu \in \mathcal{M}(\mathbb{R})\}$ be a superprocess with the same branching mechanism ψ in (1.5) and with a Brownian motion with drift λ_0 as spatial motion. Then $\langle f, X_t^{\lambda_0}\rangle = \langle f(\lambda_0 t + \cdot), X_t\rangle$ for any $f \in \mathcal{B}_b^+(\mathbb{R})$. Note that $c_{\lambda_0} = \lambda_0$, we can rewrite W_t and ∂W_t as

$$W_t = \langle e^{-\lambda_0 \cdot}, X_t^{\lambda_0}\rangle, \quad \partial W_t = \langle \cdot e^{-\lambda_0 \cdot}, X_t^{\lambda_0}\rangle.$$

Write $\mathbb{P}$ as a shorthand for $\mathbb{P}_{\delta_0}$. Throughout this paper for a probability P, we will also use P to denote expectation with respect to P. The main results of this paper are the following two theorems:

Theorem 1.1 *If* (1.6) *and* (1.7) *hold, then*

$$\lim_{t\to\infty} \sqrt{t} W_t = \sqrt{\frac{2}{\pi}} \partial W_\infty \quad \textit{in probability with respect to } \mathbb{P}.$$

The following result says that the above convergence in probability can not be strengthened to almost sure convergence.

Theorem 1.2 *If* (1.6) *and* (1.7) *hold, then on* $\mathcal{E}^c$,

$$\limsup_{t\to\infty} \sqrt{t} W_t = +\infty \quad \mathbb{P}\textit{-almost surely.} \tag{1.8}$$

We end this section with a description of the strategy of the proofs of Theorems 1.1 and 1.2, and the organization of this paper. In the remainder of this paper, we always assume that (1.6) and (1.7) hold. In Section 2, we introduce the exit measures, the $\mathbb{N}$-measures and the spine decomposition of super-Brownian motion. We also give some basic properties for Bessel-3 processes. We also use exit measures to define a variant W_t^{-y} (see (2.9)) of the additive martingale W_t and a variant V_t^{-y} (see

(2.10)) of the derivative martingale ∂W_t by killing the particles hitting $-y$ before time t. These ingredients will be used in the proof of Theorem 1.1. In this section, we also introduce the skeleton decomposition for super-Brownian motion, which is used in the proof of Theorem 1.2.

In Section 3, we prove Theorem 1.1. We will use the spine decomposition to give a copy of W_t^{-y} and a copy of V_t^{-y}, denoted as $\widetilde{W}_t^{-y}$ and $\widetilde{V}_t^{-y}$, respectively. We first prove the mean of $\frac{\sqrt{t}\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}$ converges to $\sqrt{2/\pi}$ as $t\to\infty$ in Lemma 3.2. Then in Lemma 3.3, we prove that $\frac{\sqrt{t}\widetilde{W}_t^{-y}}{W_t^{-y}+\widetilde{V}_t^{-y}}$ converges to $\sqrt{2/\pi}$ in L^2, which is the key to the proof of Theorem 1.1. Due to the weak moment condition on the Lévy measure ν in (1.7), to prove Lemma 3.3, we need to define a family of "good" sets E_t with probabilities tending to 1 as $t\to\infty$ (see Lemma 3.6). On the set E_t we prove a sharp upper bound for the ratio of these two modified martingales in Lemma 3.7. This sharp upper bound is crucial for the proof of Lemma 3.3. Although the proof of Theorem 1.1 is similar to that of the corresponding result for branching random walks given in Aïdékon and Shi [1], more efforts are need to deal with E_t since the spine decomposition of super-Brownian motion is more complicated.

In Section 4, we prove Theorem 1.2. A key for the proof of the corresponding result for branching random walks given in [1] is the asymptotic behavior for the minimal position of branching random walks given in [1, Theorem 6.1]. The fact that the spatial displacement of a branching random walk in each generation can be regarded as a point process is used crucially in the proof of [1, Theorem 1.2]. However, a super-Brownian motion in $\mathbb{R}$ has a density with respect to the Lebesgue measure and thus can not be regarded as a point process. We overcome this difficulty by using the skeleton process. Roughly speaking, we choose a sequence of random times and use the fact that the skeleton process observed at these random times is a branching random walk. In Lemmas 4.1 and 4.2, we show that this branching random walk, after a suitable translation, satisfies the conditions of [1, Theorem 6.1], i.e., conditions (1.1) (1.2) and (1.3) above. So we can apply [1, Theorem 6.1] to get the asymptotic behavior of the minimal position of this shifted branching random walk, which, in turn, is used to get the conclusion of Theorem 1.2.

2 Preliminaries

In this section, we will introduce some useful results that will be used later.

Recall that $\{(B_t)_{t\geq 0};\Pi_x, x\in\mathbb{R}\}$ is a Brownian motion. For any $x\in\mathbb{R}$, we define $\tau_x=\inf\{t>0:B_t=x\}$. It is well known that $\{e^{\lambda_0 B_t-\lambda_0^2 t/2}, t\geq 0\}$ is a positive Π_0-martingale with mean 1. We define a martingale change of measure by

$$\frac{\mathrm{d}\Pi_0^{\lambda_0}}{\mathrm{d}\Pi_0}\bigg|_{\sigma(B_s:0\leq s\leq t)}=e^{\lambda_0 B_t-\lambda_0^2 t/2}. \tag{2.1}$$

Under $\Pi_0^{\lambda_0}$, $\{B_t, t\geq 0\}$ is a Brownian motion with drift λ_0 staring from 0. For any $y>0$, we define $\widetilde{\Pi}_y$ by

$$\frac{\mathrm{d}\widetilde{\Pi}_y}{\mathrm{d}\Pi_0}\bigg|_{\sigma(B_s:s\leq t)}=\frac{y+B_t}{y}1_{(t<\tau_{-y})}. \tag{2.2}$$

Under $\widetilde{\Pi}_y$, $\{y+B_t:t\geq 0\}$ is a Bessel-3 process starting from y and the density of $y+B_t$ is

$$f_t(x)=\frac{x}{y\sqrt{2\pi t}}e^{-(x-y)^2/2t}(1-e^{-2xy/t})1_{\{x>0\}}. \tag{2.3}$$

2.1 Branching Markov exit measures

For any $r \geq 0$ and $x \in \mathbb{R}$, let $\{(B_t)_{t\geq r}; \Pi^{\lambda_0}_{r,x}\}$ be a Brownian motion with drift λ_0 started at x at time r. $\Pi^{\lambda_0}_{0,x}$ is the same as $\Pi^{\lambda_0}_x$. Let $S = [0,\infty)\times\mathbb{R}$, $\mathcal{B}(S)$ be the Borel σ-field on S, $\mathbb{O} \subset \mathcal{B}(\mathcal{S})$ the class of open subsets of S and $\mathcal{M}(S)$ the space of finite Borel measures on S. A measure $\mu \in \mathcal{M}(\mathbb{R})$ is identified with its corresponding measure on S concentrated on $\{0\}\times\mathbb{R}$. According to Dynkin [9], there exists a family of random measures $\{(X_Q, \mathbb{P}_\mu); Q \in \mathbb{O}, \mu \in \mathcal{M}(S)\}$ such that for any $Q \in \mathbb{O}$, $\mu \in \mathcal{M}(S)$ with supp $\mu \subset Q$, and bounded non-negative Borel function $f(t,x)$ on S,

$$\mathbb{P}_\mu\left[\exp\{-\langle f, X_Q\rangle\}\right] = \exp\left\{-\langle V_f^Q, \mu\rangle\right\},$$

where $V_f^Q(s,x)$ is the unique positive solution of the equation

$$V_f^Q(s,x) + \Pi_{s,x}\int_s^\tau \psi\left(V_f^Q(r,B_r)\right)\mathrm{d}r = \Pi_{s,x} f(\tau, B_\tau),$$

with $\tau := \inf\{r : (r, B_r) \notin Q\}$. By [11, (1.20)], we have the following mean formula:

$$\mathbb{P}_\mu\langle f, X_Q\rangle = \int \Pi_{s,x}\left[e^{\alpha\tau} f(\tau, B_\tau)\right]\mu(\mathrm{d}s\mathrm{d}x). \tag{2.4}$$

For $y > 0, t \geq 0$, we define $D^t_{-y} := \{(s,x) : s < t, -y < x\}$. Then the random measure $X^{\lambda_0}_{D^t_{-y}}$ is concentrated on $\partial D^t_{-y} := ([0,t)\times\{-y\}) \cup (\{t\}\times[-y,+\infty])$, and for any $\mu \in \mathcal{M}([0,\infty)\times\mathbb{R})$ with supp $\mu \subset [0,t)\times[-y,+\infty)$, and $f \in C_b(D^t_{-y})$ with $f(s,x) = f(0,x) =: f(x)$ for all $s \geq 0$,

$$\mathbb{P}_\mu\left[\exp\left\{-\langle f, X^{\lambda_0}_{D^t_{-y}}\rangle\right\}\right] = \exp\left\{-\langle U_f^{-y,t}(\cdot), \mu\rangle\right\},$$

where $U_f^{-y,t}(s,x)$ is the unique positive solution of the integral equation

$$U_f^{-y,t}(s,x) + \Pi^{\lambda_0}_{s,x}\int_s^{t\wedge\tau_{-y}} \psi\left(U_f^{-y,t}(r,B_r)\right)\mathrm{d}r = \Pi^{\lambda_0}_{s,x}[f(B_{t\wedge\tau_{-y}})], \quad (s,x) \in \overline{D^t_{-y}}, \tag{2.5}$$

with $\overline{D^t_{-y}}$ being the closure of D^t_{-y}. By (2.4) and the homogeneity of Brownian motion, for any $x \in \mathbb{R}$, we have

$$\mathbb{P}_{\delta_x}\langle f, X^{\lambda_0}_{D^t_{-y}}\rangle = \Pi^{\lambda_0}_x\left[e^{\alpha(t\wedge\tau_{-y})} f(B_{t\wedge\tau_{-y}})\right]. \tag{2.6}$$

By the time homogeneity of Brownian motion with drift λ_0, (2.5) can be written as

$$U_f^{-y,t}(s,x) + \Pi^{\lambda_0}_x\int_0^{(t-s)\wedge\tau_{-y}} \psi\left(U_f^{-y,t}(r+s,B_r)\right)\mathrm{d}r = \Pi^{\lambda_0}_x[f(B_{(t-s)\wedge\tau_{-y}})], \quad (s,x) \in \overline{D^t_{-y}}.$$

Put $u_f^{-y}(t-s,x) := U_f^{-y,t}(s,x)$. The above integral equation can be written as

$$u_f^{-y}(t-s,x) + \Pi^{\lambda_0}_x\int_0^{(t-s)\wedge\tau_{-y}} \psi\left(u_f^{-y}(t-r-s,B_r)\right)\mathrm{d}r = \Pi^{\lambda_0}_x[f(B_{(t-s)\wedge\tau_{-y}})], \quad (s,x) \in \overline{D^t_{-y}},$$

which is equivalent to

$$u_f^{-y}(s,x) + \Pi^{\lambda_0}_x\int_0^{s\wedge\tau_{-y}} \psi\left(u_f^{-y}(s-r,B_r)\right)\mathrm{d}r = \Pi^{\lambda_0}_x[f(B_{s\wedge\tau_{-y}})], \quad (s,x) \in \overline{D^t_{-y}}. \tag{2.7}$$

The special Markov property (see [11, Theorem 1.3], for example) implies that, for all $D^r_{-z} \subset D^t_{-y}$

$$\mathbb{P}_\mu \left[\langle f, X^{\lambda_0}_{D^t_{-y}} \rangle \Big| \mathcal{F}^{\lambda_0}_{D^r_{-z}} \right] = \mathbb{P}_{X^{\lambda_0}_{D^r_{-z}}} \langle f, X^{\lambda_0}_{D^t_{-y}} \rangle, \tag{2.8}$$

where $\mathcal{F}^{\lambda_0}_{D^t_{-y}} := \sigma\left(X^{\lambda_0}_{D^s_{-x}} : s \le t, x \le y \right)$.

In the proof of Theorem 1.1, we will use modifications of W_t defined below. For any $y > 0$, define

$$W^{-y}_t := \langle e^{-\lambda_0 \cdot} 1_{(-y,\infty)}(\cdot), X^{\lambda_0}_{D^t_{-y}} \rangle, \quad t \ge 0. \tag{2.9}$$

2.2 $\mathbb{N}$-measure and spine decomposition for X^{λ_0}

Without loss of generality, we assume that X is the coordinate process on $\mathbb{D} := \{w = (w_t)_{t\ge 0} : w$ is an $\mathcal{M}(\mathbb{R})$-valued càdlàg function on $[0,\infty)\}$. We assume that $(\mathcal{F}_\infty, (\mathcal{F}_t)_{t\ge0})$ is the natural filtration on $\mathbb{D}$, completed as usual with the $\mathcal{F}_\infty$-measurable and $\mathbb{P}_\mu$-negligible sets for every $\mu \in \mathcal{M}(\mathbb{R})$. Let $\mathbb{W}^+_0$ be the family of $\mathcal{M}(\mathbb{R})$-valued càdlàg functions on $(0,\infty)$ with $\mathbf{0}$ as a trap and with $\lim_{t\downarrow 0} w_t = \mathbf{0}$. $\mathbb{W}^+_0$ can be regarded as a subset of $\mathbb{D}$.

Under condition (1.6), $\mathbb{P}_{\delta_x}(X_t(1) = 0) > 0$ for any $x \in \mathbb{R}$ and $t > 0$, which implies that there exists a unique family of σ-finite measures $\{\mathbb{N}_x; x \in \mathbb{R}\}$ on $\mathbb{W}^+_0$ such that for any $\mu \in \mathcal{M}(\mathbb{R})$, if $\mathcal{N}(\mathrm{d}w)$ is a Poisson random measure on $\mathbb{W}^+_0$ with intensity measure

$$\mathbb{N}_\mu(\mathrm{d}w) := \int_{\mathbb{R}} \mathbb{N}_x(\mathrm{d}w)\mu(\mathrm{d}x),$$

then the process defined by

$$\widehat{X}_0 := \mu, \quad \widehat{X}_t := \int_{\mathbb{W}^+_0} w_t \mathcal{N}(\mathrm{d}w), \quad t > 0,$$

is a realization of the superprocess $X = \{(X_t)_{t\ge0}; \mathbb{P}_\mu, \mu \in \mathcal{M}(\mathbb{R})\}$. Furthermore, $\mathbb{N}_x(\langle f, w_t\rangle) = \mathbb{P}_{\delta_x}\langle f, X_t\rangle$ and $\mathbb{N}_x\left[1 - \exp\{-\langle f, w_t\rangle\}\right] = -\log \mathbb{P}_{\delta_x}\left[\exp\{-\langle f, X_t\rangle\}\right]$ for any $f \in \mathcal{B}^+_b(\mathbb{R})$ (see [21, Theorems 8.22 and 8.23]). $\{\mathbb{N}_x; x \in \mathbb{R}\}$ are called the $\mathbb{N}$-measures associated to $\{\mathbb{P}_{\delta_x}; x \in \mathbb{R}\}$. One can also see [13] for the definition of $\{\mathbb{N}_x; x \in \mathbb{R}\}$.

Next, we recall an important spine decomposition for super-Brownian motions. The spine decomposition is related to a martingale change of measure. Fix $y > 0$, define V^{-y}_t by

$$V^{-y}_t := \langle (y + \cdot)e^{-\lambda_0 \cdot}, X^{\lambda_0}_{D^t_{-y}} \rangle, \quad t \ge 0. \tag{2.10}$$

From [20, Section 7], we know that V^{-y}_t is a positive $\mathbb{P}$-martingale with mean y. Define $\mathbb{Q}^{-y}$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}^{-y}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := \frac{1}{y} V^{-y}_t, \quad t \ge 0. \tag{2.11}$$

We say $\{(\xi_t)_{t\ge0}, (X^{(\mathbf{n})})_{t\ge0}, (X^{(\mathbf{m})})_{t\ge0}, (X'_t)_{t\ge0}; \widetilde{\mathbb{P}}^{-y}\}$ is a *spine representation* of $\{(X_t)_{t\ge0}; \mathbb{Q}^{-y}\}$ if the following are true:

(i) The spine process is given by $\xi := \{\xi_t, t \ge 0\}$ such that $\{(\xi_t + \lambda_0 t + y)_{t\ge0}; \widetilde{\mathbb{P}}^{-y}\}$ is a Bessel-3 process starting from y.

(ii) Given $(\xi;\widetilde{\mathbb{P}}^{-y})$, let $\mathcal{N}$ be a Poisson random measure on $[0,\infty)\times\mathbb{D}$ with intensity $2\beta \mathrm{d}t\mathbb{N}_{\xi_t}(\mathrm{d}w)$. For $t\geq 0$, define $X^{(\mathbf{n})}_t=\int_{[0,t]}\int_{\mathbb{D}} w_{t-s}\mathcal{N}(\mathrm{d}t\times \mathrm{d}w)$. $X^{(\mathbf{n})}$ is referred to as the *continuous immigration.*

(iii) Given $(\xi;\widetilde{\mathbb{P}}^{-y})$, let $\{R_t:t\geq 0\}$ be a point process such that the random counting measure $\sum_{t\geq 0}\delta_{(t,R_t)}$ is a Poisson random measure on $(0,\infty)\times(0,\infty)$ with intensity $\mathrm{d}tr\nu(\mathrm{d}r)$, let $D^{\mathbf{m}}$ be the projection onto the first coordinate of the atoms $\{(s_i,r_i)\}$ of this Poisson random measure and $D^{\mathbf{m}}_t:=D^{\mathbf{m}}\cap[0,t]$. Given ξ and R, independently for each $s\in D^{\mathbf{m}}$ and $r=R_s$, a process $\{X^{\mathbf{m},s},\mathbb{P}_{r\delta_{\xi_s}}\}$ is issued at the time-space point (s,ξ_s). For $t\geq 0$, define $X^{(\mathbf{m})}_t=\sum_{s\in D^{\mathbf{m}}_t}X^{\mathbf{m},s}_{t-s}$. $X^{(\mathbf{m})}$ is referred to as the *discrete immigration.*

(iv) $(X',\widetilde{\mathbb{P}}^{-y})$ is a copy of $(X,\mathbb{P})$ and $(X',\widetilde{\mathbb{P}}^{-y})$ is independent of ξ, $X^{(\mathbf{n})}$ and $X^{\mathbf{m}}$.

For $t\geq 0$, define $\widetilde{X}_t=X'_t+X^{(\mathbf{n})}_t+X^{(\mathbf{m})}_t$. By [20, Theorem 7.2],

$$\{(\widetilde{X}_t)_{t\geq 0};\widetilde{\mathbb{P}}^{-y}\}\overset{d}{=}\{(X_t)_{t\geq 0};\mathbb{Q}^{-y}\}.$$

$\{(\widetilde{X}_t)_{t\geq 0};\widetilde{\mathbb{P}}^{-y}\}$ is called a spine representation of $\{(X_t)_{t\geq 0};\mathbb{Q}^{-y}\}$.

Now we give a *spine representation* of $\{(X^{\lambda_0}_t)_{t\geq 0};\mathbb{Q}^{-y}\}$. Define

$$\xi^{\lambda_0}:=\{\xi^{\lambda_0}_t,t\geq 0\}:=\{\xi_t+\lambda_0 t,t\geq 0\},$$

then $\{\xi^{\lambda_0}_t+y,t\geq 0;\widetilde{\mathbb{P}}^{-y}\}$ is a Bessel-3 process starting from y.

We construct $\{(\xi^{\lambda_0}_t)_{t\geq 0},(X^{(\mathbf{n}),\lambda_0})_{t\geq 0},(X^{(\mathbf{m}),\lambda_0})_{t\geq 0},((X^{\lambda_0})'_t)_{t\geq 0};\widetilde{\mathbb{P}}^{-y}\}$, called a *spine representation* of $\{(X^{\lambda_0}_t)_{t\geq 0}\}$, as follows:

(i) The spine is given by $\xi^{\lambda_0}=\{\xi_t+\lambda_0 t,t\geq 0\}$ such that $(\xi^{\lambda_0}+y,\widetilde{\mathbb{P}}^{-y})$ is a Bessel-3 process starting from y.

(ii) *Continuous immigration.* Given ξ^{λ_0}, the continuous immigration $X^{(\mathbf{n}),\lambda_0}_t$ is defined such that $\forall f\in B^+_b(\mathbb{R})$,

$$\langle f,X^{(\mathbf{n}),\lambda_0}_t\rangle=\int_{[0,t]}\int_{\mathbb{D}}\langle f(\cdot+\lambda_0(t-s)+\lambda_0 s),w_{t-s}\rangle\mathcal{N}(\mathrm{d}s\times\mathrm{d}w)=\langle f(\cdot+\lambda_0 t),X^{(\mathbf{n})}_t\rangle.$$

Define w^{λ_0} by $\langle f,w^{\lambda_0}_s\rangle=\langle f(\cdot+\lambda_0 s),w_s\rangle$. Then the random measure $\mathcal{N}^{\lambda_0}$ defined by

$$\int_{[0,t]}\int_{\mathbb{D}}\langle f(\cdot),w^{\lambda_0}_{t-s}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right):=\int_{[0,t]}\int_{\mathbb{D}}\langle f(\cdot+\lambda_0(t-s)+\lambda_0 s),w_{t-s}\rangle\mathcal{N}(\mathrm{d}s\times\mathrm{d}w),$$

is a Poisson random measure with intensity $2\beta\mathrm{d}t\mathbb{N}_{\xi^{\lambda_0}_t}(\mathrm{d}w^{\lambda_0})$.

(iii) *Discrete immigration.* Given ξ^{λ_0}, the discrete immigration $X^{\mathbf{m},s,\lambda_0}_t$ immigrated at time s is defined such that $\forall f\in B^+_b(\mathbb{R})$,

$$\langle f,X^{\mathbf{m},s,\lambda_0}_{t-s}\rangle=\langle f(\cdot+\lambda_0(t-s)+\lambda_0 s),X^{\mathbf{m},s}_{t-s}\rangle=\langle f(\cdot+\lambda_0 t),X^{\mathbf{m},s}_{t-s}\rangle.$$

The almost surely countable set of the discrete immigration times in $[0,t]$ is also given by $D^{\mathbf{m}}_t$ as in the spine decomposition of $\{(X_t)_{t\geq 0};\mathbb{Q}^{-y}\}$. Define $X^{(\mathbf{m}),\lambda_0}_t=\sum_{s\in D^{\mathbf{m}}_t}X^{\mathbf{m},s,\lambda_0}_{t-s}$.

(iv) $\{(X^{\lambda_0})'_t,t\geq 0\}$ is defined by

$$\langle f,(X^{\lambda_0})'_t\rangle=\langle f(\cdot+\lambda_0 t),X'_t\rangle,\quad f\in B^+_b(\mathbb{R}).$$

For any $t\geq 0$, define

$$\widetilde{X}^{\lambda_0}_t:=(X^{\lambda_0})'_t+X^{(\mathbf{n}),\lambda_0}_t+X^{(\mathbf{m}),\lambda_0}_t. \tag{2.12}$$

Proposition 2.1

$$\{(\widetilde{X}_t^{\lambda_0})_{t\geq 0}; \widetilde{\mathbb{P}}^{-y}\} \stackrel{d}{=} \{(X_t^{\lambda_0})_{t\geq 0}; \mathbb{Q}^{-y}\}. \tag{2.13}$$

Proof: By the definition of $\widetilde{X}_t^{\lambda_0}, X_t^{(\mathbf{n}),\lambda_0}$ and $X_{t-s}^{\mathbf{m},s,\lambda_0}$,

$$\begin{aligned}\langle f, \widetilde{X}_t^{\lambda_0}\rangle =&\langle f(\cdot+\lambda_0 t), X_t'\rangle + \langle f(\cdot+\lambda_0 t), X_t^{(\mathbf{n})}\rangle + \sum_{s\in D_t^{\mathbf{m}}}\langle f(\cdot+\lambda_0 t), X_{t-s}^{\mathbf{m},s}\rangle \\ =&\langle f(\cdot+\lambda_0 t), \widetilde{X}_t\rangle.\end{aligned}$$

This says that $\{(\widetilde{X}_t^{\lambda_0})_{t\geq 0}, \widetilde{\mathbb{P}}^{-y}\}$ is a shift of $\{(\widetilde{X}_t)_{t\geq 0}, \widetilde{\mathbb{P}}^{-y}\}$ with constant speed λ_0. Also note that

$$\mathbb{Q}^{-y}\left[\exp\left\{-\langle f, X_t^{\lambda_0}\rangle\right\}\right] = \mathbb{Q}^{-y}\left[\exp\left\{-\langle f(\cdot+\lambda_0 t), X_t\rangle\right\}\right] = \widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\langle f(\cdot+\lambda_0 t), \widetilde{X}_t\rangle\right\}\right].$$

Thus we have

$$\mathbb{Q}^{-y}\left[\exp\left\{-\langle f, X_t^{\lambda_0}\rangle\right\}\right] = \widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\langle f, \widetilde{X}_t^{\lambda_0}\rangle\right\}\right],$$

which says that $\{(\widetilde{X}_t^{\lambda_0})_{t\geq 0}, \widetilde{\mathbb{P}}^{-y}\}$ and $\{(X_t^{\lambda_0})_{t\geq 0}, \mathbb{Q}^{-y}\}$ have the same marginal distribution. By the Markov property of both processes, we have (2.13). □

2.3 Skeleton decomposition for X

In this subsection, we recall the skeleton decomposition, which is also called the backbone decomposition in some papers, see Eckhoff et al. [14] for an explanation of the terminologies. This decomposition was first proved by Duquesne and Winkel [7, Theorem 5.6], where only the genealogical structure was considered, and later generalized by Berestycki et, al [2]. This decomposition will be used in the proof of Theorem 1.2.

Recall that $X = \{(X_t)_{t\geq 0}; \mathbb{P}_\mu, \mu\in\mathcal{M}(\mathbb{R})\}$ is a supercritical super-Brownian motion and $\mathcal{E} = \{\lim_{t\to\infty}\|X_t\| = 0\}$. Under condition (1.6), $\mathcal{E} = \{\|X_t\| = 0 \text{ for some } t > 0\}$. For any $\mu\in\mathcal{M}(\mathbb{R})$, we define $\mathbb{P}_\mu^{\mathcal{E}}$ by

$$\mathbb{P}_\mu^{\mathcal{E}}(\cdot) := \mathbb{P}_\mu(\cdot|\mathcal{E}).$$

Then by [2, Lemma 2], $\{(X_t)_{t\geq 0}; \mathbb{P}_\mu^{\mathcal{E}}\}$ is a super-Brownian motion with branching mechanism

$$\psi^*(\lambda) := \psi(\lambda+\lambda^*) = -\alpha^*\lambda + \beta\lambda^2 + \int_{(0,\infty)}\left(e^{-\lambda x} - 1 + \lambda x\right)e^{-\lambda^* x}\nu(\mathrm{d}x),$$

where

$$\alpha^* = \alpha - 2\beta\lambda^* - \int_{(0,\infty)} x\left(1 - e^{-\lambda^* x}\right)\nu(\mathrm{d}x) = -\psi'(\lambda^*).$$

We denote by $\{\mathbb{N}_x^{\mathcal{E}} : x\in\mathbb{R}\}$ the $\mathbb{N}$-measures associated to $\{\mathbb{P}_{\delta_x}^{\mathcal{E}} : x\in\mathbb{R}\}$.

Let $\mathcal{M}_a(\mathbb{R})$ be the space of finite atomic measures on $\mathbb{R}$. According to Berestycki et al. [2], there exists a probability space, equipped with probability measures $\{\mathbf{P}_{(\mu,\eta)}, \mu\in\mathcal{M}(\mathbb{R}), \eta\in\mathcal{M}_a(\mathbb{R})\}$, which carries the following processes:

(i) $\{(Z_t)_{t\geq 0}, \mathbf{P}_{(\mu,\eta)}\}$, the skeleton, is a branching Brownian motion with initial configuration η, branching rate $\psi'(\lambda^*)$, and offspring distribution with generating function

$$F(s) := \frac{1}{\lambda^*\psi'(\lambda^*)}\psi\left(\lambda^*(1-s)\right) + s, \quad s\in(0,1). \tag{2.14}$$

The law of this offspring, denoted by $\{p_n : n \geq 0\}$, satisfies $p_0 = p_1 = 0$ and for $n \geq 2$,

$$p_n = \frac{1}{\lambda^*\psi'(\lambda^*)} \left\{ \beta(\lambda^*)^2 1_{\{n=2\}} + (\lambda^*)^n \int_{(0,\infty)} \frac{x^n}{n!} e^{-\lambda^* x} \nu(\mathrm{d}x) \right\}.$$

For the individuals in Z, we will use the classical Ulam-Harris notation. Let $\mathcal{T}^Z$ denote the set labels realized in Z and let $N_t^Z \subset \mathcal{T}^Z$ denote the set of individuals alive at time t, for $u \in N_t^Z$, we use $z_u(t)$ to denote the position of u at time t. The birth time and the death time of a particle u are denoted by b_u and d_u respectively.

(ii) $\{(X_t^{\mathcal{E}})_{t\geq 0}, \mathbf{P}_{(\mu,\eta)}\}$ is a copy of $\{(X_t)_{t\geq 0}; \mathbb{P}_\mu^{\mathcal{E}})$.

(iii) Three different types of immigration on Z: $I^{\mathbb{N}^{\mathcal{E}}} = \left\{ I_t^{\mathbb{N}^{\mathcal{E}}}, t \geq 0 \right\}, I^{\mathbb{P}^{\mathcal{E}}} = \left\{ I_t^{\mathbb{P}^{\mathcal{E}}}, t \geq 0 \right\}$ and $I^B = \left\{ I_t^B, t \geq 0 \right\}$, which are independent of $X^{\mathcal{E}}$ and, conditioned on Z, are independent of each other. The three processes are described as follows:

- Given Z, independently for each $u \in \mathcal{T}^Z$, let $\mathcal{N}^{\mathcal{E},u}$ be a Poisson random measure on $(b_u, d_u] \times \mathbb{D}$ with intensity $2\beta \mathrm{d}t \times \mathbb{N}_{z_u(t)}^{\mathcal{E}}(\mathrm{d}w)$. The continuous immigration $I^{\mathbb{N}^{\mathcal{E}}}$ is a measure-valued process on $\mathbb{R}$ such that
$$I_t^{\mathbb{N}^{\mathcal{E}}} := \sum_{u \in \mathcal{T}^Z} \int_{(b_u,d_u]\cap[0,t]} \int_{\mathbb{D}} w_{t-s} \mathcal{N}^{\mathcal{E},u}(\mathrm{d}s \times \mathrm{d}w).$$

- Given Z, independently for each $u \in \mathcal{T}^Z$, let $\{R_t^u : t \in (b_u, d_u]\}$ be a point process such that the random counting measure $\sum_{t\in(b_u,d_u]} \delta_{(t,R_t^u)}$ is a Poisson random measure on $(b_u, d_u] \times (0,\infty)$ with intensity $\mathrm{d}tre^{-\lambda^* r}\nu(\mathrm{d}r)$ and let $\{(s_i^{2,u}, r_i) : i \geq 1\}$ be the atoms of this Poisson random measure. The discrete immigration $I^{\mathbb{P}^{\mathcal{E}}}$ is a measure-valued process on $\mathbb{R}$ such that
$$I_t^{\mathbb{P}^{\mathcal{E}}} := \sum_{u\in\mathcal{T}^Z} \sum_{i: s_i^{2,u} \leq t} X^{(2,u,i)}_{t-s_i^{2,u}},$$
where $X^{(2,u,i)}$ is a measure-valued process with law $\mathbb{P}^{\mathcal{E}}_{r_i z_u(s_i^{2,u})}$.

- The branching point immigration I^B is a measure-valued process on $\mathbb{R}$ such that
$$I_t^B := \sum_{u\in\mathcal{T}^Z} 1_{\{d_u \leq t\}} X^{(3,u)}_{t-d_u},$$
here, given Z, independently for each $u \in \mathcal{T}^Z$ with $d_u \leq t$, $X^{(3,u)}$ is an independent copy of X issued at time d_u with law $\mathbb{P}^{\mathcal{E}}_{Y_u \delta_{z_u(d_u)}}$, where Y_u is an independent random variable with distribution $\pi_{O_u}(\mathrm{d}y), O_u$ is the number of the offspring of u and $\{\pi_n(\mathrm{d}y), n \geq 2\}$ is a sequence of probability measures such that
$$\pi_n(\mathrm{d}y) := \frac{1}{p_n \lambda^* \psi'(\lambda^*)} \left\{ \beta(\lambda^*)^2 \delta_0(\mathrm{d}y) 1_{\{n=2\}} + (\lambda^*)^n \frac{y^n}{n!} e^{-\lambda^* y} \nu(\mathrm{d}y) \right\}.$$

We define $\Lambda_t = \{\Lambda_t : t \geq 0\}$ on $\mathbb{R}$ by

$$\Lambda_t := X_t^{\mathcal{E}} + I_t^{\mathbb{N}^{\mathcal{E}}} + I_t^{\mathbb{P}^{\mathcal{E}}} + I_t^B, \quad t \geq 0.$$

For $\mu \in \mathcal{M}(\mathbb{R})$, we denote the law of a Poisson random measure with intensity $\lambda^* \mathrm{d}\mu$ by $\mathfrak{P}_\mu$, and define $\mathbf{P}_\mu$ by

$$\mathbf{P}_\mu := \int \mathbf{P}_{(\mu,\eta)} \mathfrak{P}_\mu(\mathrm{d}\eta).$$

According to [2, Theorem 2], for any $\mu \in \mathcal{M}(\mathbb{R})$, $\{(\Lambda_t)_{t\geq 0}; \mathbf{P}_\mu\}$ is equal in law to $\{(X_t)_{t\geq 0}; \mathbb{P}_\mu\}$. The branching Brownian motion $\{Z_t, t \geq 0\}$ is referred to as the skeleton process, and $\{(\Lambda_t)_{t\geq 0}; \mathbf{P}_\mu\}$ is called a *skeleton decomposition* of $\{(X_t)_{t\geq 0}; \mathbb{P}_\mu\}$.

2.4 Properties of Brownian motion and Bessel-3 process

Recall $B = \{(B_t)_{t\geq 0}; \Pi_x, x \in \mathbb{R}\}$ is a Brownian motion and $\tau_{-y} = \inf\{t > 0 : B_t = -y\}$ for $y \in \mathbb{R}$.

Lemma 2.2 *For $x \geq -y$,*

$$\Pi_x(t < \tau_{-y}) = 2\int_0^{(y+x)/\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z, \quad t \geq 0.$$

Proof: This can be easily obtained by the reflection principle of Brownian motion. □

Proposition 2.3 *There exists a constant C such that*

$$\int_0^\infty \Pi_z \left(B_s < x, \ \min_{r\in[0,s]} B_r > 0 \right) \mathrm{d}s \leq C(1+x)(1+\min\{x,z\}), \quad x, z \geq 0.$$

Proof: First note that, for any $h, t > 0$ and $y \in \mathbb{R}$, we have

$$\sup_{r\in\mathbb{R}} \Pi_y(r \leq B_t \leq r+h) = \sup_{r\in\mathbb{R}} \int_r^{r+h} \frac{1}{\sqrt{2\pi t}} e^{-(u-y)^2/(2t)} \mathrm{d}u \leq \sup_{r\in\mathbb{R}} \int_r^{r+h} \frac{\mathrm{d}u}{\sqrt{2\pi t}} = \frac{h}{\sqrt{2\pi t}}. \tag{2.15}$$

Next, for any $0 \leq a < b$, $z \geq 0, t > 0$, by the Markov property, we have

$$\begin{aligned} &\Pi_z \left(B_t \in [a,b], \min_{r\in[0,t]} B_r > 0 \right) \\ &\leq \Pi_z \left(\min_{r\in[0,t/3]} B_r > 0 \right) \sup_{y>0} \Pi_y \left(B_{2t/3} \in [a,b], \min_{r\in[0,2t/3]} B_r > 0 \right). \end{aligned} \tag{2.16}$$

It follows from Lemma 2.2 that

$$\Pi_z \left(\min_{r\in[0,t/3]} B_r > 0 \right) \leq \sqrt{\frac{2}{\pi}} \frac{z}{\sqrt{t/3}} = \sqrt{\frac{6}{\pi}} \frac{z}{\sqrt{t}}. \tag{2.17}$$

The second term of right-hand of (2.16) is bounded by

$$\begin{aligned} &\Pi_y \left(B_{2t/3} \in [a,b], \min_{r\in[0,2t/3]} B_r > 0 \right) \\ &\leq \Pi_y \left(\min_{s\in[t/3,2t/3]} (B_s - B_{2t/3}) > -b, B_0 - B_{2t/3} \in [y-b, y-a] \right) \\ &= \Pi_0 \left(\min_{s\in[0,t/3]} \tilde{B}_s > -b, \tilde{B}_{2t/3} \in [y-b, y-a] \right) \end{aligned}$$

$$\le \Pi_0\left(\min_{s\in[0,t/3]}\tilde{B}_s > -b\right)\sup_{v\in\mathbb{R}}\Pi_v(\tilde{B}_{t/3}\in[y-b,y-a])$$

$$\le \sqrt{\frac{6}{\pi}}\frac{b}{\sqrt{t}}\frac{b-a}{\sqrt{2\pi t/3}} = \frac{3}{\pi}\frac{b(b-a)}{t}, \tag{2.18}$$

where $\tilde{B}_s = B_{2t/3-s} - B_{2t/3}$ is a Brownian motion for $s\in[0,2t/3]$; we used the Markov property of $\tilde{B}$ at time $t/3$ in the second inequality of (2.18), and the last inequality of (2.18) is due to (2.17) and (2.15). Combining (2.16)-(2.18), we obtain

$$\Pi_z\left(B_t\in[a,b],\ \min_{r\in[0,t]}B_r>0\right)\le\sqrt{\frac{54}{\pi^3}}\frac{zb(b-a)}{\sqrt{t^3}},\quad z\ge 0. \tag{2.19}$$

If $x<z$, by the strong Markov property at τ_x, we have

$$\int_0^\infty \Pi_z\left(B_s<x,\ \min_{r\in[0,s]}B_r>0\right)\mathrm{d}s = \Pi_z\left[\int_0^\infty 1_{\{B_s<x,\ \min_{r\in[0,s]}B_r>0\}}\mathrm{d}s\right]$$
$$\le \Pi_z\left[\int_{\tau_x}^\infty 1_{\{B_s<x,\ \min_{r\in[\tau_x,s]}B_r>0\}}\mathrm{d}s\right] = \Pi_x\left[\int_0^\infty 1_{\{B_s<x,\ \min_{r\in[0,s]}B_r>0\}}\mathrm{d}s\right]$$
$$= \int_0^\infty \Pi_x\left(B_s<x,\ \min_{r\in[0,s]}B_r>0\right)\mathrm{d}s. \tag{2.20}$$

Using (2.19) and (2.20), we obtain that

$$\int_0^\infty \Pi_z\left(B_s<x,\ \min_{r\in[0,s]}B_r>0\right)\mathrm{d}s \le x^2+\int_{x^2}^\infty \Pi_x\left(B_s<x,\ \min_{r\in[0,s]}B_r>0\right)\mathrm{d}s$$
$$\le x^2+\int_{x^2}^\infty\sqrt{\frac{54}{\pi^3}}\frac{x^3}{\sqrt{s^3}}\mathrm{d}s\le C_1(1+x)^2 \tag{2.21}$$

for some constant $C_1>0$. If $x\ge z$, by (2.17) and (2.19), we also have

$$\int_0^\infty \Pi_z\left(B_s<x,\ \min_{r\in[0,s]}B_r>0\right)\mathrm{d}s$$
$$\le \int_0^{x^2}\Pi_z\left(\min_{r\in[0,s]}B_r>0\right)\mathrm{d}s+\int_{x^2}^\infty \Pi_z\left(B_s<x,\ \min_{r\in[0,s]}B_r>0\right)\mathrm{d}s$$
$$\le \int_0^{x^2}\sqrt{\frac{6}{\pi}}\frac{z}{\sqrt{s}}\mathrm{d}s+\int_{x^2}^\infty\sqrt{\frac{54}{\pi^3}}\frac{zx^2}{\sqrt{s^3}}\mathrm{d}s\le C_2(1+x)(1+z) \tag{2.22}$$

for some constant $C_2>0$. Combining (2.21) and (2.22), we arrive at the assertion of the proposition. □

The following is a direct consequence of [18, (3.1)]. From now on, we use $\mathbb{R}_+$ to denote $[0,\infty)$.

Lemma 2.4 *Suppose that* $\{(\eta_t)_{t\ge0};\widetilde{\Pi}_x, x\in\mathbb{R}_+\}$ *is a Bessel-3 process. If* F *is a non-negative function on* $C([0,t],\mathbb{R})$*, then*

$$\Pi_x\left[F\left(B_s,s\in[0,t]\right)1_{\{\forall s\in[0,t],B_s>0\}}\right]=\widetilde{\Pi}_x\left[\frac{x}{\eta_t}F\left(\eta_s,s\in[0,t]\right)\right],\quad x\in\mathbb{R}_+.$$

Lemma 2.5 *If* $\{(\eta_t)_{t\ge0};\widetilde{\Pi}_y, y\in\mathbb{R}_+\}$ *is a Bessel-3 process, then*

$$\widetilde{\Pi}_y\left[\eta_t^{-2}\right]\le\frac{2}{t},\quad t>0,\ y\ge0.$$

Proof: Using the inequality $1-e^{-x}\le x$ and the density of η_t given by (2.3), we have

$$\widetilde{\Pi}_y\left[\eta_t^{-2}\right]=\int_0^\infty x^{-2}f_{\eta_t}(x)\mathrm{d}x\le\int_{-\infty}^\infty x^{-2}\cdot\frac{2x^2}{t\sqrt{2\pi t}}e^{-(x-y)^2/2t}\mathrm{d}x=\frac{2}{t}.$$

□

Lemma 2.6 *Suppose that $\{(\eta_t)_{t\ge0};\widetilde{\Pi}_y,y\in\mathbb{R}_+\}$ is a Bessel-3 process. Then for any event A_t with $\lim_{t\to\infty}\widetilde{\Pi}_y(A_t)=1$, we have*

$$\lim_{t\to\infty}t\widetilde{\Pi}_y\left[\eta_t^{-2}1_{A_t^c}\right]=0.\tag{2.23}$$

Proof: For any $\varepsilon>0$, we have

$$\begin{aligned}\widetilde{\Pi}_y\left[\eta_t^{-2}1_{A_t^c}\right]&\le\widetilde{\Pi}_y\left[\eta_t^{-2}1_{A_t^c}1_{\{\eta_t\ge\varepsilon\sqrt{t}\}}\right]+\widetilde{\Pi}_y\left[\eta_t^{-2}1_{\{\eta_t<\varepsilon\sqrt{t}\}}\right]\\&\le\widetilde{\Pi}_y(A_t^c)\cdot\frac{1}{\varepsilon^2t}+\widetilde{\Pi}_y\left[\eta_t^{-2}1_{\{\eta_t<\varepsilon\sqrt{t}\}}\right].\end{aligned}\tag{2.24}$$

By the same estimate for the density of η_t in Lemma 2.5,

$$\begin{aligned}&\widetilde{\Pi}_y\left[\eta_t^{-2}1_{\{\eta_t<\varepsilon\sqrt{t}\}}\right]=\int_0^{\varepsilon\sqrt{t}}x^{-2}f_{\eta_t}(x)\mathrm{d}x\\&\le\frac{2}{t}\int_0^{\varepsilon\sqrt{t}}\frac{1}{\sqrt{2\pi t}}e^{-(x-y)^2/2t}\mathrm{d}x\le\frac{2}{t}\int_0^{\varepsilon\sqrt{t}}\frac{1}{\sqrt{2\pi t}}\mathrm{d}t=\frac{2\varepsilon}{\sqrt{2\pi}}\frac{1}{t}.\end{aligned}\tag{2.25}$$

Combining (2.24) and (2.25), letting $t\to\infty$, we get

$$\limsup_{t\to\infty}t\widetilde{\Pi}_y\left[\eta_t^21_{A_t^c}\right]\le\frac{2\varepsilon}{\sqrt{2\pi}}.$$

Since ε is arbitrary, we get (2.23). □

3 Proof of Theorem 1.1

Proposition 3.1 *For any $y>0$, we have*

$$\widetilde{\mathbb{P}}^{-y}\left[\xi_t^{\lambda_0}\in\mathrm{d}x\middle|\widetilde{X}^{\lambda_0}_{D^t_{-y}}\right]=\frac{e^{-\lambda_0x}(x+y)\widetilde{X}^{\lambda_0}_{D^t_{-y}}(\mathrm{d}x)}{\widetilde{V}_t^{-y}},$$

where

$$\widetilde{V}_t^{-y}:=\langle(y+\cdot)e^{-\lambda_0\cdot},\widetilde{X}^{\lambda_0}_{D^t_{-y}}\rangle.$$

Proof: The main idea comes from [20, Theorem 5.1]. Let $C_b^+(\partial D^t_{-y})$ be the set of bounded non-negative continuous functions on ∂D^t_{-y}. We only need to show that for any $g\in C_b^+(\partial D^t_{-y})$,

$$\widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\theta\xi_t^{\lambda_0}-\langle g,\widetilde{X}^{\lambda_0}_{D^t_{-y}}\rangle\right\}\right]=\widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\langle g,\widetilde{X}^{\lambda_0}_{D^t_{-y}}\rangle\right\}\frac{\langle e^{-(\lambda_0+\theta)\cdot}(\cdot+y),\widetilde{X}^{\lambda_0}_{D^t_{-y}}\rangle}{\widetilde{V}_t^{-y}}\right].\tag{3.1}$$

By (2.13) and the definition (2.11) of $\mathbb{Q}^{-y}$, the right hand side of (3.1) is equal to

$$\frac{1}{y}\mathbb{P}\left[\exp\left\{-\langle g, X^{\lambda_0}_{D^t_{-y}}\rangle\right\}\cdot\langle e^{-(\lambda_0+\theta)\cdot}(\cdot+y), X^{\lambda_0}_{D^t_{-y}}\rangle\right] = -\frac{1}{y}\mathbb{P}\left[\frac{\partial}{\partial\gamma}\left[\exp\left\{-\langle g_\gamma, X^{\lambda_0}_{D^t_{-y}}\rangle\right\}\right]\Big|_{\gamma=0^+}\right]$$

with $g_\gamma(t,x) = g(t,x)+\gamma e^{-(\lambda_0+\theta)x}(x+y)$. Interchanging the order of expectation and differentiation, we get that

$$\text{the right hand side of (3.1)} = -\frac{1}{y}\frac{\partial}{\partial\gamma}e^{-u^{-y}_{g_\gamma}(t,0)}\Big|_{\gamma=0^+},$$

where $u^{-y}_{g_\gamma}$ satisfies (2.7) and $u^{-y}_{g_0} = u^{-y}_g$. Thus,

$$\text{the right hand side of (3.1)} = \frac{1}{y}e^{-u^{-y}_g(t,0)}\frac{\partial}{\partial\gamma}u^{-y}_{g_\gamma}(t,0)\Big|_{\gamma=0^+}. \tag{3.2}$$

Let $m^{-y}_g(t,x) := \frac{\partial}{\partial\gamma}u^{-y}_{g_\gamma}(t,x)|_{\gamma=0^+}$. Replacing f by g_γ in (2.7), taking derivative with respect to γ, and then letting $\gamma\to 0+$, we get that m^{-y}_g is the solution to the equation

$$m^{-y}_g(t,x) + \Pi^{\lambda_0}_x\int_0^{t\wedge\tau_{-y}}\psi'\left(u^{-y}_g(t-r,B_r)\right)m^{-y}_g(t-r,B_r)\mathrm{d}r = \Pi^{\lambda_0}_x\left[e^{-(\lambda_0+\theta)B_{t\wedge\tau_{-y}}}(B_{t\wedge\tau_{-y}}+y)\right].$$

Note that $B_{t\wedge\tau_{-y}}+y=0$ when $t\geq\tau_{-y}$. The solution to the above integral equation is given by

$$m^{-y}_g(t,x) = \Pi^{\lambda_0}_x\left[e^{-(\lambda_0+\theta)B_t}(B_t+y)\exp\left\{-\int_0^t\psi'\left(u^{-y}_g(s,B_{t-s})\right)\mathrm{d}s\right\}, t<\tau_{-y}\right]. \tag{3.3}$$

By the definitions (2.1) and (2.2), we have

$$\begin{aligned}m^{-y}_g(t,0) =&\Pi_0\left[e^{-\frac{1}{2}\lambda_0^2t-\theta B_t}(B_t+y)\exp\left\{-\int_0^t\psi'\left(u^{-y}_g(s,B_{t-s})\right)\mathrm{d}s\right\}, t<\tau_{-y}\right]\\ =&y\widetilde{\Pi}_y\left[e^{-\frac{1}{2}\lambda_0^2t-\theta B_t}\exp\left\{-\int_0^t\psi'\left(u^{-y}_g(s,B_{t-s})\right)\mathrm{d}s\right\}\right].\end{aligned}$$

Using (3.2) and (3.3), we have

$$\text{the right hand side of (3.1)} = e^{-u^{-y}_g(0,t)}\widetilde{\Pi}_y\left[e^{-\lambda_0^2t/2-\theta B_t}\exp\left\{-\int_0^t\psi'\left(u^{-y}_g(s,B_{t-s})\right)\mathrm{d}s\right\}\right]. \tag{3.4}$$

Next we deal with the left-hand of (3.1). Applying Campbell's formula, we get

$$\begin{aligned}&\widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\langle g, X^{(\mathbf{n}),\lambda_0}_{D^t_{-y}}\rangle\right\}\Big|\xi^{\lambda_0}\right] = \widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\int_{[0,t]}\int_{\mathbb{D}}\langle g, w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right)\right\}\Big|\xi^{\lambda_0}\right]\\ &= \exp\left\{-2\beta\int_0^t\int_{\mathbb{D}}\left(1-\exp\left\{-\langle g, w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\right\}\right)\mathrm{d}\mathbb{N}_{\xi^{\lambda_0}_s}\mathrm{d}s\right\}\\ &= \exp\left\{-2\beta\int_0^t-\log\mathbb{P}_{\delta_{\xi^{\lambda_0}_s}}\left[\exp\left\{-\langle g, X^{\lambda_0}_{D^{t-s}_{-y}}\rangle\right\}\right]\mathrm{d}s\right\}\\ &= \exp\left\{-2\beta\int_0^t u^{-y}_g(t-s,\xi^{\lambda_0}_s)\mathrm{d}s\right\} = \exp\left\{-2\beta\int_0^t u^{-y}_g(s,\xi^{\lambda_0}_{t-s})\mathrm{d}s\right\}.\end{aligned} \tag{3.5}$$

For $X^{(\mathbf{m}),\lambda_0}$, let $m_s := \|X^{\mathbf{m},s,\lambda_0}_{D^0_{-y}}\|$ denote by the initial mass of the discrete immigration for $s \in D^{\mathbf{m}}$. Then $\{m_s : s \geq 0\}$ is a Poisson point process on $(0,\infty)^2$ with intensity $\mathrm{d}tr\nu(\mathrm{d}r)$. We similarly have

$$\widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\langle g, X^{(\mathbf{m}),\lambda_0}_{D^t_{-y}}\rangle\right\}\Big|\xi^{\lambda_0}\right] = \widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\sum_{s\in D^{\mathbf{m}}_t} m_s u^{-y}_g(t-s,\xi^{\lambda_0}_s)\right\}\Big|\xi^{\lambda_0}\right]$$

$$= \exp\left\{-\int_0^t\int_{(0,\infty)}\left(1-\exp\left\{-ru^{-y}_g(s,\xi^{\lambda_0}_{t-s})\right\}\right)r\nu(\mathrm{d}r)\mathrm{d}s\right\}. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\langle g, X^{(\mathbf{n}),\lambda_0}_{D^t_{-y}} + X^{(\mathbf{m}),\lambda_0}_{D^t_{-y}}\rangle\right\}\Big|\xi^{\lambda_0}\right] = \exp\left\{-\int_0^t\left[\psi'\left(u^{-y}_g(s,\xi^{\lambda_0}_{t-s})\right)-\psi'(0)\right]\mathrm{d}s\right\}. \tag{3.7}$$

Note that $(X^{\lambda_0})'$ is independent of ξ and has the same law as X^{λ_0}. So by (3.7),

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\theta\xi^{\lambda_0}_t - \langle g, \widetilde{X}^{\lambda_0}_{D^t_{-y}}\rangle\right\}\right]\\
&= \widetilde{\mathbb{P}}^{-y}\left[e^{-\theta\xi^{\lambda_0}_t}\widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\langle g, (X^{\lambda_0})'_{D^t_{-y}} + X^{(\mathbf{n}),\lambda_0}_{D^t_{-y}} + X^{(\mathbf{m}),\lambda_0}_{D^t_{-y}}\rangle\right\}\Big|\xi^{\lambda_0}\right]\right]\\
&= \widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\langle g, (X^{\lambda_0})'_{D^t_{-y}}\rangle\right\}\right]\widetilde{\mathbb{P}}^{-y}\left[e^{-\theta\xi^{\lambda_0}_t}\widetilde{\mathbb{P}}^{-y}\left[\exp\left\{-\langle g, X^{(\mathbf{n}),\lambda_0}_{D^t_{-y}} + X^{(\mathbf{m}),\lambda_0}_{D^t_{-y}}\rangle\right\}\Big|\xi^{\lambda_0}\right]\right]\\
&= e^{-u^{-y}_g(t,0)}\widetilde{\mathbb{P}}^{-y}\left[e^{-\theta\xi^{\lambda_0}_t}\exp\left\{-\int_0^t\left[\psi'\left(u^{-y}_g(s,\xi^{\lambda_0}_{t-s})\right)-\psi'(0^+)\right]\mathrm{d}s\right\}\right].
\end{aligned} \tag{3.8}$$

Recall that $-\psi'(0^+) = \lambda_0^2/2$, $\{y+B_t, t\geq 0; \widetilde{\Pi}_y\}$ is a Bessel-3 process starting from y and $\{\xi^{\lambda_0}_t + y, t \geq 0; \widetilde{\mathbb{P}}^{-y}\}$ is also a Bessel-3 process starting from y. Thus, by (3.4) and (3.8), (3.1) holds. □

For $t \geq 0$, define

$$\widetilde{W}^{-y}_t := (W^{-y}_t)' + \int_{[0,t]}\int_{\mathbb{D}}\langle e^{-\lambda_0\cdot}1_{(-y,\infty)}(\cdot), w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right) + \sum_{s\in D^{\mathbf{m}}_t} W^{\mathbf{m},s,-y}_{t-s}, \tag{3.9}$$

where

$$(W^{-y}_t)' := \langle e^{-\lambda_0\cdot}1_{(-y,\infty)}(\cdot), (X^{\lambda_0})'_{D^t_{-y}}\rangle, \quad W^{\mathbf{m},s,-y}_{t-s} := \langle e^{-\lambda_0\cdot}1_{(-y,\infty)}(\cdot), X^{\mathbf{m},s,\lambda_0}_{D^{t-s}_{-y}}\rangle.$$

By the spine decomposition (2.12), $(W^{-y}_t, t\geq 0; \mathbb{Q}^{-y})$ has the same law as $(\widetilde{W}^{-y}_t, t \geq 0; \widetilde{\mathbb{P}}^{-y})$. Recall the definition (2.10) of V^{-y}_t and that $(V^{-y}_t, t\geq 0; \mathbb{Q}^{-y})$ has the same law as $(\widetilde{V}^{-y}_t, t\geq 0; \widetilde{\mathbb{P}}^{-y})$. Note also that

$$\widetilde{V}^{-y}_t = (V^{-y}_t)' + \int_{[0,t]}\int_{\mathbb{D}}\langle (y+\cdot)e^{-\lambda_0\cdot}, w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right) + \sum_{s\in D^{\mathbf{m}}_t} V^{\mathbf{m},s,-y}_{t-s},$$

where

$$(V^{-y}_t)' := \langle (y+\cdot)e^{-\lambda_0\cdot}, (X^{\lambda_0})'_{D^t_{-y}}\rangle, \quad V^{\mathbf{m},s,-y}_{t-s} := \langle (y+\cdot)e^{-\lambda_0\cdot}, X^{\mathbf{m},s,\lambda_0}_{D^{t-s}_{-y}}\rangle.$$

Lemma 3.2 *For any $y > 0$ fixed, we have*

$$\lim_{t\to\infty}\sqrt{t}\,\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}^{-y}_t}{\widetilde{W}^{-y}_t + \widetilde{V}^{-y}_t}\right] = \sqrt{\frac{2}{\pi}}.$$

Proof: First notice that

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}\right]=\mathbb{Q}^{-y}\left[\frac{W_t^{-y}}{V_t^{-y}}\right]=\frac{1}{y}\mathbb{P}[W_t^{-y}].$$

Using (2.6), and noting that $\lambda_0^2/2=\alpha$, we have that for any $f\in\mathcal{B}_b^+(\mathbb{R})$,

$$\mathbb{P}_{\delta_x}\left[\langle f, X^{\lambda_0}_{D^t_{-y}}\rangle\right]=\Pi_x^{\lambda_0}\left[e^{\lambda_0^2(t\wedge\tau_{-y})/2}f(B_{t\wedge\tau_{-y}})\right].$$

Using the mean formula above with $f(x)=e^{-\lambda_0 x}1_{(-y,\infty)}(x)$, we obtain that

$$\begin{aligned}\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}\right]&=\frac{1}{y}\mathbb{P}[W_t^{-y}]=\frac{1}{y}\Pi_0^{\lambda_0}\left[e^{\lambda_0^2(t\wedge\tau_{-y})/2}e^{-\lambda_0 B_{t\wedge\tau_{-y}}}1_{(-y,\infty)}(B_{t\wedge\tau_{-y}})\right]\\&=\frac{1}{y}\Pi_0^{\lambda_0}\left[e^{\lambda_0^2 t/2}e^{-\lambda_0 B_t}1_{\{t<\tau_{-y}\}}\right]=\frac{1}{y}\Pi_0(t<\tau_{-y})=\frac{2}{y}\int_0^{y/\sqrt{t}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,\mathrm{d}x.\end{aligned}$$

Thus

$$\lim_{t\to\infty}\sqrt{t}\,\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}\right]=\sqrt{\frac{2}{\pi}}.\tag{3.10}$$

To complete the proof of the lemma, it suffices to show that

$$\limsup_{t\to\infty}\sqrt{t}\widetilde{\mathbb{P}}^{-y}\left[\frac{(\widetilde{W}_t^{-y})^2}{(\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y})\widetilde{V}_t^{-y}}\right]=\limsup_{t\to\infty}\sqrt{t}\left\{\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}\right]-\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\right]\right\}=0.$$

It follows from Proposition 3.1 that

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{1}{\xi_t^{\lambda_0}+y}\Big|\widetilde{X}^{\lambda_0}_{D^t_{-y}}\right]=\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}.\tag{3.11}$$

Under $\widetilde{\mathbb{P}}^{-y}$, $\xi^{\lambda_0}+y$ is a Bessel-3 process starting from y. So by Lemma 2.5, (3.11) and Jensen's inequality, we have

$$\begin{aligned}\widetilde{\mathbb{P}}^{-y}\left[\frac{(\widetilde{W}_t^{-y})^2}{(\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y})\widetilde{V}_t^{-y}}\right]&\le\widetilde{\mathbb{P}}^{-y}\left[\left(\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}\right)^2\right]=\widetilde{\mathbb{P}}^{-y}\left[\left(\widetilde{\mathbb{P}}^{-y}\left[\frac{1}{\xi_t^{\lambda_0}+y}\Big|\widetilde{X}^{\lambda_0}_{D^t_{-y}}\right]\right)^2\right]\\&\le\widetilde{\mathbb{P}}^{-y}\left[\left(\frac{1}{\xi_t^{\lambda_0}+y}\right)^2\right]\le\frac{2}{t}.\end{aligned}\tag{3.12}$$

Therefore

$$\sqrt{t}\widetilde{\mathbb{P}}^{-y}\left[\frac{(\widetilde{W}_t^{-y})^2}{(\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y})\widetilde{V}_t^{-y}}\right]=o(1),\qquad\text{as } t\to\infty.$$

This concludes the proof. □

Next we prove the following result:

Proposition 3.3

$$\lim_{t\to\infty} \widetilde{\mathbb{P}}^{-y}\left[\left(\frac{\sqrt{t}\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}-\sqrt{\frac{2}{\pi}}\right)^2\right]=0. \tag{3.13}$$

To prove (3.13), we first prove some lemmas. Let E_t be events with $\lim_{t\to\infty}\widetilde{\mathbb{P}}^{-y}(E_t)=1$. Combining (3.11) and the estimate $\widetilde{\mathbb{P}}^{-y}\left[\left(\widetilde{W}_t^{-y}/\widetilde{V}_t^{-y}\right)^2\right]\le \frac{2}{t}$ in (3.12), we get

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[\left(\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y}}\right)^2\right]\le \widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y}}\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}\right]\\
&=\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\widetilde{\mathbb{P}}^{-y}\left[\frac{1}{\xi_t^{\lambda_0}+y}\Big|\widetilde{X}^{\lambda_0}_{D^t_{-y}}\right]\right]\\
&=\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1}{\xi_t^{\lambda_0}+y}\right]\le \widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}\frac{1_{E_t^c}}{\xi_t^{\lambda_0}+y}\right]+\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]\\
&\le\sqrt{\widetilde{\mathbb{P}}^{-y}\left[\left(\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}\right)^2\right]\widetilde{\mathbb{P}}^{-y}\left[\left(\frac{1_{E_t^c}}{\xi_t^{\lambda_0}+y}\right)^2\right]}+\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]\\
&\le\sqrt{\frac{2}{t}}\sqrt{\widetilde{\mathbb{P}}^{-y}\left[\frac{1_{E_t^c}}{(\xi_t^{\lambda_0}+y)^2}\right]}+\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right].
\end{aligned}\tag{3.14}$$

Note that, under $\widetilde{\mathbb{P}}^{-y}$, $\xi_t^{\lambda_0}+y$ is a Bessel-3 process starting from y. Using Lemma 2.6 and the assumption that $\widetilde{\mathbb{P}}^{-y}(E_t)\to 1$ as $t\to\infty$, we have

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{1_{E_t^c}}{(\xi_t^{\lambda_0}+y)^2}\right]=o\left(\frac{1}{t}\right). \tag{3.15}$$

By (3.14) and (3.15), we conclude that

$$\widetilde{\mathbb{P}}^{-y}\left[\left(\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y}}\right)^2\right]\le o\left(\frac{1}{t}\right)+\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]. \tag{3.16}$$

Next, we need to construct E_t such that the right-hand side of (3.16) is bounded by $2/(\pi t)+o(1/t)$. Let $[0,\infty)\ni t\mapsto k_t$ be a positive function such that $\lim_{t\to\infty}k_t/(\log t)^6=\infty$ and $\lim_{t\to\infty}k_t/\sqrt{t}=0$. For instance, we can take $k_t=(\log t)^7$ for large t. For $t>0$ large, we define

$$\begin{aligned}
\widetilde{W}_t^{-y,[0,k_t)}&:=(W_t^{-y})'+\int_{[0,k_t)}\int_{\mathbb{D}}\langle e^{-\lambda_0\cdot}1_{(-y,\infty)}(\cdot),w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right)+\sum_{s\in D^{\mathbf{m}}\cap[0,k_t)}W_{t-s}^{\mathbf{m},s,-y},\\
\widetilde{W}_t^{-y,[k_t,t]}&:=\int_{[k_t,t]}\int_{\mathbb{D}}\langle e^{-\lambda_0\cdot}1_{(-y,\infty)}(\cdot),w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right)+\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}W_{t-s}^{\mathbf{m},s,-y},\\
\widetilde{V}_t^{-y,[0,k_t)}&:=(V_t^{-y})'+\int_{[0,k_t)}\int_{\mathbb{D}}\langle(y+\cdot)e^{-\lambda_0\cdot},w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right)+\sum_{s\in D^{\mathbf{m}}\cap[0,k_t)}V_{t-s}^{\mathbf{m},s,-y},
\end{aligned}$$

$$\widetilde{V}_t^{-y,[k_t,t]} := \int_{[k_t,t]}\int_{\mathbb{D}} \langle (y+\cdot)e^{-\lambda_0\cdot}, w^{\lambda_0}_{D^{t-s}_{-y}}\rangle \mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times \mathrm{d}w^{\lambda_0}\right) + \sum_{s\in D^{\mathbf{m}}\cap[k_t,t]} V^{\mathbf{m},s,-y}_{t-s}.$$

Recall that $m_s = \|X^{\mathbf{m},s,\lambda_0}_{D^0_{-y}}\|$. Define

$$E_{t,1} := \{k_t^{1/3}\le \xi^{\lambda_0}_{k_t}\le k_t\}\bigcap\left\{\inf_{s\in[k_t,t]}\xi^{\lambda_0}_s \ge k_t^{1/6}\right\}, \qquad E_{t,2} := \bigcap_{s\in D^{\mathbf{m}}\cap[k_t,t]}\left\{m_s\le e^{\lambda_0\xi_s^{\lambda_0}/2}\right\},$$

$$E_{t,3} := \left\{\widetilde{V}_t^{-y,[k_t,t]} + \widetilde{W}_t^{-y,[k_t,t]}\le \frac{1}{t^2}\right\}, \qquad E_t := E_{t,1}\cap E_{t,2}\cap E_{t,3}.$$

Lemma 3.4 *For any fixed $y>0$, it holds that*

$$\lim_{t\to\infty}\sup_{u\in[k_t^{1/3},k_t]}\widetilde{\mathbb{P}}^{-y}\left[E^c_{t,2}\middle|\xi^{\lambda_0}_{k_t}=u\right]=0.$$

Proof: First, by Campbell's formula, we have

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[E^c_{t,2}\middle|\xi^{\lambda_0}_{k_t}=u\right] = \widetilde{\mathbb{P}}^{-y}\left[\bigcup_{s\in D^{\mathbf{m}}\cap[k_t,t]}\{m_s> e^{\lambda_0\xi_s^{\lambda_0}/2}\}\middle|\xi^{\lambda_0}_{k_t}=u\right]\\
&\le \widetilde{\mathbb{P}}^{-y}\left[\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]} 1_{\{m_s>e^{\lambda_0\xi_s^{\lambda_0}/2}\}}\middle|\xi^{\lambda_0}_{k_t}=u\right] \le \widetilde{\mathbb{P}}^{-y}\left[\sum_{s\in D^{\mathbf{m}}\cap[k_t,\infty)} 1_{\{m_s>e^{\lambda_0\xi_s^{\lambda_0}/2}\}}\middle|\xi^{\lambda_0}_{k_t}=u\right]\\
&= \widetilde{\mathbb{P}}^{-y}\left[\int_{k_t}^\infty \mathrm{d}s\int_0^\infty 1_{\{\xi_s^{\lambda_0}<2\log r/\lambda_0\}} r\nu(\mathrm{d}r)\middle|\xi^{\lambda_0}_{k_t}=u\right]. \qquad (3.17)
\end{aligned}$$

Since under $\widetilde{\mathbb{P}}^{-y}, \xi_s^{\lambda_0}>-y$ for all $s\ge 0$, it holds that

$$\begin{aligned}
1_{\{\xi_s^{\lambda_0}<2\log r/\lambda_0\}} &= 1_{\{\xi_s^{\lambda_0}<2\log r/\lambda_0\}}\cdot 1_{\{-y<2\log r/\lambda_0\}} + 1_{\{\xi_s^{\lambda_0}<2\log r/\lambda_0\}}\cdot 1_{\{-y\ge 2\log r/\lambda_0\}}\\
&= 1_{\{\xi_s^{\lambda_0}<2\log r/\lambda_0\}}\cdot 1_{\{-y<2\log r/\lambda_0\}} + 1_{\{\xi_s^{\lambda_0}<2\log r/\lambda_0\le -y\}}\\
&= 1_{\{\xi_s^{\lambda_0}<2\log r/\lambda_0\}}\cdot 1_{\{-y<2\log r/\lambda_0\}}. \qquad (3.18)
\end{aligned}$$

Plugging (3.18) into (3.17) and noting that $-y<2\log r/\lambda_0 \Leftrightarrow r>e^{-\lambda_0 y/2}$, we get that

$$\begin{aligned}
\widetilde{\mathbb{P}}^{-y}\left[E^c_{t,2}\middle|\xi^{\lambda_0}_{k_t}=u\right] &\le \widetilde{\mathbb{P}}^{-y}\left[\int_{k_t}^\infty \mathrm{d}s\int_0^\infty 1_{\{\xi_s^{\lambda_0}<2\log r/\lambda_0\}} r\nu(\mathrm{d}r)\middle|\xi^{\lambda_0}_{k_t}=u\right]\\
&= \widetilde{\mathbb{P}}^{-y}\left[\int_{k_t}^\infty \mathrm{d}s\int_{e^{-\lambda_0 y/2}}^\infty 1_{\{\xi_s^{\lambda_0}<2\log r/\lambda_0\}} r\nu(\mathrm{d}r)\middle|\xi^{\lambda_0}_{k_t}=u\right]\\
&= \int_{k_t}^\infty \mathrm{d}s\int_{e^{-\lambda_0 y/2}}^\infty r\nu(\mathrm{d}r)\widetilde{\mathbb{P}}^{-y}\left[\xi_s^{\lambda_0}<2\log r/\lambda_0\middle|\xi^{\lambda_0}_{k_t}=u\right]. \qquad (3.19)
\end{aligned}$$

By the Markov property, when $s\ge k_t$,

$$\widetilde{\mathbb{P}}^{-y}\left[\xi_s^{\lambda_0}<2\log r/\lambda_0\middle|\xi^{\lambda_0}_{k_t}=u\right] = \widetilde{\mathbb{P}}^{-(y+u)}\left[\xi^{\lambda_0}_{s-k_t}+u<2\log r/\lambda_0\right]. \qquad (3.20)$$

So (3.19) and (3.20) yield that

$$\widetilde{\mathbb{P}}^{-y}\left[E^c_{t,2}\middle|\xi^{\lambda_0}_{k_t}=u\right] \le \int_{k_t}^\infty \mathrm{d}s\int_{e^{-\lambda_0 y/2}}^\infty r\nu(\mathrm{d}r)\widetilde{\mathbb{P}}^{-(y+u)}\left[\xi^{\lambda_0}_{s-k_t}+u<2\log r/\lambda_0\right]$$

$$= \int_0^\infty \mathrm{d}s \int_{e^{-\lambda_0 y/2}}^\infty r\nu(\mathrm{d}r)\widetilde{\mathbb{P}}^{-(y+u)}\left[\xi_s^{\lambda_0} + u < 2\log r/\lambda_0\right]. \tag{3.21}$$

Now by Lemma 2.4 and Proposition 2.3, (3.21) is bounded above by

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[E_{t,2}^c\Big|\xi_{k_t}^{\lambda_0} = u\right] \le \int_0^\infty \mathrm{d}s \int_{e^{-\lambda_0 y/2}}^\infty r\nu(\mathrm{d}r)\widetilde{\mathbb{P}}^{-(y+u)}\left[\xi_s^{\lambda_0} + u < 2\log r/\lambda_0\right] \\
&= \int_{e^{-\lambda_0 y/2}}^\infty r\nu(\mathrm{d}r)\int_0^\infty \mathrm{d}s \frac{1}{u+y}\Pi_{u+y}\left(B_s 1_{\{B_s < y + 2\log r/\lambda_0, s<\tau_0\}}\right) \\
&\le \int_{e^{-\lambda_0 y/2}}^\infty r\nu(\mathrm{d}r)\int_0^\infty \mathrm{d}s \frac{y + 2\log r/\lambda_0}{u+y}\Pi_{u+y}\left(B_s < y + 2\log r/\lambda_0, s<\tau_0\right) \\
&\le \frac{C}{u+y}\int_{e^{-\lambda_0 y/2}}^\infty r(1+y+2\log r/\lambda_0)^2\left(1+\min\{y+2\log r/\lambda_0, u+y\}\right)\nu(\mathrm{d}r).
\end{aligned} \tag{3.22}$$

For any fixed $\varepsilon > 0$, note that $2\log r/\lambda_0 \le \varepsilon u \iff r \le e^{\varepsilon\lambda_0 u/2}$. We suppose that t is large enough such that for any $u \in [k_t^{1/3}, k_t]$, $u + y > 1$ and $1 + \varepsilon u + y \le 2\varepsilon(u + y)$. Thus,

$$\begin{aligned}
\widetilde{\mathbb{P}}^{-y}\left[E_{t,2}^c\Big|\xi_{k_t}^{\lambda_0} = u\right] \le & \frac{C}{u+y}\int_{e^{-\lambda_0 y/2}}^{e^{\varepsilon\lambda_0 u/2}} r(1+y+2\log r/\lambda_0)^2(1+y+2\log r/\lambda_0)\nu(\mathrm{d}r) \\
&+ \frac{C(1+u+y)}{u+y}\int_{e^{\varepsilon\lambda_0 u/2}}^\infty r(1+y+2\log r/\lambda_0)^2\nu(\mathrm{d}r) \\
\le & \frac{C}{u+y}\int_{e^{-\lambda_0 y/2}}^{e^{\varepsilon\lambda_0 u/2}} r(1+y+2\log r/\lambda_0)^2\left(1+y+\varepsilon u\right)\nu(\mathrm{d}r) \\
&+ \frac{C(1+u+y)}{u+y}\int_{e^{\varepsilon\lambda_0 u/2}}^\infty r(1+y+2\log r/\lambda_0)^2\nu(\mathrm{d}r) \\
\le & 2C\varepsilon\int_{e^{-\lambda_0 y/2}}^\infty r(1+y+2\log r/\lambda_0)^2\nu(\mathrm{d}r) \\
&+ 2C\int_{e^{\varepsilon\lambda_0 k_t^{1/3}/2}}^\infty r(1+y+2\log r/\lambda_0)^2\nu(\mathrm{d}r).
\end{aligned} \tag{3.23}$$

Using condition (1.6) and taking $t \to \infty$, (3.23) yields that

$$\limsup_{t\to\infty} \sup_{u\in[k_t^{1/3}, k_t]} \widetilde{\mathbb{P}}^{-y}\left[E_{t,2}^c\Big|\xi_{k_t}^{\lambda_0} = u\right] \le C\varepsilon\int_{e^{-\lambda_0 y/2}}^\infty r(1+y+2\log r/\lambda_0)^2\nu(\mathrm{d}r).$$

Since ε is arbitrary, the desired assertion is valid. □

Lemma 3.5 *For any fixed $y > 0$, there exist constants $T, C' > 0$ such that for any $t \ge T$,*

$$\widetilde{\mathbb{P}}^{-y}\left[E_{t,1}\cap E_{t,2}\cap E_{t,3}^c\big|\xi^{\lambda_0}\right] \le \frac{C'}{t}, \quad \widetilde{\mathbb{P}}^{-y}\text{-a.s.}$$

Proof: Recall that W_t^{-y} is defined in (2.9). Define $\mathcal{W}_t^{-y}$ by

$$\mathcal{W}_t^{-y} := \langle e^{-\lambda_0\cdot}, X_{D_{-y}^t}^{\lambda_0}\rangle.$$

By (2.6), for any $t, r > 0$ and $z \ge -y$, $\mathbb{P}_{r\delta_z}\left[\mathcal{W}_t^{-y}\right] = re^{-\lambda_0 z}$, which does not depend on t. By this and the special Markov property (2.8), we see that $\mathcal{W}_t^{-y}$ is a non-negative $\mathbb{P}_{r\delta_z}$-martingale. Note that $W_t^{-y} \le \mathcal{W}_t^{-y}$. Similarly to (3.9), we define

$$\mathcal{W}_{t-s}^{\mathbf{m},s,-y} := \langle e^{-\lambda_0\cdot}, X_{D_{-y}^{t-s}}^{\mathbf{m},s,\lambda_0}\rangle.$$

Because $E_{t,1}\in\sigma(\xi_t:t\geq 0)$, by the martingale property of $\mathcal{W}_t^{-y}$, we obtain that

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}}\int_{[k_t,t]}\int_{\mathbb{D}}\langle e^{-\lambda_0\cdot}1_{(-y,\infty)}(\cdot),w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right)\Big|\xi^{\lambda_0}\right]\\
&\leq\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}}\int_{[k_t,t]}\int_{\mathbb{D}}\langle e^{-\lambda_0\cdot},w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right)\Big|\xi^{\lambda_0}\right]\\
&=2\beta 1_{E_{t,1}}\int_{k_t}^t\mathbb{N}_{\xi_s^{\lambda_0}}\left(\langle e^{-\lambda_0\cdot},w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\Big|\xi^{\lambda_0}\right)\mathrm{d}s=2\beta 1_{E_{t,1}}\int_{k_t}^t\mathbb{P}_{\delta_{\xi_s^{\lambda_0}}}\left(\mathcal{W}_{t-s}^{-y}\Big|\xi^{\lambda_0}\right)\mathrm{d}s\\
&=2\beta 1_{E_{t,1}}\int_{k_t}^t e^{-\lambda_0\xi_s^{\lambda_0}}\mathrm{d}s\leq 2\beta te^{-\lambda_0k_t^{1/6}}\leq 2\beta te^{-\lambda_0k_t^{1/6}/4},
\end{aligned}\tag{3.24}$$

where the second to the last inequality of (3.24) holds because on $E_{t,1}$ we have $\xi_s\geq k_t^{1/6}$ for all $k_t\leq s\leq t$. Next, for $s\in D^{\mathbf{m}}$ and recall that $m_s=\|X^{\mathbf{m},s}_{D^0_{-y}}\|$, by the martingale property of $\mathcal{W}_t^{-y}$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}\cap E_{t,2}}\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}W_{t-s}^{\mathbf{m},s,-y}\Big|\xi^{\lambda_0},\mathbf{m}\right]\leq\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}\cap E_{t,2}}\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}\mathcal{W}_{t-s}^{\mathbf{m},s,-y}\Big|\xi^{\lambda_0},\mathbf{m}\right]\\
&=1_{E_{t,1}\cap E_{t,2}}\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}\mathbb{P}_{m_s\delta_{\xi_s^{\lambda_0}}}\left(\mathcal{W}_{t-s}^{\mathbf{m},s,-y}\Big|\xi^{\lambda_0},\mathbf{m}\right)=1_{E_{t,1}\cap E_{t,2}}\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}e^{-\lambda_0\xi_s^{\lambda_0}}m_s\\
&\leq 1_{E_{t,1}}\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}e^{-\lambda_0\xi_s^{\lambda_0}/2}1_{\{m_s>1\}}+1_{E_{t,1}}\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}e^{-\lambda_0\xi_s^{\lambda_0}}m_s1_{\{m_s\leq 1\}}\\
&\leq e^{-\lambda_0k_t^{1/6}/2}\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}1_{\{m_s>1\}}+e^{-\lambda_0k_t^{1/6}}\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}m_s1_{\{m_s\leq 1\}}.
\end{aligned}\tag{3.25}$$

Taking expectation with respect to $\mathbf{m}$ in (3.25), we get

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}\cap E_{t,2}}\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}W_{t-s}^{\mathbf{m},s,-y}\Big|\xi^{\lambda_0}\right]\\
&\leq e^{-\lambda_0k_t^{1/6}/2}\widetilde{\mathbb{P}}^{-y}\left[\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}1_{\{m_s>1\}}\Big|\xi^{\lambda_0}\right]+e^{-\lambda_0k_t^{1/6}}\widetilde{\mathbb{P}}^{-y}\left[\sum_{s\in D^{\mathbf{m}}\cap[k_t,t]}m_s1_{\{m_s\leq 1\}}\Big|\xi^{\lambda_0}\right]\\
&=e^{-\lambda_0k_t^{1/6}/2}\int_{k_t}^t\mathrm{d}s\int_1^\infty r\nu(\mathrm{d}r)+e^{-\lambda_0k_t^{1/6}}\int_{k_t}^t\mathrm{d}s\int_0^1 r^2\nu(\mathrm{d}r)\\
&\leq te^{-\lambda_0k_t^{1/6}/2}\int_1^\infty r\nu(\mathrm{d}r)+te^{-\lambda_0k_t^{1/6}}\int_0^1 r^2\nu(\mathrm{d}r)\leq C_3te^{-\lambda_0k_t^{1/6}/4}
\end{aligned}\tag{3.26}$$

for some constant C_3. Similarly, for large t such that for all $u\geq k_t^{1/3}$, $(y+u)\leq e^{\lambda_0u/4}$, we have

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}}\int_{[k_t,t]}\int_{\mathbb{D}}\langle(y+\cdot)e^{-\lambda_0\cdot},w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\mathcal{N}^{\lambda_0}\left(\mathrm{d}s\times\mathrm{d}w^{\lambda_0}\right)\Big|\xi^{\lambda_0}\right]\\
&=2\beta 1_{E_{t,1}}\int_{k_t}^t\mathbb{N}_{\xi_s^{\lambda_0}}\left(\langle(y+\cdot)e^{-\lambda_0\cdot},w^{\lambda_0}_{D^{t-s}_{-y}}\rangle\Big|\xi^{\lambda_0}\right)\mathrm{d}s\\
&=2\beta 1_{E_{t,1}}\int_{k_t}^t\mathbb{P}_{\delta_{\xi_s^{\lambda_0}}}\left(V_{t-s}^{-y}\Big|\xi^{\lambda_0}\right)\mathrm{d}s=2\beta 1_{E_{t,1}}\int_{k_t}^t e^{-\lambda_0\xi_s^{\lambda_0}}(y+\xi_s^{\lambda_0})\mathrm{d}s
\end{aligned}$$

$$\leq 2\beta t e^{-3\lambda_0 k_t^{1/6}/4} \leq 2\beta t e^{-\lambda_0 k_t^{1/6}/4}. \tag{3.27}$$

For large t such that for all $u \geq k_t^{1/3}$, $(y+u) \leq e^{\lambda_0 u/4}$, we also have

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}\cap E_{t,2}} \sum_{s\in D^{\mathbf{m}}\cap[k_t,t]} V_{t-s}^{\mathbf{m},s,-y}\middle|\xi^{\lambda_0},\mathbf{m}\right] = 1_{E_{t,1}\cap E_{t,2}} \sum_{s\in D^{\mathbf{m}}\cap[k_t,t]} \mathbb{P}_{m_s\delta_{\xi_s^{\lambda_0}}}\left(V_{t-s}^{\mathbf{m},s,-y}\middle|\xi^{\lambda_0},\mathbf{m}\right)\\
&= 1_{E_{t,1}\cap E_{t,2}} \sum_{s\in D^{\mathbf{m}}\cap[k_t,t]} e^{-\lambda_0\xi_s^{\lambda_0}}(y+\xi_s^{\lambda_0})m_s \leq 1_{E_{t,1}\cap E_{t,2}} \sum_{s\in D^{\mathbf{m}}\cap[k_t,t]} e^{-3\lambda_0\xi_s^{\lambda_0}/4} m_s\\
&\leq e^{-\lambda_0 k_t^{1/6}/4} \sum_{s\in D^{\mathbf{m}}\cap[k_t,t]} 1_{\{m_s>1\}} + e^{-3\lambda_0 k_t^{1/6}/4} \sum_{s\in D^{\mathbf{m}}\cap[k_t,t]} m_s 1_{\{m_s\leq 1\}}.
\end{aligned} \tag{3.28}$$

Taking expectation with respect to $\mathbf{m}$ in (3.28), we obtain that for some constant C_4,

$$\begin{aligned}
\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}\cap E_{t,2}} \sum_{s\in D^{\mathbf{m}}\cap[k_t,t]} V_{t-s}^{\mathbf{m},s,-y}\middle|\xi^{\lambda_0}\right] &\leq t e^{-\lambda_0 k_t^{1/6}/4}\int_1^\infty r\nu(\mathrm{d}r) + t e^{-3\lambda_0 k_t^{1/6}/4}\int_0^1 r^2\nu(\mathrm{d}r)\\
&\leq C_4 t e^{-\lambda_0 k_t^{1/6}/4}.
\end{aligned} \tag{3.29}$$

Combining (3.24), (3.26), (3.27) and (3.29), we get that

$$\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}\cap E_{t,2}}\left(\widetilde{V}_t^{-y,[k_t,t]} + \widetilde{W}_t^{-y,[k_t,t]}\right)\middle|\xi^{\lambda_0}\right] \leq (C_3+C_4+4\beta) t e^{-\lambda_0 k_t^{1/6}/4}.$$

On $E_{t,3}^c$ we have $\widetilde{V}_t^{-y,[k_t,t]} + \widetilde{W}_t^{-y,[k_t,t]} > 1/t^2$. Then for t large enough such that $k_t^{1/6} > 16\log t/\lambda_0$, we have

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}\cap E_{t,2}\cap E_{t,3}^c}\middle|\xi^{\lambda_0}\right] \leq t^2 \widetilde{\mathbb{P}}^{-y}\left[1_{E_{t,1}\cap E_{t,2}}\left(\widetilde{V}_t^{-y,[k_t,t]} + \widetilde{W}_t^{-y,[k_t,t]}\right)\middle|\xi^{\lambda_0}\right]\\
&\leq (C_3+C_4+4\beta) t^3 e^{-\lambda_0 k_t^{1/6}/4} \leq (C_3+C_4+4\beta) t^{-1}.
\end{aligned}$$

The proof is complete. □

Lemma 3.6 *For any $y > 0$, we have*

$$\lim_{t\to\infty} \widetilde{\mathbb{P}}^{-y}[E_t] = 1 \tag{3.30}$$

and

$$\lim_{t\to\infty} \inf_{k_t^{1/3}\leq u\leq k_t} \widetilde{\mathbb{P}}^{-y}[E_t|\xi_{k_t}^{\lambda_0} = u] = 1. \tag{3.31}$$

Proof: First, by Lemma 3.4,

$$\lim_{t\to\infty} \sup_{u\in[k_t^{1/3},k_t]} \widetilde{\mathbb{P}}^{-y}\left[E_{t,2}^c\middle|\xi_{k_t}^{\lambda_0} = u\right] = 0. \tag{3.32}$$

By Lemma 3.5, we have

$$\lim_{t\to\infty} \sup_{u\in[k_t^{1/3},k_t]} \widetilde{\mathbb{P}}^{-y}\left[E_{t,1}\cap E_{t,2}\cap E_{t,3}^c|\xi_{k_t}^{\lambda_0} = u\right] = 0.$$

Note that

$$\Omega = E_t \cup E_{t,2}^c \cup E_{t,1}^c \cup (E_{t,1} \cap E_{t,2} \cap E_{t,3}^c). \tag{3.33}$$

To prove (3.31), we only need to prove that

$$\inf_{u\in[k_t^{1/3},k_t]} \widetilde{\mathbb{P}}^{-y}[E_{t,1}|\xi_{k_t}^{\lambda_0} = u] \to 1, \quad \text{as } t\to\infty. \tag{3.34}$$

Recall that under $\widetilde{\mathbb{P}}^{-y}, y+\xi_t^{\lambda_0}$ is a Bessel-3 process starting from y. Now let $\eta_t := \xi_t^{\lambda_0} + y$. Then $(\eta, \widetilde{\mathbb{P}}^{-y})$ is equal in law with $(\eta, \widetilde{\Pi}_y)$. For any $u \in [k_t^{1/3}, k_t]$, by the Markov property and Lemma 2.4, we have

$$\begin{aligned}\widetilde{\mathbb{P}}^{-y}[E_{t,1}|\xi_{k_t}^{\lambda_0} = u] &\ge \widetilde{\Pi}_{y+u}\left(\min_{r\in[0,t-k_t]} \eta_r \ge k_t^{1/6} + y\right)\\ &= \frac{1}{y+u}\Pi_0\left[(B_{t-k_t} + y + u)\,1_{\{\min_{r\in[0,t-k_t]} B_r \ge k_t^{1/6} - u\}}\right].\end{aligned} \tag{3.35}$$

Set $a = u - k_t^{1/6} \ge 0$. Then using the fact that $\Pi_0 B_{t\wedge\tau_{-a}} = 0$ for any $t\ge 0$, we have

$$0 = \Pi_0 B_{(t-k_t)\wedge\tau_{-a}} = -a\Pi_0(\tau_{-a} < t-k_t) + \Pi_0(B_{t-k_t}1_{\{\tau_{-a}\ge t-k_t\}}).$$

Also note that by Lemma 2.2,

$$\Pi_0(\tau_{-a} \le t-k_t) = 2\int_{a/\sqrt{t-k_t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\mathrm{d}x.$$

Then the right-hand of (3.35) is equal to

$$\begin{aligned}&\frac{1}{y+u}\Pi_0\left[B_{t-k_t}1_{\{\tau_{-a}\ge t-k_t\}} + (y+u)1_{\{\tau_{-a}\ge t-k_t\}}\right]\\ &= 1 - \frac{2(y+k_t^{1/6})}{y+u}\int_{(u-k_t^{1/6})/\sqrt{t-k_t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\mathrm{d}x.\end{aligned} \tag{3.36}$$

By (3.35) and (3.36), we get

$$\widetilde{\mathbb{P}}^{-y}[E_{t,1}|\xi_{k_t}^{\lambda_0} = u] \ge 1 - \frac{2(y+k_t^{1/6})}{y+k_t^{1/3}}\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\mathrm{d}x.$$

By the assumption on k_t, we get (3.34).

Now we prove (3.30). We claim that

$$\widetilde{\mathbb{P}}^{-y}[k_t^{1/3} \le \xi_{k_t}^{\lambda_0} \le k_t] = \widetilde{\Pi}_y[k_t^{1/3} + y \le \eta_{k_t} \le k_t + y] \to 1, \quad \text{as } t\to\infty. \tag{3.37}$$

In fact, by Theorem 3.2 of [26], $\lim_{t\to\infty} \log(\eta_t)/\log t = 1/2$, $\widetilde{\Pi}_y$-a.s. Using the fact that $k_t\to\infty$ as $t\to\infty$, we get (3.37) holds. Combining (3.37) and (3.32), we have

$$\lim_{t\to\infty} \widetilde{\mathbb{P}}^{-y}[E_{t,2}^c] = 0. \tag{3.38}$$

Combining (3.37) and (3.34), we have

$$\lim_{t\to\infty} \widetilde{\mathbb{P}}^{-y}[E_{t,1}] = 1. \tag{3.39}$$

It follows from Lemma 3.5 that

$$\lim_{t\to\infty} \widetilde{\mathbb{P}}^{-y}\left[E_{t,1}\cap E_{t,2}\cap E_{t,3}^c\right]=0. \tag{3.40}$$

Using (3.33), and combining (3.38)-(3.40), we obtain (3.30). □

Lemma 3.7 *For any $y>0$, it holds that*

$$\limsup_{t\to\infty} t\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]\le \frac{2}{\pi}.$$

Proof: First note that

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]=\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[k_t,t]}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]+\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right].$$

For the first term on the right hand, we have

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[k_t,t]}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]\le \widetilde{\mathbb{P}}^{-y}\left[\frac{1/t^2}{\widetilde{V}_t^{-y}(y+k_t^{1/6})}\right]=\frac{1}{yt^2(k_t^{1/6}+y)},$$

here we used the property that $E_t\subset\{\xi_t\ge k_t^{1/6}\}$, $E_t\subset E_{t,3}$ and the equality $\widetilde{\mathbb{P}}^{-y}\left[\frac{1}{\widetilde{V}_t^{-y}}\right]=\mathbb{Q}^{-y}\left[\frac{1}{V_t^{-y}}\right]=\frac{1}{y}$. Hence,

$$\lim_{t\to\infty} t\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[k_t,t]}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]=0.$$

Therefore, we only need to prove that

$$\limsup_{t\to\infty} t\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]\le \frac{2}{\pi}. \tag{3.41}$$

Note that

$$\begin{aligned}\widetilde{\mathbb{P}}^{-y}&\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]\le \widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right]\\ &\le \widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)}}1_{\{\xi_{k_t}^{\lambda_0}\in[k_t^{1/3},k_t]\}}\right]\times\sup_{u\in[k_t^{1/3},k_t]}\widetilde{\mathbb{P}}^{-y}\left[\frac{1}{\xi_t^{\lambda_0}+y}\middle|\xi_{k_t}^{\lambda_0}=u\right].\end{aligned} \tag{3.42}$$

In the last inequality we used the Markov property of ξ. Let $\{(\eta_t)_{t\ge0},\widetilde{\Pi}_{u+y}\}$ be a Bessel-3 process starting from $u+y$. By Lemmas 2.4 and 2.2, we have

$$\begin{aligned}\widetilde{\mathbb{P}}^{-y}\left[\frac{1}{\xi_t^{\lambda_0}+y}\middle|\xi_{k_t}^{\lambda_0}=u\right]&=\widetilde{\Pi}_{u+y}\left[\frac{1}{\eta_{t-k_t}}\right]=\frac{1}{u+y}\Pi_{u+y}\left[1_{\{\min_{r\in[0,t-k_t]}B_r>0\}}\right]\\ &=\frac{1}{u+y}\Pi_0(\tau_{-(y+u)}>t-k_t)=\frac{2}{y+u}\int_0^{(y+u)/\sqrt{t-k_t}}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\mathrm{d}x.\end{aligned} \tag{3.43}$$

By (3.42) and (3.43), we get

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right] \leq \widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)}}1_{\{\xi_{k_t}^{\lambda_0}\in[k_t^{1/3},k_t]\}}\right]$$
$$\times \sup_{u\in[k_t^{1/3},k_t]}\frac{2}{y+u}\int_0^{(y+u)/\sqrt{t-k_t}}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\mathrm{d}x. \qquad (3.44)$$

Because $\lim_{\varepsilon\to 0^+}\frac{2}{\varepsilon}\int_0^\varepsilon e^{-x^2/2}/\sqrt{2\pi}\mathrm{d}x = \sqrt{2/\pi}$ and $(y+u)/\sqrt{t-k_t}$ converges to 0 uniformly on $u\in[k_t^{1/3},k_t]$ as $t\to\infty$, we have

$$\sup_{u\in[k_t^{1/3},k_t]}\frac{2\sqrt{t}}{y+u}\int_0^{(y+u)/\sqrt{t-k_t}}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\mathrm{d}x \to \sqrt{\frac{2}{\pi}}. \qquad (3.45)$$

Using the Markov property at time k_t again, we get

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)}}1_{E_t}\right]$$
$$\geq \widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)}}1_{\{\xi_{k_t}^{\lambda_0}\in[k_t^{1/3},k_t]\}}\right]\cdot \inf_{u\in[k_t^{1/3},k_t]}\widetilde{\mathbb{P}}^{-y}[E_t|\xi_{k_t}^{\lambda_0}=u]. \qquad (3.46)$$

Because $\widetilde{W}_t^{-y,[0,k_t)}/(\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)})\cdot 1_{E_t}\leq 1$, the left-hand of (3.46) is bounded above by

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)}}1_{E_t}\right] \leq \widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)}}1_{E_t}1_{\{\widetilde{V}_t^{-y}>1/t\}}\right]+\widetilde{\mathbb{P}}^{-y}\left[\widetilde{V}_t^{-y}\leq\frac{1}{t}\right]$$
$$\leq \widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{V}_t^{-y,[0,k_t)}}1_{E_t}1_{\{\widetilde{V}_t^{-y}>1/t\}}\right]+\frac{1}{t}\widetilde{\mathbb{P}}^{-y}\left[\frac{1}{\widetilde{V}_t^{-y}}\right] = \widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{V}_t^{-y,[0,k_t)}}1_{E_t}1_{\{\widetilde{V}_t^{-y}>1/t\}}\right]+\frac{1}{ty}, \qquad (3.47)$$

where in the last inequality we used the Markov inequality for $\left(\widetilde{V}_t^{-y}\right)^{-1}$. Fix a constant $\eta\in(0,1)$, on $E_t\cap\{\widetilde{V}_t^{-y}>1/t\}$, we have, for large t such that $t>\eta^{-1}$, $\widetilde{V}_t^{-y,[k_t,t]}\leq\eta\widetilde{V}_t^{-y}$. So when t is large, using (3.47), we have

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)}}1_{E_t}\right] \leq \frac{1}{ty}+\frac{1}{1-\eta}\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}}\right].$$

By (3.10), we have

$$\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y,[0,k_t)}+\widetilde{V}_t^{-y,[0,k_t)}}1_{E_t}\right] \leq \frac{\sqrt{2/\pi}}{(1-\eta)\sqrt{t}}+o\left(\frac{1}{\sqrt{t}}\right), \quad \text{as } t\to\infty. \qquad (3.48)$$

By (3.31), (3.44), (3.45), (3.46) and (3.48), we finally get that

$$\limsup_{t\to\infty} t\widetilde{\mathbb{P}}^{-y}\left[\frac{\widetilde{W}_t^{-y,[0,k_t)}}{\widetilde{W}_t^{-y}+\widetilde{V}_t^{-y}}\frac{1_{E_t}}{\xi_t^{\lambda_0}+y}\right] \leq \frac{2}{\pi(1-\eta)}.$$

Since the above holds for any small $\eta\in(0,1)$, (3.41) holds. The proof is complete. □

Proof of Proposition 3.3: Applying Lemmas 3.2 and 3.7, and (3.16), we get

$$\limsup_{t\to\infty}\widetilde{\mathbb{P}}^{-y}\left[\left(\frac{\sqrt{t}\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y}}-\sqrt{\frac{2}{\pi}}\right)^2\right]$$
$$=\limsup_{t\to\infty}\left\{\widetilde{\mathbb{P}}^{-y}\left[\left(\frac{\sqrt{t}\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y}}\right)^2\right]-\frac{2}{\pi}\right\}-2\sqrt{\frac{2}{\pi}}\lim_{t\to\infty}\left\{\widetilde{\mathbb{P}}^{-y}\left[\frac{\sqrt{t}\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y}}\right]-\sqrt{\frac{2}{\pi}}\right\}\le 0,$$

which means that (3.13) holds. □

Proof of Theorem 1.1: Let $\mathcal{R}^{\lambda_0}$ and $\widetilde{\mathcal{R}}^{\lambda_0}$ be the smallest closed set containing $\bigcup_{t\ge 0}\mathrm{supp}X_t^{\lambda_0}$ and $\bigcup_{t\ge 0}\mathrm{supp}\widetilde{X}_t^{\lambda_0}$, respectively. Then by [20, Corollary 3.2], under condition (1.6), $\mathbb{P}(\inf\mathcal{R}^{\lambda_0}>-\infty)=1$. So for any $0<\eta<\mathbb{P}(\mathcal{E}^c)$, there exists $K>0$ such that $\mathbb{P}(\inf\mathcal{R}^{\lambda_0}>-K)>1-\eta$. Let $y:=K$ be fixed and define $\Omega_k:=\{\inf\mathcal{R}^{\lambda_0}>-K\}$ and $\widetilde{\Omega}_k:=\{\inf\widetilde{\mathcal{R}}^{\lambda_0}>-K\}$. Then

$$\mathbb{P}(\Omega_K\cap\mathcal{E}^c)\ge\mathbb{P}(\Omega_K)+\mathbb{P}(\mathcal{E}^c)-1>1-\eta+\mathbb{P}(\mathcal{E}^c)-1>0.$$

For any $\varepsilon>0$, put

$$G_t=\left\{\Big|\frac{\sqrt{t}W_t^{-y}}{V_t^{-y}+W_t^{-y}}-\sqrt{\frac{2}{\pi}}\Big|>\varepsilon\right\},\quad \widetilde{G}_t=\left\{\Big|\frac{\sqrt{t}\widetilde{W}_t^{-y}}{\widetilde{V}_t^{-y}+\widetilde{W}_t^{-y}}-\sqrt{\frac{2}{\pi}}\Big|>\varepsilon\right\}.$$

Define $\mathbb{P}^{**}(\cdot)=\mathbb{P}(\cdot|\Omega_K\cap\mathcal{E}^c)$. By (3.13) we have $\lim_{t\to\infty}\widetilde{\mathbb{P}}^{-y}[\widetilde{G}_t]=0$. Thus,

$$\frac{\mathbb{P}(\Omega_K\cap\mathcal{E}^c)}{y}\lim_{t\to\infty}\mathbb{P}^{**}[V_t^{-y}1_{G_t}]=\lim_{t\to\infty}\widetilde{\mathbb{P}}^{-y}[\widetilde{G}_t\cap\widetilde{\Omega}_K\cap\widetilde{\mathcal{E}}^c]=\lim_{t\to\infty}\widetilde{\mathbb{P}}^{-y}[\widetilde{G}_t]=0,$$

where $\widetilde{\mathcal{E}}:=\{\exists t\ge 0 \text{ such that } \|\widetilde{X}_t^{\lambda_0}\|=0\}$ with $\widetilde{\mathbb{P}}^{-y}$-probability 0. Then by Proposition 3.3, we have

$$V_t^{-y}1_{G_t}\xrightarrow[t\to\infty]{}0\quad\text{in probability with respect to }\mathbb{P}^{**}.\tag{3.49}$$

Notice that on the event $\Omega_K:=\{\inf\mathcal{R}^{\lambda_0}>-K\}$, we have

$$V_t^{-y}=V_t^{-K}=\partial W_t+KW_t>0,\qquad W_t^{-y}=W_t^{-K}=W_t,$$

and $\lim_{t\to\infty}V_t^{-y}=\partial W_\infty>0$ $\mathbb{P}^{**}$-a.s.. Together with (3.49) we get $\lim_{t\to\infty}\mathbb{P}^{**}[G_t]=0$ for any $\varepsilon>0$, which says

$$\frac{\sqrt{t}W_t^{-y}}{V_t^{-y}+W_t^{-y}}=\frac{\sqrt{t}W_t}{\partial W_t+(K+1)W_t}\xrightarrow[t\to\infty]{}\sqrt{\frac{2}{\pi}}\quad\text{in probability with respect to }\mathbb{P}^{**}.\tag{3.50}$$

Recall that $\mathbb{P}(\mathcal{E}^c)=1-e^{-\lambda^*}>0$ and $\mathbb{P}^{**}(W_t>0,\forall t>0)=\mathbb{P}^{**}(\lim_{t\to\infty}W_t>0)=1$. According to (3.50) we get

$$\frac{\partial W_t}{\sqrt{t}W_t}\xrightarrow[t\to\infty]{}\sqrt{\frac{\pi}{2}}\quad\text{in probability with respect to }\mathbb{P}^{**}.$$

For any $\gamma>0$, define

$$A_t=\left\{\Big|\frac{\partial W_t}{\sqrt{t}W_t}-\sqrt{\frac{\pi}{2}}\Big|>\gamma\right\}.$$

Then $\lim_{t\to\infty}\mathbb{P}^{**}[1_{A_t}] = 0$. Noticing that $\mathbb{P}^*(\cdot) = \mathbb{P}(\cdot|\mathcal{E}^c)$ and $\mathbb{P}^*[1_{A_t}1_{\Omega_K}] = \mathbb{P}^{**}[1_{A_t}]\mathbb{P}(\Omega_K \cap \mathcal{E}^c)/\mathbb{P}(\mathcal{E}^c)$, we obtain that

$$1_{A_t}1_{\Omega_K} \xrightarrow[t\to\infty]{} 0 \quad \text{in probability with respect to } \mathbb{P}^*,$$

which means $\limsup_{t\to\infty}\mathbb{P}^*(A_t) \le \lim_{t\to\infty}\mathbb{P}^*(A_t\cap\Omega_K) + \mathbb{P}^*(\Omega_K^c) \le \eta/\mathbb{P}(\mathcal{E}^c)$. Since η is arbitrary, we deduce that $\lim_{t\to\infty}\mathbb{P}^*(A_t) = 0$ for any $\gamma > 0$, which says

$$\frac{\partial W_t}{\sqrt{t}W_t} \xrightarrow[t\to\infty]{} \sqrt{\frac{\pi}{2}} \quad \text{in probability with respect to } \mathbb{P}^*.$$

This is also equivalent to say that, on the event $\mathcal{E}^c$, we have

$$\sqrt{t}W_t \xrightarrow[t\to\infty]{} \sqrt{\frac{2}{\pi}}\partial W_\infty \quad \text{in probability with respect to } \mathbb{P} \tag{3.51}$$

On $\mathcal{E}$, (3.51) holds obviously. The proof is now complete. □

4 Proof of Theorem 1.2

Recall the definitions of the process $\{(Z_t,\Lambda_t)_{t\ge0}\}$ and the probability measures $\mathbf{P}_{(\mu,\eta)}$ and $\mathbf{P}_\mu$ with $\mu\in\mathcal{M}(\mathbb{R})$ and $\eta\in\mathcal{M}_a(\mathbb{R})$, defined in Subsection 2.3. Set $\mathbf{P} := \mathbf{P}_{\delta_0}$. By the skeleton decomposition for X, $(\Lambda_t,\mathbf{P})$ is equal in law to $(X,\mathbb{P})$. To prove Theorem 1.2, we only need to prove that on survival event $\left(\mathcal{E}^\Lambda\right)^c$ where $\mathcal{E}^\Lambda := \{\lim_{t\to\infty}\|\Lambda_t\| = 0\}$,

$$\limsup_{t\to\infty}\sqrt{t}\langle e^{-\lambda_0(\cdot+\lambda_0 t)},\Lambda_t\rangle = +\infty \quad \mathbf{P}\text{-almost surely.} \tag{4.1}$$

The intuitive idea for proving the limit above is that the behaviour of Λ is determined by the skeleton Z. By branching property of Z we only consider the law $\mathbf{P}_{(\delta_0,\delta_0)}$. Let $\{\mathbf{e}_n : n\ge1\}$ be iid exponential random variables independent of Z. Let $T_0 := 0$ and $T_n = \sum_{i=1}^n \mathbf{e}_i$ for $n\ge1$. If we look at Z at independent times $\{T_n : n = 1,2,...\}$, then $\{Z_{T_n}, n\ge1\}$ is a branching random walk. We expect the behavior of this branching random walk to dominate the behavior of Λ. Let $\{\mathcal{Z}_n, n\ge1\}$ be the translation of $\{Z_{T_n}, n\ge1\}$ defined in (4.4) below. We will show that $\{\mathcal{Z}_n, n\ge1\}$ satisfies conditions of Aidekon and Shi [1]. Then by [1, Theorem 6.1],

$$\liminf_{n\to\infty}\left(L_n^{\mathcal{Z}} - \frac{1}{2}\log n\right) = -\infty \quad \mathbf{P}_{(\delta_0,\delta_0)}\text{-almost surely,}$$

where $L_n^{\mathcal{Z}}$ is minimum of the support of $\mathcal{Z}_n$. Let L_t^Z be minimum of the support of Z_t. By definition (4.4), $L_n^{\mathcal{Z}} = \lambda_0(L_{T_n}^Z + \lambda_0 T_n)$, and then we have

$$\liminf_{n\to\infty}\left(\lambda_0(L_{T_n}^Z + \lambda_0 T_n) - \frac{1}{2}\log T_n\right) = -\infty \quad \mathbf{P}_{(\delta_0,\delta_0)}\text{-almost surely.} \tag{4.2}$$

We will bound $\langle e^{-\lambda_0(\cdot+\lambda_0 T_n)},\Lambda_{T_n}\rangle$ from below by immigrations along the path of $L_\cdot^Z$, and then use the limit result (4.2) for $L_{T_n}^Z$ to get (4.1).

Now we prove the above rigorously. Note that

$$\mathbf{P}(\cdot) = \sum_{k=0}^\infty \frac{(\lambda^*)^k}{k!}e^{-\lambda^*}\mathbf{P}_{(\delta_0,k\delta_0)}(\cdot), \tag{4.3}$$

and $\mathbf{P}(\mathcal{E}^\Lambda) = \mathbb{P}(\mathcal{E}) = e^{-\lambda^*}$. It is obvious that $\mathbf{P}_{(\delta_0,0\delta_0)}(\mathcal{E}^\Lambda) = 1$. Together with (4.3), we know that for $k \geq 1, \mathbf{P}_{(\delta_0,k\delta_0)}(\mathcal{E}^\Lambda) = 0$. Thus, to prove Theorem 1.2, it suffices to show that, for any $k \geq 1$, the limsup in (1.8) is valid $\mathbf{P}_{(\delta_0,k\delta_0)}$-almost surely. By the branching property, without loss of generality, we only need to deal with the case of $k = 1$.

Let $\{\mathbf{e}_n : n \geq 1\}$ be iid exponential random variables with parameter $\kappa \in (0, \infty)$, independent of Z. Put $T_0 := 0$ and $T_n = \sum_{i=1}^n \mathbf{e}_i$ for $n \geq 1$. Now for $n \geq 1$, we define $\mathcal{Z}_n$ so that, for any $f \in \mathcal{B}_b^+(\mathbb{R})$,

$$\langle f, \mathcal{Z}_n\rangle = \langle f\left(\lambda_0(\cdot + \lambda_0 T_n)\right), Z_{T_n}\rangle. \tag{4.4}$$

Then $\{(\mathcal{Z}_n)_{n\geq 1}, \mathbf{P}_{(\delta_0,\delta_0)}\}$ is a branching random walk. Define $m := \sum_{n\geq 0} np_n = F'(1-)$, where we used (2.14). It is easy to check that $\lambda_0 = \sqrt{2\psi'(\lambda^*)(m-1)}$. We first check that the conditions of [1, Theorem 6.1] for $\mathcal{Z}$ are satisfied. More precisely, under assumption (1.7), (1.1) (1.2) and (1.3) hold. For simplicity, we define

$$W_n^{\mathcal{Z}} := \langle e^{-\cdot}, \mathcal{Z}_n\rangle, \quad D_n^{\mathcal{Z}} := \langle \cdot e^{-\cdot}, \mathcal{Z}_n\rangle, \quad D_n^{\mathcal{Z},2} := \langle (\cdot)^2 e^{-\cdot}, \mathcal{Z}_n\rangle, \quad D_n^{\mathcal{Z},+} := \langle (\cdot)_+ e^{-\cdot}, \mathcal{Z}_n\rangle.$$

The additive martingale associated to Z with parameter λ is defined as

$$W_s^Z(\lambda) := e^{-\lambda c_\lambda s}\langle e^{-\lambda\cdot}, Z_s\rangle = e^{-(\lambda-\lambda_0)^2 s/2}\langle e^{-\lambda(\cdot+\lambda_0 s)}, Z_s\rangle, \tag{4.5}$$

where $c_\lambda := \lambda/2 + \psi'(\lambda^*)(m-1)/\lambda = (\lambda^2+\lambda_0^2)/(2\lambda)$ and $\lambda c_\lambda = (\lambda-\lambda_0)^2/2 + \lambda\lambda_0$.

Lemma 4.1 *If $\sum_{n\geq 1} n(\log n)^2 p_n < \infty$, then*

$$\mathbf{P}_{(\delta_0,\delta_0)}\left[W_1^{\mathcal{Z}}\right] = 1, \quad \mathbf{P}_{(\delta_0,\delta_0)}\left[D_1^{\mathcal{Z}}\right] = 0, \quad \mathbf{P}_{(\delta_0,\delta_0)}\left[D_1^{\mathcal{Z},2}\right] < \infty \tag{4.6}$$

and

$$\mathbf{P}_{(\delta_0,\delta_0)}\left[W_1^{\mathcal{Z}}\log_+^2 W_1^{\mathcal{Z}}\right] < \infty, \quad \mathbf{P}_{(\delta_0,\delta_0)}\left[D_1^{\mathcal{Z},+}\log_+ D_1^{\mathcal{Z},+}\right] < \infty. \tag{4.7}$$

Proof : *Step* 1 : Define W_s^Z and D_s^Z by

$$W_s^Z := \langle e^{-\lambda_0(\cdot+\lambda_0 s)}, Z_s\rangle, \quad D_s^Z := \langle (\cdot+\lambda_0 s)e^{-\lambda_0(\cdot+\lambda_0 s)}, Z_s\rangle.$$

Then by [19], W_s^Z and D_s^Z are the additive martingale and the derivative martingale associated to the branching Brownian motion Z in the critical case $\lambda = \lambda_0$ respectively.

By some direct calculation and the martingale property, we have

$$\mathbf{P}_{(\delta_0,\delta_0)}\left[W_1^{\mathcal{Z}}\right] = \int_0^\infty \kappa e^{-\kappa s}\mathbf{P}_{(\delta_0,\delta_0)}\left[W_s^Z\right]\mathrm{d}s = \int_0^\infty \kappa e^{-\kappa s}\mathrm{d}s = 1,$$
$$\mathbf{P}_{(\delta_0,\delta_0)}\left[D_1^{\mathcal{Z}}\right] = \int_0^\infty \kappa e^{-\kappa s}\mathbf{P}_{(\delta_0,\delta_0)}\left[D_s^Z\right]\mathrm{d}s = 0.$$

Now define

$$D_s^{Z,2} := \lambda_0^2\langle(\cdot+\lambda_0 s)^2 e^{-\lambda_0(\cdot+\lambda_0 s)}, Z_s\rangle.$$

Using the many-to-one formula, we get

$$\mathbf{P}_{(\delta_0,\delta_0)}\left[D_1^{\mathcal{Z},2}\right] = \int_0^\infty \kappa e^{-\kappa s}\mathbf{P}_{(\delta_0,\delta_0)}\left[D_s^{Z,2}\right]\mathrm{d}s = \int_0^\infty \kappa e^{-\kappa s}\lambda_0^2 e^{\lambda_0^2 s/2}\Pi_0\left[(B_s+\lambda_0 s)^2 e^{-\lambda_0(B_s+\lambda_0 s)}\right]\mathrm{d}s$$

$$= \lambda_0^2 \int_0^\infty \kappa e^{-\kappa s} \Pi_0^{-\lambda_0} \left[(B_s + \lambda_0 s)^2 \right] \mathrm{d}s = \lambda_0^2 \int_0^\infty \kappa s e^{-\kappa s} \mathrm{d}s < \infty.$$

Thus, (4.6) holds.

Step 2 : In this step we prove the first inequality of (4.7). Define a new probability $\mathbb{Q}^Z$ by

$$\left. \frac{\mathrm{d}\mathbb{Q}^Z}{\mathrm{d}\mathbf{P}_{(\delta_0,\delta_0)}} \right|_{\sigma(Z_r^1, r\le s)} := W_s^Z, \quad s \ge 0.$$

Then under $\mathbb{Q}^Z$, Z has the following spine decomposition:

(i) There is a initial marked particle moving as a Brownian motion with drift $-\lambda_0$ starting from 0, we denote the trajectory of this particle by w_s.

(ii) The branching rate of this marked particle is $\psi'(\lambda^*)m$ and the offspring distribution of the marked particle is given by $\widetilde{p}_n := np_n/m, n = 1, 2, \ldots$.

(iii) When the marked particle dies, given the number of the offspring, mark one of its offspring uniformly.

(iv) The unmarked individuals evolve independently as Z under $\mathbb{P}_{(\delta_0,\delta_0)}$.

Note that

$$\mathbf{P}_{(\delta_0,\delta_0)} \left[W_1^{\mathcal{Z}} \log_+^2 W_1^{\mathcal{Z}} \right] = \int_0^\infty \kappa e^{-\kappa s} \mathbf{P}_{(\delta_0,\delta_0)} \left[W_s^Z \log_+^2 W_s^Z \right] \mathrm{d}s. \tag{4.8}$$

By a change of measure, we have

$$\mathbf{P}_{(\delta_0,\delta_0)} \left[W_s^Z \log_+^2 W_s^Z \right] = \mathbb{Q}^Z \left[\log_+^2 W_s^Z \right].$$

Let $A > 4$ be a constant such that

$$\log A(\log A - 2\log 2) \ge \sup_{a\ge 1} \left(\log^2(a+1) - \log^2 a \right). \tag{4.9}$$

There exists such an A since for all $a \ge 1$, by inequality $\ln(x+1) \le x$, we have

$$\log_+^2(a+1) - \log_+^2 a = (\log(a+1) + \log a) \left(\log \left(1 + a^{-1} \right) \right) \le (2a-1) \times a^{-1} < 2.$$

Now let $b, c \ge A$. Using (4.9), it is easy to check that the inequality

$$\log^2(b+c) \le \log^2 b + \log^2 c \tag{4.10}$$

holds by assuming $b \ge c$ and $b = ac$. For $\ell \ge 1$, we use Γ_ℓ to denote the ℓ-th fission time of the spine under $\mathbb{Q}^Z$, and O_ℓ the number of offspring at the fission time Γ_ℓ. Then

$$\begin{aligned} W_s^Z =& \sum_{\ell\ge 1} 1_{\{\Gamma_\ell \le s\}} e^{-\lambda_0^2 \Gamma_\ell} W_{s-\Gamma_\ell}^{Z,\Gamma_\ell} 1_{\left\{ e^{-\lambda_0^2 \Gamma_\ell} W_{s-\Gamma_\ell}^{Z,\Gamma_\ell} < A \right\}} \\ &+ \sum_{\ell\ge 1} 1_{\{\Gamma_\ell \le s\}} e^{-\lambda_0^2 \Gamma_\ell} W_{s-\Gamma_\ell}^{Z,\Gamma_\ell} 1_{\left\{ e^{-\lambda_0^2 \Gamma_\ell} W_{s-\Gamma_\ell}^{Z,\Gamma_\ell} \ge A \right\}} + e^{-\lambda_0(w_s + \lambda_0 s)} \\ =&: H_1 + H_2 + H_3, \end{aligned} \tag{4.11}$$

where, given the information along the spine, W^{Z,Γ_ℓ} is the additive martingale associated with the branching Brownian motion starting from the $O_\ell - 1$ unmarked individuals. Note that for any

$x, y, z > 0$, we have $\log_+^2(x+y+z) \le \log_+^2(3x) + \log_+^2(3y) + \log_+^2(3z)$ and $\log_+^2 x \le 4x$. Then (4.11) implies that

$$\log_+^2 W_s^Z \le \log_+^2(3H_1) + \log_+^2(3H_2) + \log_+^2(3H_3) \le 12H_1 + \log_+^2(3H_2) + \log_+^2(3H_3). \tag{4.12}$$

Since $H_1 \le A \sum_{\ell \ge 1} 1_{\{\Gamma_\ell \le s\}}$, we have

$$\mathbb{Q}^Z[H_1] \le A \int_0^s \psi'(\lambda^*) m \mathrm{d}r = A\psi'(\lambda^*) ms. \tag{4.13}$$

Also, note that $w_s + \lambda_0 s$ under $\mathbb{Q}^Z$ is a standard Brownian motion, so

$$\begin{aligned} &\mathbb{Q}^Z\left[\log_+^2(3H_3)\right] \le 2(\log 3)^2 + 2\mathbb{Q}^Z\left[\log_+^2(H_3)\right] \\ &\le 2(\log 3)^2 + 2\lambda_0^2 \mathbb{Q}^Z(w_s + \lambda_0 s)^2 = 2(\log 3)^2 + 2\lambda_0^2 s. \end{aligned} \tag{4.14}$$

Here in the first inequality above we used inequality

$$\log_+^2(ab) \le (\log_+ a + \log_+ b)^2 \le 2\log_+^2 a + 2\log_+^2 b. \tag{4.15}$$

Define

$$\overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell} := e^{\lambda_0 w_{\Gamma_\ell}} W^{Z,\Gamma_\ell}_{s-\Gamma_\ell}.$$

Using (4.10) and (4.15) again, we deduce that

$$\begin{aligned} &\log_+^2(3H_2) \le 2(\log 3)^2 + 2\log_+^2(H_2) \\ &\le 2(\log 3)^2 + 2\sum_{\ell \ge 1} 1_{\{\Gamma_\ell \le s\}} 1_{\left\{e^{-\lambda_0^2 \Gamma_\ell} W^{Z,\Gamma_\ell}_{s-\Gamma_\ell} \ge A\right\}} \log_+^2\left[e^{-\lambda_0^2 \Gamma_\ell} W^{Z,\Gamma_\ell}_{s-\Gamma_\ell}\right] \\ &\le 2(\log 3)^2 + 4\sum_{\ell \ge 1} 1_{\{\Gamma_\ell \le s\}} \log_+^2 \overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell} + 4\sum_{\ell \ge 1} 1_{\{\Gamma_\ell \le s\}} \log_+^2\left(e^{-\lambda_0(w_{\Gamma_\ell} + \lambda_0 \Gamma_\ell)}\right) \\ &\le 2(\log 3)^2 + 4\sum_{\ell \ge 1} 1_{\{\Gamma_\ell \le s\}} \log_+^2 \overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell} + 4\lambda_0^2 \sum_{\ell \ge 1} 1_{\{\Gamma_\ell \le s\}} (w_{\Gamma_\ell} + \lambda_0 \Gamma_\ell)^2. \end{aligned} \tag{4.16}$$

Similarly, we have

$$\mathbb{Q}^Z\left[\sum_{\ell \ge 1} 1_{\{\Gamma_\ell \le s\}} (w_{\Gamma_\ell} + \lambda_0 \Gamma_\ell)^2\right] = \psi'(\lambda^*) m \int_0^s \mathbb{Q}^Z\left[(w_r + \lambda_0 r)^2\right] \mathrm{d}r = \psi'(\lambda^*) m s^2/2. \tag{4.17}$$

Now given w, Γ_ℓ and O_ℓ, by the spatial homogeneity of branching Brownian motion, we have that $\mathbb{Q}^Z\left[\overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell} \middle| w, \Gamma_\ell, O_\ell\right] = O_\ell - 1$. By the branching property of Z, we have $\overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell} = \sum_{j=1}^{O_\ell - 1} \overline{W}^{Z,\Gamma_\ell,j}_{s-\Gamma_\ell}$, where $\overline{W}^{Z,\Gamma_\ell,j}_{s-\Gamma_\ell}$, $j = 1, \cdots, O_\ell - 1$, are independent and have the same distribution given w, Γ_ℓ and O_ℓ. Thus,

$$\mathbb{Q}^Z\left[\log_+^2 \overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell} \middle| w, \Gamma_\ell, O_\ell\right] \le 2\log_+^2(O_\ell - 1) + 2\mathbb{Q}^Z\left[\log_+^2\left(\max_{j \le O_\ell - 1} \overline{W}^{Z,\Gamma_\ell,j}_{s-\Gamma_\ell}\right) \middle| w, \Gamma_\ell, O_\ell\right]. \tag{4.18}$$

By the Markov inequality,

$$\mathbb{Q}^Z\left[\log_+^2\left(\max_{j \le O_\ell - 1} \overline{W}^{Z,\Gamma_\ell,j}_{s-\Gamma_\ell}\right) \middle| w, \Gamma_\ell, O_\ell\right] = \int_0^\infty 2y \mathrm{d}y \mathbb{Q}^Z\left[\max_{j \le O_\ell - 1} \overline{W}^{Z,\Gamma_\ell,j}_{s-\Gamma_\ell} > e^y \middle| w, \Gamma_\ell, O_\ell\right]$$

$$= \int_0^\infty 2y\mathrm{d}y \left[1 - \prod_{j\le O_\ell-1}\left(1 - \mathbb{Q}^Z\left[\overline{W}^{Z,\Gamma_\ell,j}_{s-\Gamma_\ell} > e^y \middle| w,\Gamma_\ell,O_\ell\right]\right)\right]$$

$$\le \int_0^\infty 2y\mathrm{d}y \left[1 - \prod_{j\le O_\ell-1}(1-e^{-y})\right] = \int_0^\infty 2y\left[1-(1-e^{-y})^{O_\ell-1}\right]\mathrm{d}y. \tag{4.19}$$

When $O_\ell - 1 < e^{y/2}$, using the fact that $(1-x)^k \ge 1-kx$ for all $x \le 1$, we get

$$2y\left[1-(1-e^{-y})^{O_\ell-1}\right] \le 2y(O_\ell-1)e^{-y} \le 2ye^{-y/2};$$

while when $O_\ell - 1 \ge e^{y/2}$, which is equivalent to $y \le 2\log(O_\ell-1)$, we have

$$2y\left[1-(1-e^{-y})^{O_\ell-1}\right] \le 2y \le 4\log(O_\ell-1).$$

Hence, combining (4.18) and (4.19), we get

$$\mathbb{Q}^Z\left[\log_+^2 \overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell}\middle| w,\Gamma_\ell,O_\ell\right] \le 18\log^2(O_\ell-1) + \int_0^\infty 4ye^{-y/2}\mathrm{d}y. \tag{4.20}$$

By (4.16), (4.17) and (4.20), we obtain

$$\begin{aligned}\mathbb{Q}^Z\left[\log_+^2(3H_2)\right] \le & 2(\log 3)^2 + 2\lambda_0^2\psi'(\lambda^*)ms^2 + 4\mathbb{Q}^Z\left[\sum_{\ell\ge1} 1_{\{\Gamma_\ell\le s\}} 18\log^2(O_\ell-1)\right] \\ & + 4\int_0^\infty 4ye^{-y/2}\mathrm{d}y\,\mathbb{Q}^Z\left[\sum_{\ell\ge1}1_{\{\Gamma_\ell\le s\}}\right] = K_1 + K_2 s + K_3 s^2, \end{aligned} \tag{4.21}$$

here

$$K_1 = 2(\log 3)^2, \quad K_2 = 4\psi'(\lambda^*)m\int_0^\infty 4ye^{-y/2}\mathrm{d}y + 72\psi'(\lambda^*)\sum_{k\ge2} k\log^2(k-1)p_k,$$
$$K_3 = 2\lambda_0^2\psi'(\lambda^*)m.$$

By (4.8), (4.12), (4.13), (4.14) and (4.21), we deduce that $\mathbf{P}_{(\delta_0,\delta_0)}\left[W_1^{\mathcal{Z}}\log_+^2 W_1^{\mathcal{Z}}\right] < \infty$.

Step 3 : In this step we prove the second inequality of (4.7). We use similar arguments as in Step 2. First we have

$$\mathbf{P}_{(\delta_0,\delta_0)}\left[D_1^{\mathcal{Z},+}\log_+ D_1^{\mathcal{Z},+}\right] = \int_0^\infty \kappa e^{-\kappa s}\mathrm{d}s\,\mathbf{P}_{(\delta_0,\delta_0)}\left[D_s^{Z,+}\log_+ D_s^{Z,+}\right], \tag{4.22}$$

here

$$D_s^{Z,+} := \lambda_0\langle(\cdot+\lambda_0 s)_+ e^{-\lambda_0(\cdot+\lambda_0 s)}, Z_s\rangle.$$

For any $\epsilon > 0$, there exists a constant $K_\epsilon > 0$ such that $\sup_{x\in\mathbb{R}}\left[(x)_+e^{-\epsilon x}\right] \le K_\epsilon$. Using the definition (4.5) of the additive martingale $W_t^Z(\lambda)$, one can easily get that

$$D_s^{Z,+} \le K_\epsilon\lambda_0\langle e^{-(\lambda_0-\epsilon)(\cdot+\lambda_0 s)}, Z_s\rangle = K_\epsilon\lambda_0 e^{\epsilon^2 s/2}W_s^Z(\lambda_0-\epsilon).$$

By the inequality $\log_+(xy) \le \log_+ x + \log_+ y$ and the equality $\mathbf{P}_{(\delta_0,\delta_0)}\left[W_s^Z(\lambda_0-\epsilon)\right] = 1$, we get

$$\begin{aligned}&\mathbf{P}_{(\delta_0,\delta_0)}\left[D_s^{Z,+}\log_+ D_s^{Z,+}\right]\\ &\le K_\epsilon\lambda_0 e^{\epsilon^2 s/2}\log_+\left(K_\epsilon\lambda_0 e^{\epsilon^2 s/2}\right) + K_\epsilon\lambda_0 e^{\epsilon^2 s/2}\mathbf{P}_{(\delta_0,\delta_0)}\left[W_s^Z(\lambda_0-\epsilon)\log_+ W_s^Z(\lambda_0-\epsilon)\right]. \qquad (4.23)\end{aligned}$$

By (4.22) and (4.23), to complete the proof, it suffices to prove that, for fixed $\epsilon^2/2 < \kappa$, we have

$$\int_0^\infty e^{-(\kappa-\epsilon^2/2)s}\mathrm{d}s\mathbf{P}_{(\delta_0,\delta_0)}\left[W_s^Z(\lambda_0-\epsilon)\log_+ W_s^Z(\lambda_0-\epsilon)\right] < \infty. \qquad (4.24)$$

As in Step 2, we define $\mathbb{Q}^{Z,\epsilon}$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}^{Z,\epsilon}}{\mathrm{d}\mathbf{P}_{(\delta_0,\delta_0)}}\right|_{\sigma(Z_r,r\le s)} := W_s^Z(\lambda_0-\epsilon), \quad s\ge 0.$$

Then Z has another spine decomposition, which is the same as the spine decomposition at the beginning of Step 2 except with λ_0 replaced by $\lambda_0-\epsilon$, also see [19, page 59–60]. Set $g(t) = e^{-\epsilon^2 t/2-(\lambda_0-\epsilon)\lambda_0 t}$. Using the same notation as in Step 2, we have

$$\begin{aligned}W_s^Z(\lambda_0-\epsilon) =&\sum_{\ell\ge1} 1_{\{\Gamma_\ell\le s\}} g(\Gamma_\ell) W_{s-\Gamma_\ell}^{Z,\Gamma_\ell}(\lambda_0-\epsilon) 1_{\left\{g(\Gamma_\ell)W_{s-\Gamma_\ell}^{Z,\Gamma_\ell}(\lambda_0-\epsilon)<A\right\}}\\ &+\sum_{\ell\ge1} 1_{\{\Gamma_\ell\le s\}} g(\Gamma_\ell) W_{s-\Gamma_\ell}^{Z,\Gamma_\ell}(\lambda_0-\epsilon) 1_{\left\{g(\Gamma_\ell)W_{s-\Gamma_\ell}^{Z,\Gamma_\ell}(\lambda_0-\epsilon)\ge A\right\}} + g(s)e^{-(\lambda_0-\epsilon)w_s}\\ =:&H_1+H_2+H_3,\end{aligned}$$

where $A>1$ is a constant such that $\log A > 1 \ge \sup_{a\ge1}\left[\log(1+a)-\log a\right]$, which means that $\log(b+c)\le \log b+\log c$ for all $b,c\ge A$. Also note that (4.12) and $H_1 \le A\sum_{\ell\ge1} 1_{\{\Gamma_\ell\le s\}}$ still hold. And we have

$$\begin{aligned}\mathbb{Q}^{Z,\epsilon}[\log_+(3H_3)] &\le \log 3 + s\epsilon(\lambda_0-\epsilon/2) + (\lambda_0-\epsilon)\mathbb{Q}^{Z,\epsilon}|w_s+(\lambda_0-\epsilon)s|\\ &= \log 3 + s\epsilon(\lambda_0-\epsilon/2) + (\lambda_0-\epsilon)\sqrt{\frac{2}{\pi}}\sqrt{s}.\end{aligned}$$

Similarly we define $\overline{W}_{s-\Gamma_\ell}^{Z,\Gamma_\ell}(\lambda_0-\epsilon)$ by

$$\overline{W}_{s-\Gamma_\ell}^{Z,\Gamma_\ell}(\lambda_0-\epsilon) := e^{(\lambda_0-\epsilon)w_{\Gamma_\ell}} W_{s-\Gamma_\ell}^{Z,\Gamma_\ell}(\lambda_0-\epsilon).$$

Then using an argument similar to (4.16), we have

$$\begin{aligned}&\log_+(3H_2) \le \log 3 + \log_+ H_2\\ &\le \log 3 + \sum_{\ell\ge1} 1_{\{\Gamma_\ell\le s\}}\log_+\left(g(\Gamma_\ell)e^{-(\lambda_0-\epsilon)w_{\Gamma_\ell}}\right) + \sum_{\ell\ge1} 1_{\{\Gamma_\ell\le s\}}\log_+ \overline{W}_{s-\Gamma_\ell}^{Z,\Gamma_\ell}(\lambda_0-\epsilon)\end{aligned}$$

and

$$\mathbb{Q}^{Z,\epsilon}\left[\sum_{\ell\ge1} 1_{\{\Gamma_\ell\le s\}}\log_+\left(g(\Gamma_\ell)e^{-(\lambda_0-\epsilon)w_{\Gamma_\ell}}\right)\right] \le \psi'(\lambda^*)m\int_0^s \left[\mathbb{Q}^{Z,\epsilon}|w_r+(\lambda_0-\epsilon)r| + \epsilon\left(\lambda_0-\frac{\epsilon}{2}\right)r\right]\mathrm{d}r.$$

Since (4.18) and (4.19) hold with $\overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell}$ replaced by $\overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell}(\lambda_0-\epsilon)$ (we only use the martingale property and branching property), (4.20) holds for $\overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell}(\lambda_0-\epsilon)$. Applying Jensen's inequality for $\overline{W}^{Z,\Gamma_\ell}_{s-\Gamma_\ell}(\lambda_0-\epsilon)$ in (4.20), we finally deduce that there exist constants $K^\epsilon_j, j=1,2,3,4,5$, such that for all $s\geq 0$,

$$\mathbf{P}_{(\delta_0,\delta_0)}\left[W^Z_s(\lambda_0-\epsilon)\log_+ W^Z_s(\lambda_0-\epsilon)\right]\leq K^\epsilon_1+K^\epsilon_2\sqrt{s}+K^\epsilon_3 s+K^\epsilon_4 s^{3/2}+K^\epsilon_5 s^2. \tag{4.25}$$

Combining (4.23), (4.24) and (4.25), we obtain $\mathbf{P}_{(\delta_0,\delta_0)}\left[D^{\mathcal{Z},+}_1\log_+ D^{\mathcal{Z},+}_1\right]<\infty$. □

Lemma 4.2 *If* (1.7) *holds, then* $\sum_{n\geq 1} n(\log n)^2 p_n<\infty$.

Proof : By the definition of $\{p_n: n\geq 2\}$, we only need to prove that

$$\int_{(0,\infty)}\sum_{n\geq 2} n(\log n)^2\frac{(\lambda^* x)^n}{n!}e^{-\lambda^* x}\nu(\mathrm{d}x)<\infty. \tag{4.26}$$

Define $h(x):=(\log(1+x))^2$, then $h''(x)=\frac{2}{(1+x)^2}(1-\log(1+x))$. When $x\geq 2>e-1, h''(x)<0$, which implies h is concave in $[2,\infty)$. By Jensen's inequality,

$$\begin{aligned}
&\sum_{n\geq 3} n(\log n)^2\frac{(\lambda^* x)^n}{n!}e^{-\lambda^* x}=\lambda^* x\sum_{n\geq 2}(\log(1+n))^2\,\frac{(\lambda^* x)^n}{n!}e^{-\lambda^* x}\\
&\leq(\lambda^* x)\left[\sum_{n\geq 2}\frac{(\lambda^* x)^n}{n!}e^{-\lambda^* x}\right]\left\{\log\left[\frac{\sum_{n\geq 2} n(\lambda^* x)^n e^{-\lambda^* x}/n!}{\sum_{n\geq 2}(\lambda^* x)^n e^{-\lambda^* x}/n!}+1\right]\right\}^2\\
&\leq \lambda^* x\left\{\log\left[\frac{\lambda^* x(1-e^{-\lambda^* x})}{1-e^{-\lambda^* x}-e^{-\lambda^* x}\lambda^* x}+1\right]\right\}^2.
\end{aligned} \tag{4.27}$$

Since

$$\lim_{x\to\infty}\log\left[\frac{\lambda^* x(1-e^{-\lambda^* x})}{1-e^{-\lambda^* x}-e^{-\lambda^* x}\lambda^* x}+1\right]/\log x=1,$$

there exists $K>0$ such that when $x\geq K$, we have

$$\log\left[\frac{\lambda^* x(1-e^{-\lambda^* x})}{1-e^{-\lambda^* x}-e^{-\lambda^* x}\lambda^* x}+1\right]\leq 2\log x. \tag{4.28}$$

Together with (4.26), (4.27) and (4.28), we complete the proof. □

Proof of Theorem 1.2: By the first two paragraphs of this section, to prove Theorem 1.2, it suffices to show that, the limsup in (1.8) is valid $\mathbf{P}_{(\delta_0,\delta_0)}$-almost surely.

Case 1 : $\beta\neq 0$. Let L^Z_t be the left-most point of Z_t. Note that, for any $x\in\mathbb{R}$,

$$\begin{aligned}
\mathbb{N}^{\mathcal{E}}_x\left(\langle 1,w_1\rangle>0\right)&=\lim_{\theta\to+\infty}\mathbb{N}^{\mathcal{E}}_x\left(1-e^{-\theta\langle 1,w_1\rangle}\right)=\lim_{\theta\to+\infty}-\log\mathbb{P}^{\mathcal{E}}_{\delta_x}\left[e^{-\theta\langle 1,X_1\rangle}\right]\\
&=-\log\mathbb{P}^{\mathcal{E}}_{\delta_0}\left[\|X_1\|=0\right]=\mathbb{N}^{\mathcal{E}}_0\left(\langle 1,w_1\rangle>0\right)\in(0,\infty).
\end{aligned}$$

Suppose that the continuous immigrations in the skeleton decomposition of X along the trajectory of L^Z_t such that $\langle 1,w_1\rangle>0$ are given by $\left\{(\tau_n,\bar{X}^{(1,\tau_n)}):n=1,2,...\right\}$. Then it is obvious that

$\{\tau_n - \tau_{n-1} : n = 1, 2, ...\}$ are iid and independent of Z. The law of $\tau_n - \tau_{n-1}$ is exponential with parameter $2\beta\mathbb{N}_0^{\mathcal{E}}(\langle 1, w_1\rangle > 0)$ and the law of the immigration is $\frac{\mathbb{N}^{\mathcal{E}}_{L^Z_{\tau_n}}(\cdot\cap\{\langle 1,w_1\rangle>0\})}{\mathbb{N}_0^{\mathcal{E}}(\langle 1,w_1\rangle>0)}$.

Since (1.7) holds, using Lemmas 4.1 and 4.2 with $T_n = \tau_n$, we know that $\mathcal{Z}_n$ satisfies (1.1), (1.2) and (1.3). Noticing that the left support of $\mathcal{Z}_n$ is $\lambda_0(L^Z_{\tau_n} + \lambda_0\tau_n)$, by [1, Theorem 6.1],

$$\liminf_{n\to\infty}\left(\lambda_0(L^Z_{\tau_n} + \lambda_0\tau_n) - \frac{1}{2}\log n\right) = -\infty, \quad \mathbf{P}_{(\delta_0,\delta_0)}\text{-a.s.} \tag{4.29}$$

By the strong law of large numbers, $\tau_n/n \to (2\beta)^{-1}$ as $n \to \infty$. Hence, (4.29) is equivalent to

$$\liminf_{n\to\infty}\left(\lambda_0(L^Z_{\tau_n} + \lambda_0\tau_n) - \frac{1}{2}\log \tau_n\right) = -\infty, \quad \mathbf{P}_{(\delta_0,\delta_0)}\text{-a.s.} \tag{4.30}$$

Define W_t^Λ by

$$W_t^\Lambda := \langle e^{-\lambda_0(\cdot+\lambda_0 t)}, \Lambda_t\rangle.$$

Then

$$\sqrt{\tau_n+1}\langle e^{-\lambda_0(\cdot+\lambda_0(\tau_n+1))}, \Lambda_{\tau_n+1}\rangle \geq \sqrt{\tau_n}\langle e^{-\lambda_0(\cdot+\lambda_0(\tau_n+1))}, \bar{X}_1^{(1,\tau_n)}\rangle =: H_nJ_n. \tag{4.31}$$

Here H_n and J_n are defined as

$$H_n := \sqrt{\tau_n}e^{-\lambda_0(L^Z_{\tau_n}+\lambda_0\tau_n)}, \quad J_n := e^{-\lambda_0^2}\langle e^{-\lambda_0(\cdot-L^Z_{\tau_n})}, \bar{X}_1^{(1,\tau_n)}\rangle.$$

Then by the construction of the continuous immigration in the skeleton decomposition and the spatial homogeneity of super-Brownian motion, we deduce that $\{J_n : n = 1, 2, ...\}$ are iid and for every n, J_n is independent of $\sigma(H_\ell, \ell \geq 1)$. Define $\mathcal{G}_n := \sigma(H_\ell, J_\ell : 1 \leq \ell \leq n)$. By (4.30), we have $\limsup_{n\to\infty} H_n = +\infty$, $\mathbf{P}_{(\delta_0,\delta_0)}$-a.s., which together with the second Borel-Cantelli lemma (see e.g. [8, Theorem 5.3.2]) is equivalent to that, for any $K > 0$,

$$\sum_{n=1}^\infty \mathbf{P}_{(\delta_0,\delta_0)}\left[H_n > K\middle|\mathcal{G}_{n-1}\right] = +\infty, \quad \mathbf{P}_{(\delta_0,\delta_0)}\text{-a.s.} \tag{4.32}$$

Now it is clear that $\mathbf{P}_{(\delta_0,\delta_0)}(J_n > 0) = 1$, so there exists a constant $\varepsilon > 0$ such that for all $n \geq 1$, $\mathbf{P}_{(\delta_0,\delta_0)}(J_n > \varepsilon) > 0$. By (4.32) and the independence between J_n and $\mathcal{G}_{n-1}$, we deduce that, for any $K > 0$,

$$\begin{aligned}\sum_{n=1}^\infty \mathbf{P}_{(\delta_0,\delta_0)}\left[H_nJ_n > K\middle|\mathcal{G}_{n-1}\right] &\geq \sum_{n=1}^\infty \mathbf{P}_{(\delta_0,\delta_0)}\left[J_n > \varepsilon,\ H_n > K/\varepsilon\middle|\mathcal{G}_{n-1}\right] \\ &= \mathbf{P}_{(\delta_0,\delta_0)}[J_1 > \varepsilon]\sum_{n=1}^\infty \mathbf{P}_{(\delta_0,\delta_0)}\left[H_n > K/\varepsilon\middle|\mathcal{G}_{n-1}\right] = +\infty. \quad \mathbf{P}_{(\delta_0,\delta_0)}\text{-a.s.,}\end{aligned}$$

which is, according to the second Borel-Cantelli lemma, equivalent to

$$\limsup_{n\to\infty} H_nJ_n = +\infty, \quad \mathbf{P}_{(\delta_0,\delta_0)}\text{-a.s.} \tag{4.33}$$

In view of (4.31) and (4.33), we get

$$\limsup_{t\to\infty}\sqrt{t}W_t^\Lambda \geq \limsup_{n\to\infty}\sqrt{\tau_n+1}\langle e^{-\lambda_0(\cdot+\lambda_0(\tau_n+1))}, \Lambda_{\tau_n+1}\rangle = +\infty, \quad \mathbf{P}_{(\delta_0,\delta_0)}\text{-a.s.,}$$

which implies the desired result.

Case $2 : \nu \neq 0$. Suppose that $\nu((\varepsilon, +\infty)) > 0$ for some $\epsilon > 0$. Then $\nu((\varepsilon, +\infty)) < \infty$. Suppose that the times and masses of the discrete immigration along the trajectory of L_t^Z in the skeleton decomposition with initial immigration mass large than ε are $\{(\tilde{\tau}_n, \mathfrak{m}_n) : n = 1, 2, ...\}$. Then $\{\tilde{\tau}_n - \tilde{\tau}_{n-1} : n = 1, 2, ...\}$ are iid exponential random variables with parameter $\kappa = \int_{(\varepsilon,\infty)} ye^{-\lambda^* y}\nu(\mathrm{d}y)$, $\mathfrak{m}_n > \varepsilon$ for all $n \geq 1$ with law $ye^{-\lambda^* y}1_{\{y>\varepsilon\}}\nu(\mathrm{d}y)/\int_{(\varepsilon,\infty)} ye^{-\lambda^* y}\nu(\mathrm{d}y)$, and $\{\tilde{\tau}_n : n = 1, 2, ...\}$ is independent of Z. Applying Lemmas 4.1 and 4.2 with $T_n = \tilde{\tau}_n$, we get

$$\liminf_{n\to\infty} \left(\lambda_0(L^Z_{\tilde{\tau}_n} + \lambda_0\tilde{\tau}_n) - \frac{1}{2}\log\tilde{\tau}_n \right) = -\infty, \quad \mathbf{P}_{(\delta_0,\delta_0)}\text{-a.s.} \tag{4.34}$$

By the same argument as Case 1, we have

$$\sqrt{\tilde{\tau}_n}\langle e^{-\lambda_0(\cdot+\lambda_0\tilde{\tau}_n)}, \Lambda_{\tilde{\tau}_n}\rangle \geq \sqrt{\tilde{\tau}_n}e^{-\lambda_0(L^Z_{\tilde{\tau}_n}+\lambda_0\tilde{\tau}_n)}\mathfrak{m}_n > \varepsilon\sqrt{\tilde{\tau}_n}e^{-\lambda_0(L^Z_{\tilde{\tau}_n}+\lambda_0\tilde{\tau}_n)}. \tag{4.35}$$

Combining (4.34) and (4.35), we also get the desired results. □

A byproduct of the proof of Theorem 1.2 is the following result:

Corollary 4.3 *Let L_t be the minimum of the support of X_t, i.e., $L_t := \inf\{y \in \mathbb{R} : X_t((-\infty, y)) > 0\}$. If* (1.6) *and* (1.7) *hold, then on $\mathcal{E}^c$, it holds that*

$$\liminf_{t\to\infty} \left(L_t + \lambda_0 t - \frac{1}{2\lambda_0}\log t \right) = -\infty \quad \mathbb{P}\text{-almost surely.} \tag{4.36}$$

Proof: Let L_t^Λ be the minimum of the support of Λ_t. We keep the notation in the proof of Theorem 1.2.

If $\nu \neq 0$, by the definition of $L^\Lambda_{\tilde{\tau}_n}$, we have $L^\Lambda_{\tilde{\tau}_n} \leq L^Z_{\tilde{\tau}_n}$, $\forall n \geq 1$, $\mathbf{P}_{(\delta_0,\delta_0)}$-a.s. By the branching property, we deduce that on $\left(\mathcal{E}^\Lambda\right)^c$, $L^\Lambda_{\tilde{\tau}_n} \leq L^Z_{\tilde{\tau}_n}$, $\forall n \geq 1$, $\mathbf{P}_{(\delta_0,\delta_0)}$-a.s. Together with (4.34), we get (4.36).

If $\beta \neq 0$, for a fixed constant A, define $\mathcal{J}_n$ by

$$\mathcal{J}_n := \langle 1_{(-\infty, A+L^Z_{\tau_n})}(\cdot), \bar{X}_1^{(1,\tau_n)}\rangle = \langle 1_{(-\infty,A)}(\cdot - L^Z_{\tau_n}), \bar{X}_1^{(1,\tau_n)}\rangle.$$

Put $\mathcal{H}_n := \lambda_0(L^Z_{\tau_n} + \lambda_0\tau_n) - \frac{1}{2}\log\tau_n$. By the spatial homogeneity of super-Brownian motion, $\{\mathcal{J}_n\}$ are iid and for every n, $\mathcal{J}_n$ is independent of $\sigma(\mathcal{H}_\ell, \ell \geq 1)$. We also define $\widetilde{\mathcal{G}}_n := \sigma(\mathcal{H}_\ell, \mathcal{J}_\ell, 1 \leq \ell \leq n)$. Since $\mathbf{P}_{(\delta_0,\delta_0)}\left(\|\bar{X}_1^{(1,\tau_n)}\| > 0\right) = \mathbf{P}_{(\delta_0,\delta_0)}\left(\|\bar{X}_1^{(1,\tau_1)}\| > 0\right) = 1$ and $\lim_{A\to+\infty}\mathcal{J}_n = \|\bar{X}_1^{(1,\tau_n)}\|, \mathbf{P}_{(\delta_0,\delta_0)}$-a.s., there exists an A such that $\mathbf{P}_{(\delta_0,\delta_0)}(\mathcal{J}_n > 0) = \mathbf{P}_{(\delta_0,\delta_0)}(\mathcal{J}_1 > 0) > 0$. We see that for any $K > 0$,

$$\sum_{n=1}^\infty \mathbf{P}_{(\delta_0,\delta_0)}\left[\mathcal{J}_n > 0, \mathcal{H}_n < -K \middle| \widetilde{\mathcal{G}}_{n-1}\right] = \mathbf{P}_{(\delta_0,\delta_0)}\left[\mathcal{J}_1 > 0\right]\sum_{n=1}^\infty \mathbf{P}_{(\delta_0,\delta_0)}\left[\mathcal{H}_n < -K \middle| \widetilde{\mathcal{G}}_{n-1}\right] = +\infty,$$

$\mathbf{P}_{(\delta_0,\delta_0)}$-a.s., where in the last equality we used (4.30) and the second Borel-Cantelli lemma. Therefore, for all $K > 0, \mathbf{P}_{(\delta_0,\delta_0)}(\mathcal{J}_n > 0, \mathcal{H}_n < -K \text{ i.o.}) = 1$. Note that

$$\{\mathcal{J}_n > 0, \mathcal{H}_n < -K\} \subset \left\{\lambda_0(L^\Lambda_{\tau_n+1} + \lambda_0\tau_n) - \frac{1}{2}\log\tau_n < -K + \lambda_0 A\right\},$$

we get

$$\mathbf{P}_{(\delta_0,\delta_0)}\left(\lambda_0(L^\Lambda_{\tau_n+1} + \lambda_0\tau_n) - \frac{1}{2}\log\tau_n < -K + \lambda_0 A \text{ i.o.}\right) = 1.$$

Since $(\tau_n+1)/\tau_n \to 1$ as $n \to \infty$ and K is arbitrary, we get that (4.36) holds $\mathbf{P}_{(\delta_0,\delta_0)}$-almost surely. By the branching property argument, we get the desired result. □

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