Tail probability of maximal displacement in critical branching Lévy process with stable branching

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Consider a critical branching Lévy process $\{X_t, t \ge 0\}$ with branching rate $\beta > 0$, offspring distribution $\{p_k : k \ge 0\}$ and spatial motion $\{\xi_t, \mathcal{P}_x\}$. For any $t \ge 0$, let N_t be the collection of particles alive at time t, and, for any $u \in N_t$, let $X_u(t)$ be the position of u at time t. We study the tail probability of the maximal displacement $M := \sup_{t>0}\sup_{u\in N_t} X_u(t)$ under the assumption $\lim_{n\to\infty} n^{\alpha}\sum_{k=n}^{\infty} p_k = \kappa \in (0,\infty)$ for some $\alpha \in (1,2)$, $\mathcal{E}_0(\xi_1) = 0$ and $\mathcal{E}_0((\xi_1^+)^r) \in (0,\infty)$ for some $r > 2\alpha/(\alpha-1)$. Our main result is a generalization of the main result of Sawyer and Fleischman (1979) for branching Brownian motions and that of Lalley and Shao (2015) for branching random walks, both of these results are proved under the assumption $\sum_{k=0}^{\infty} k^3 p_k < \infty$.

Keywords: Branching Lévy process; critical branching process; Feynman-Kac representation

1. Introduction and notation

1.1. Introduction

Consider a system, in which at time $n = 0$, there is a particle at $0 \in \mathbb{R}$. At time $n = 1$, this particle dies and gives birth to a collection of particles whose configuration relative to their parent is given by a copy of a point process $\mathcal{L}$. At time $n = 2$, the individuals alive at time 1 repeat their parent's behavior and the process goes on. We will use N_n to denote the set of particles alive at time n and for $u \in N_n$, the position of u is denoted by $X_u(n)$. Define random measures $\mathcal{X}_n := \sum_{u\in N_n} \delta_{X_u(n)}, n \ge 0$. Then $\{\mathcal{X}_n, n \ge 0\}$ is a Markov process, and called a branching random walk (BRW). We denote the law of the BRW by $\mathbb{P}$.

Now we consider the special case $\mathcal{L} = \sum_{i=1}^{B} \delta_{X_i}$, where B is a non-negative integer valued random variable with $\mathbb{P}(B = k) = p_k$ and $X_1, X_2, \ldots$ are iid $\mathbb{Z}$-valued random variables independent of B with common distribution $\{\mu_k, k \in \mathbb{Z}\}$. We say that this process is critical if

$$\mathbb{E}(B) = \sum_{k=0}^{\infty} k p_k = 1.$$

Since the total mass of the branching random walk is a Galton-Waston process, a critical branching random walk must extinct in finite time, which implies that the following maximal displacement M is a finite random variable:

$$M := \sup_{n\in\mathbb{N}}\sup_{u\in N_n} X_u(n) \tag{1.1}$$

with the convention $\sup_{u\in N_n} X_u(n) = -\infty$ if $N_n = \emptyset$. Lalley and Shao (2015) proved that if

$$\sum_{k=0}^{\infty} k^3 p_k < \infty, \quad \sum_{k\in\mathbb{Z}} k\mu_k = 0, \quad \sum_{k\in\mathbb{Z}} |k|^{4+\varepsilon} \mu_k < \infty \tag{1.2}$$

for some $\varepsilon > 0$, then

$$\lim_{x\to+\infty} x^2 \mathbb{P}(M \geq x) = \frac{6\eta^2}{\sigma^2},$$

where $\eta^2 := \sum_{k\in\mathbb{Z}} k^2\mu_k$ and $\sigma^2 := \sum_{k=0}^{\infty} k^2 p_k - 1$.

Now we turn to the continuous time and space case: branching Lévy processes in the sense of Kyprianou (1999). Let $(\xi_t, \mathcal{P}_x)$ be a Lévy process with $\xi_0 = x$. A branching Lévy process is defined as follows: initially there is a particle at $x \in \mathbb{R}$ and it moves according to $(\xi_t, \mathcal{P}_x)$. After an exponential time with parameter $\beta > 0$, independent of the motion, it dies and produces k offsprings with probability p_k, $k \geq 0$. The offsprings move independently according to ξ from the place where they are born and obey the same branching mechanism as their parent. Denote the law by $\mathbb{P}_x$ and $\mathbb{P} := \mathbb{P}_0$. In this paper we focus on the critical case, i.e., we always assume that $\{p_k : k \geq 0\}$ satisfies $\sum_{k=0}^{\infty} k p_k = 1$.

Similarly, we define the maximal position by

$$M := \sup_{t\geq 0} \sup_{u\in N_t} X_u(t),$$

where N_t is the set of particles alive at time t and $X_u(t)$ is the position of $u \in N_t$. When the spatial motion ξ is a standard Brownian motion, Sawyer and Fleischman (1979) proved that under the assumption $\sum_{k=0}^{\infty} k^3 p_k < \infty$,

$$\lim_{x\to+\infty} x^2 \mathbb{P}(M \geq x) = \frac{6}{\sigma^2} \tag{1.3}$$

with $\sigma^2 = \sum_{k=0}^{\infty} k^2 p_k - 1$. Profeta (2024) extended (1.3) to the case when ξ is a spectrally negative Lévy process and $\sum_{k=0}^{\infty} k^3 p_k < \infty$. When the spatial motion is a γ-stable process with index $\gamma \in (0, 2)$, $\sum_{k=0}^{\infty} k^3 p_k < \infty$ and $\beta = 1$, Lalley and Shao (2016) and Profeta (2022) proved that

$$\lim_{x\to+\infty} x^{\gamma/2} \mathbb{P}(M \geq x) = \kappa,$$

where κ is an explicit constant depending on the normalization of ξ and on the offspring distribution. For results where the spatial motion is a general spectrally negative Lévy process, see Profeta (2024).

1.2. Main result

The main aim of this paper is to study the tail probability of M when the offspring distribution $\{p_k : k \geq 0\}$ is in the domain of attraction of an α-stable distribution with index $\alpha \in (1, 2)$ and the spatial motion has lighter tails. Suppose that there exist constants $\kappa > 0$ and $\alpha \in (1, 2)$ such that

$$\lim_{n\to\infty} n^{\alpha} \sum_{k=n}^{\infty} p_k = \kappa. \tag{1.4}$$

We denote $x^+ := \max(x, 0)$ and $x^- := \max(-x, 0)$. Assume that

$$\mathcal{E}_0(\xi_1) = 0, \quad \eta^2 := \mathcal{E}_0(\xi_1^2) \in (0, \infty). \tag{1.5}$$

Our main result is as follows:

Theorem 1.1. *If*

$$\mathcal{E}_0\left(\left(\xi_1^+\right)^r\right) < \infty \quad \textit{for some } r > \frac{2\alpha}{\alpha - 1}, \tag{1.6}$$

then

$$\lim_{x\to\infty} x^{\frac{2}{\alpha-1}} \mathbb{P}(M \geq x) = \left(\frac{(\alpha+1)\eta^2}{\beta\kappa(\alpha-1)\Gamma(2-\alpha)}\right)^{\frac{1}{\alpha-1}}, \tag{1.7}$$

where $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$ *is the Gamma function.*

Note that $\frac{2\alpha}{\alpha-1} > 4$, so (1.6) requires finiteness of at least the 4th moment of the positive part of ξ_1. Also, if the Lévy process is spectrally negative, then (1.6) automatically holds by (Sato, 1999, Theorem 25.3) (or see (2.4) below). Therefore, only (1.5) is needed for the spectrally negative case. This is also discussed in (Profeta, 2024, Theorem 3).

Our argument of proving the above main result is an adaptation of that of Lalley and Shao (2015). Our assumption (1.4) on the branching mechanism is weaker than the assumption (1.2) in Lalley and Shao (2015). Under our assumption that the positive part of the spatial motion has finite moments of order $r > 2\alpha/(\alpha - 1)$, the weaker assumption above on the branching mechanism does not cause too much trouble. The assumption (1.4) only changes the behavior of f, defined in (2.9) below, from $f(v) = Cv(1 + o(1))$ to $f(v) = Cv^{\alpha-1}(1 + o(1))$ for some constant $C > 0$.

We end this section by giving a brief sketch of the proof of Theorem 1.1. Define $v(x) := \mathbb{P}(M \geq x), x \in \mathbb{R}$. We first give a Feynman-Kac formula for $v(x)$, see Lemma 2.2 below. Then we prove that there exists a sequence $\{x_k \in [0, \infty)\}$ with $\lim_{k\to\infty} x_k = +\infty$ such that for all $y \geq 0$, the following limit exists:

$$\phi(y) := \lim_{k\to\infty} \frac{v\left(x_k + yv(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)},$$

and ϕ is the unique bounded solution to the following problem:

$$\begin{cases} \phi''(y) = C\left(\phi(y)\right)^\alpha, & y > 0, \\ \phi(0) = 1, \end{cases} \tag{1.8}$$

with C being some positive constant. In Lalley and Shao (2015), $\phi(y)$ is defined as the limit of $\frac{v(x_k+yv(x_k)^{-1/2})}{v(x_k)}$ as $k \to \infty$. The above problem is replaced by

$$\begin{cases} \phi''(y) = \frac{\sigma^2}{\eta^2}\left(\phi(y)\right)^2, & y > 0, \\ \phi(0) = 1, \end{cases}$$

and the explicit solution is given by $\left(\frac{\sigma}{\sqrt{6}\eta}y + 1\right)^{-2}$, which plays an important role and leads to the limit behavior (1.3). In our case, the solution to (1.8) is $(\theta y + 1)^{-\frac{2}{\alpha-1}}$ with some constant $\theta > 0$ (see the proof of Corollary 3.1), which leads to the limit behavior (1.7).

2. Preliminaries

Set $\widetilde{\xi}_t := -\xi_t$. Consider a branching Lévy process $\{\widetilde{X}_u(t), u \in \widetilde{N}_t, t > 0\}$ with spatial motion $\widetilde{\xi}$, branching rate $\beta > 0$ and offspring distribution $\{p_k : k \geq 0\}$. Then

$$\mathbb{P}(M < x) = \mathbb{P}\left(\inf_{t\geq 0}\inf_{u\in N_t} \widetilde{X}_u(t) > -x\right) = \mathbb{P}_x\left(\inf_{t\geq 0}\inf_{u\in N_t} \widetilde{X}_u(t) > 0\right), \tag{2.1}$$

with the convention $\inf_{u\in N_t} \widetilde{X}_u(t) = +\infty$ when $\widetilde{N}_t = \emptyset$. Recall that $v(x) = \mathbb{P}(M \geq x)$. Since under $\mathbb{P}$, the initial ancestor is located at 0, we have $v(x) = \mathbb{P}(M \geq x) = 1$ for $x \leq 0$. Also, $v(x)$ is left-continuous since $v(x) = 1 - \mathbb{P}(M < x)$. Define

$$\widetilde{\tau}_y := \inf\left\{t > 0 : \widetilde{\xi}_t \leq y\right\}.$$

2.1. Moment for overshoot of Lévy process

For integer-valued random walks, the following result can be found in (Lalley and Shao, 2015, Lemma 10). We now prove that it also holds for some Lévy processes.

Lemma 2.1. *Let $\widetilde{\xi}$ be a Lévy process, which satisfies $\mathcal{E}_0(\widetilde{\xi}_1) = 0$ and is not spectrally positive. If $\mathcal{E}_0((\widetilde{\xi}_1^-)^r) < \infty$ for some $r > 2$, then*

$$\sup_{x>0} \mathcal{E}_x\left(\left|\widetilde{\xi}_{\widetilde{\tau}_0}\right|^{r-2}\right) < \infty. \tag{2.2}$$

Proof. By the Lévy-Khintchine formula, $\mathcal{E}_0(e^{\mathrm{i}\theta\widetilde{\xi}_1}) = e^{-\Psi(\mathrm{i}\theta)}$, where

$$\Psi(\mathrm{i}\theta) = -\mathrm{i}\gamma\theta + \frac{\nu^2}{2}\theta^2 + \int_{x\neq 0}\left(1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta x 1_{\{|x|\in(0,1]\}}\right)\pi(\mathrm{d}x)$$

with π being the Lévy measure.

(i) If $\pi(\{|x| > 1\}) = 0$, then $\mathcal{E}_0((\widetilde{\xi}_1^-)^s) < \infty$ for all $s > 0$. Since $\widetilde{\xi}$ oscillates and $\widetilde{\tau}_0 < \infty$ $\mathcal{P}_x$ a.s., we get

$$\sup_{x>0} \mathcal{E}_x\left(\left|\widetilde{\xi}_{\widetilde{\tau}_0}\right|^{r-2}\right) \leq 1 < \infty.$$

(ii) If $\pi(\{|x| > 1\}) > 0$, let σ_n be the n-th time that $\widetilde{\xi}$ has a jump of magnitude larger than 1, and put $\sigma_0 = 0$, then $\{\sigma_n - \sigma_{n-1}, n \geq 1\}$ are iid exponential random variables with parameter $\pi(\{|x| > 1\})$. Similar to (Doney and Maller, 2002, p.208), for $j \geq 1$, define $W_j = \widetilde{\xi}_{\sigma_j-} - \widetilde{\xi}_{\sigma_{j-1}}$ and $V_j = \widetilde{\xi}_{\sigma_j} - \widetilde{\xi}_{\sigma_j-}$. Then $\{W_j : j \geq 1\}$ and $\{V_j : j \geq 1\}$ are both iid families of random variables and independent of each other. Let the random walk $Z = (Z_n, n \geq 0)$ be defined by

$$Z_n := \widetilde{\xi}_{\sigma_n} = \sum_{j=1}^{n}\left(W_j + V_j\right) + \widetilde{\xi}_0 \quad \text{for } n \geq 1,$$

and $Z_0 = \widetilde{\xi}_{\sigma_0} = x$ under $\mathcal{P}_x$. Furthermore,

$$\mathcal{P}_0(V_1 \in \mathrm{d}x) = \frac{\pi(\mathrm{d}x)}{\pi(\{|x| > 1\})} 1_{\{|x|>1\}} \tag{2.3}$$

and $W_1 \overset{\mathrm{d}}{=} \widetilde{\xi}^{(1)}_{\mathbf{e}}$ where $\widetilde{\xi}^{(1)}$ is a Lévy process with

$$\mathcal{E}_0\left(e^{\mathrm{i}\theta\widetilde{\xi}^{(1)}_1}\right) = \exp\left\{\mathrm{i}\gamma\theta - \frac{\nu^2}{2}\theta^2 - \int_{|x|\in(0,1]}\left(1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta x 1_{\{|x|\in(0,1]\}}\right)\pi(\mathrm{d}x)\right\}$$

and $\mathbf{e}$ is an independent exponential random variable with parameter $\pi(\{|x| > 1\})$. Therefore, by (2.3) and (Sato, 1999, Theorem 25.3) with $g(x) = \max(-x, 1)$,

$$\mathcal{E}_0\left(\left(\widetilde{\xi}^-_1\right)^r\right) < \infty \iff \int_{(-\infty,-1)} |x|^r \pi(\mathrm{d}x) < \infty \iff \mathcal{E}_0\left((V_1^-)^r\right) < \infty. \tag{2.4}$$

Using $\mathcal{E}_0\left(|W_1|^s\right) < \infty$ for all $s > 0$, we infer

$$\mathcal{E}_0\left(\left(\widetilde{\xi}^-_1\right)^r\right) < \infty \iff \mathcal{E}_0\left((Z_1^-)^r\right) < \infty. \tag{2.5}$$

By (Doney and Maller, 2002, p.209), for all $z > 1$ and $x \geq 0$,

$$\mathcal{P}_x\left(\left|\widetilde{\xi}_{\widetilde{\tau}_0}\right| > z\right) \leq \mathcal{P}_x\left(\left|Z_{\widehat{\tau}_0}\right| > z\right),$$

where $\widehat{\tau}_0 := \inf\{n : Z_n \leq 0\}$. Then we get

$$\begin{aligned}\sup_{x>0}\mathcal{E}_x\left(\left|\widetilde{\xi}_{\widetilde{\tau}_0}\right|^{r-2}\right) &= (r-2)\sup_{x>0}\int_0^\infty z^{r-3}\mathcal{P}_x\left(\left|\widetilde{\xi}_{\widetilde{\tau}_0}\right| > z\right)\mathrm{d}z\\ &\leq 2^{r-2} + (r-2)\sup_{x>0}\int_2^\infty z^{r-3}\mathcal{P}_x\left(|Z_{\widehat{\tau}_0}| > z\right)\mathrm{d}z,\end{aligned} \tag{2.6}$$

where in the last inequality we used the fact that $(r-2)\int_0^2 z^{r-3}\mathrm{d}z = 2^{r-2}$. On the other hand, define

$$\begin{aligned}&T_1 := \min\{n > 0 : Z_n < Z_0\}, \quad T_k := \inf\{n > T_{k-1} : Z_n < Z_{T_{k-1}}\}, \quad k \geq 1,\\ &S_0 := Z_0, \quad S_n := Z_{T_n}, \quad n \geq 1,\end{aligned}$$

then $S_1 - S_0, S_2 - S_1, S_3 - S_2, \ldots,$ are iid with $\mathcal{E}_x(|S_1 - S_0|^{r-1}) < \infty$ if $\mathcal{E}_0\big((Z_1^-)^r\big) < \infty$ (see (Doney, 1980, Corollary 1)). Note that for $z > 1$,

$$\begin{aligned}\mathcal{P}_x\left(|Z_{\widehat{\tau}_0}| > z\right) &= \sum_{k=0}^\infty \mathcal{P}_x\left(S_k > 0, S_{k+1} < -z\right)\\ &\leq \sum_{\ell=0}^{[x]}\left(\sum_{k=0}^\infty \mathcal{P}_x\left(S_k \in [\ell, \ell+1)\right)\right)\mathcal{P}_0\left(|S_1| > z + \ell\right).\end{aligned} \tag{2.7}$$

Define renewal function $U(y) := \sum_{k=0}^\infty \mathcal{P}_0\{-S_k \leq y\}$, $y \in \mathbb{R}$. By renewal theory, we know that U is subadditive on $\mathbb{R}$, and $U(1) < \infty$ if and only if $\mathcal{P}_0\{S_1 = 0\} < 1$, which is the case here. Thus,

$$\begin{aligned}\sum_{k=0}^\infty \mathcal{P}_x\{S_k \in [\ell, \ell+1)\} &= \sum_{k=0}^\infty \mathcal{P}_0\{x - \ell - 1 < -S_k \leq x - \ell\}\\ &= U(x-\ell) - U(x-\ell-1) \leq U(1) < \infty.\end{aligned} \tag{2.8}$$

Combining (2.6), (2.7) and (2.8), we get that

$$\sup_{x>0} \mathcal{E}_x \left(\left| \widetilde{\xi}_{\widetilde{\tau}_0} \right|^{r-2} \right) \le 2^{r-2} + (r-2)U(1) \sup_{x>0} \int_2^\infty z^{r-3} \sum_{\ell=0}^{[x]} \mathcal{P}_0\left(|S_1| > z + \ell\right) \mathrm{d}z$$

$$\le 2^{r-2} + (r-2)U(1) \int_2^\infty z^{r-3} \int_0^\infty \mathcal{P}_0\left(|S_1| > z + \ell - 1\right) \mathrm{d}\ell \mathrm{d}z$$

$$\le 2^{r-2} + (r-2)U(1) \int_2^\infty z^{r-3} \mathcal{E}_0\left(|S_1| 1_{\{|S_1| > z-1\}}\right) \mathrm{d}z$$

$$\le 2^{r-2} + U(1)\mathcal{E}_0(|S_1|(|S_1|+1)^{r-2}) < \infty,$$

which completes the proof of the lemma. □

Remark 2.1. Combining (2.5) and (Chow and Lai, 1979, Theorem 1), we see that

$$\mathcal{E}_0\left((\tilde{\xi}_1^-)^r\right) < \infty \quad \Longrightarrow \quad \mathcal{E}_0\left(\left|Z_{\widetilde{\tau}_0}\right|^{r-1}\right) < \infty \quad \Longrightarrow \quad \mathcal{E}_0\left(\left|\widetilde{\xi}_{\widetilde{\tau}_0}\right|^{r-1}\right) < \infty,$$

Also, (Chow, 1986, Theorem 1) provides necessary and sufficient conditions for $\mathcal{E}_0(|\widetilde{\xi}_{\widetilde{\tau}_0}|^{r-1}) < \infty$. But here we need the supremum over all starting points $x \in (0, \infty)$ to be finite, see (2.2). Lemma 2.1 gives a sufficient condition for (2.2). We will not explore the converse implication here.

2.2. Feynman-Kac representation for $v(x)$

Define a function $f : [0, 1] \mapsto \mathbb{R}$ by

$$f(v) := \beta \frac{\sum_{k=0}^\infty p_k (1-v)^k - (1-v)}{v}, \quad v \in (0, 1], \tag{2.9}$$

and $f(0) := f(0+) = 0$. Since for any nonnegative integer-valued random variable X with $\mathbb{E}X = 1$, $\mathbb{E}s^X \ge s$ for all $s \in [0, 1]$, we get $f(v) \ge 0$ for $v \in [0, 1]$. Also, define

$$F(v) = \frac{1}{v}\left(1 - \sum_{k=0}^\infty p_k (1-v)^k\right), \quad v \in (0, 1].$$

Note that $\beta(F(v) - 1) = -f(v)$. Recall that $v(x) = \mathbb{P}(M \ge x)$.

Lemma 2.2. *For any* $0 \le y < x$,

$$v(x) = \mathcal{E}_x \left(\exp\left\{ -\int_0^{\widetilde{\tau}_y} f\left(v\left(\widetilde{\xi}_s\right)\right) \mathrm{d}s \right\} v\left(\widetilde{\xi}_{\widetilde{\tau}_y}\right) \right).$$

Proof. Put $u(x) = 1 - v(x)$. Since the first branching time is an independent exponential random variable of parameter β, by Fubini's theorem, we have

$$u(x) = \mathbb{P}_x\left(\inf_{t\ge 0} \inf_{u \in N_t} \widetilde{X}_u(t) > 0 \right) = \int_0^\infty \beta e^{-\beta s} \sum_{k=0}^\infty p_k \mathcal{E}_x \left(1_{\{\widetilde{\tau}_0 > s\}} \left(u(\widetilde{\xi}_s)\right)^k \right) \mathrm{d}s$$

$$= \mathcal{E}_x\left(\int_0^{\widetilde{\tau}_0} \beta e^{-\beta s} \sum_{k=0}^{\infty} p_k \left(u(\widetilde{\xi}_s)\right)^k \mathrm{d}s\right).$$

According to (Dynkin, 2001, Lemma 4.1), we have

$$u(x) + \beta\mathcal{E}_x\left(\int_0^{\widetilde{\tau}_0} u(\widetilde{\xi}_s)\mathrm{d}s\right) = \beta\mathcal{E}_x\left(\int_0^{\widetilde{\tau}_0} \sum_{k=0}^{\infty} p_k \left(u(\widetilde{\xi}_s)\right)^k \mathrm{d}s\right),$$

which is equivalent to

$$\begin{aligned} v(x) &= 1 - \beta\mathcal{E}_x\left(\int_0^{\widetilde{\tau}_0} \sum_{k=0}^{\infty} p_k \left(1 - v(\widetilde{\xi}_s)\right)^k - \left(1 - v(\widetilde{\xi}_s)\right)\mathrm{d}s\right) \\ &= 1 - \mathcal{E}_x\left(\int_0^{\widetilde{\tau}_0} f(v(\widetilde{\xi}_s))v(\widetilde{\xi}_s)\mathrm{d}s\right), \end{aligned}$$

which in turn can be written as

$$v(x) + \mathcal{E}_x\left(\int_0^{\widetilde{\tau}_0} f(v(\widetilde{\xi}_s))v(\widetilde{\xi}_s)\mathrm{d}s\right) = 1.$$

Therefore, v is a solution of the equation: $v(x) + \mathcal{E}_x(\int_0^{\widetilde{\tau}_0} c(\widetilde{\xi}_s)v(\widetilde{\xi}_s)\mathrm{d}s) = 1$ in $(0,\infty)$ with $c(x) := f(v(x)) \geq 0$. Successively iterating the equation above, we get

$$v(x) = \mathcal{E}_x\left(\exp\left\{-\int_0^{\widetilde{\tau}_0} f\left(v\left(\widetilde{\xi}_s\right)\right)\mathrm{d}s\right\}\right).$$

The desired result follows by conditioning on $\mathcal{F}_{\widetilde{\tau}_y}$ and applying the strong Markov property of $\widetilde{\xi}$. □

2.3. An invariance principle for Lévy process

The following invariance principle is given in (Skorokhod, 1957, Theorem 2.7)

Lemma 2.3. *Suppose that $\widetilde{\xi}_t$ is a Lévy process with $\mathcal{E}_0(\widetilde{\xi}_1) = 0, \eta^2 = \mathcal{E}_0(\widetilde{\xi}_1^2) \in (0,\infty)$. Then the processes*

$$\frac{\widetilde{\xi}_{nt}}{\eta\sqrt{n}}, \quad t \in [0,\infty)$$

converge weakly to a standard Brownian motion $\{B_t, t \geq 0\}$ in the J_1-topology as $n \to \infty$.

3. Proof of the main result

Lemma 3.1. *Under the assumption (1.4), the function f defined in (2.9) satisfies that*

$$\lim_{v\to 0+} \frac{f(v)}{v^{\alpha-1}} = \frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}.$$

Proof. Let L be a random variable with distribution equal to the offspring distribution $\{p_k; k \geq 0\}$. It follows from (Bingham, Goldie and Teugels, 1989, Theorem 8.1.6) that $\mathbb{P}(L > x) \overset{x\to+\infty}{\sim} x^{-\alpha}c$ is equivalent to $\mathbb{E}(e^{-sL}) - 1 + \mathbb{E}(L)s \overset{s\to 0+}{\sim} s^{\alpha}\frac{\Gamma(2-\alpha)}{\alpha-1}c$, which is in turn equivalent to $\mathbb{E}(e^{-sL}) - e^{-s\mathbb{E}(L)} \overset{s\to 0+}{\sim} s^{\alpha}\frac{\Gamma(2-\alpha)}{\alpha-1}c$. Therefore, letting $1 - v = e^{-s}$, (1.4) is equivalent to

$$\lim_{v\to 0+} \frac{vf(v)}{(-\ln(1-v))^{\alpha}} = \frac{\beta\kappa\Gamma(2-\alpha)}{\alpha - 1},$$

which completes the proof of the lemma since $\lim_{v\to 0+} \frac{v^{\alpha}}{(-\ln(1-v))^{\alpha}} = 1$. □

For any fixed $y \geq 0$, the function

$$[0,\infty) \ni x \mapsto \frac{v\left(x + yv(x)^{-\frac{\alpha-1}{2}}\right)}{v(x)}$$

is bounded between 0 and 1. Therefore, by a diagonalization argument, we can find a subsequence $\{x_k \in [0,\infty)\}$ with $\lim_{k\to\infty} x_k = +\infty$ such that for all $y \geq 0, y \in \mathbb{Q}$, the following limits exist:

$$\phi(y) := \lim_{k\to\infty} \frac{v\left(x_k + yv(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)}. \tag{3.1}$$

Using the fact that $v(x)$ is decreasing, we see that $\phi(0) = 1$ and $\phi(y) \in [0,1]$ for any $y \in \mathbb{Q} \cap [0,\infty)$. Moreover, for non-negative rational numbers $y_1 < y_2$, it holds that $\phi(y_1) \geq \phi(y_2)$. Therefore, for any $y \geq 0$, we can define

$$\phi(y) := \sup_{z\in\mathbb{Q}, z\geq y} \phi(z) = \lim_{z\in\mathbb{Q}, z\downarrow y} \phi(y). \tag{3.2}$$

Proposition 3.1. *The function $\phi(y)$ is a continuous decreasing function in $[0,\infty)$ and*

$$\phi(y) = \lim_{k\to\infty} \frac{v\left(x_k + yv(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)}, \quad \text{for all } y \geq 0. \tag{3.3}$$

Moreover, for any $K > 0$, we have uniformly for $y \in [0,K]$,

$$\lim_{k\to\infty} \frac{v\left(x_k + yv(x_k)^{-\frac{\alpha-1}{2}}\right)}{\phi(y)v(x_k)} = 1. \tag{3.4}$$

Proof. Fix two non-negative rational numbers $y_1 < y_2$. By Lemma 3.1, there exists a constant $C_1 > 0$ such that $f(v) \leq C_1 v^{\alpha-1}$ for all $v \in [0,1]$. Set $z_i(k) = y_i v(x_k)^{-\frac{\alpha-1}{2}}$. It follows from Lemma 2.2 that

$$\begin{aligned}
\phi(y_1) \geq \phi(y_2) &= \lim_{k\to\infty} \frac{v(x_k + z_2(k))}{v(x_k)} \\
&= \lim_{k\to\infty} \mathcal{E}_{x_k+z_2(k)}\left(\exp\left\{-\int_0^{\widetilde{\tau}_{x_k+z_1(k)}} f\left(v\left(\widetilde{\xi}_s\right)\right) \mathrm{d}s\right\} \frac{v\left(\widetilde{\xi}_{\widetilde{\tau}_{x_k+z_1(k)}}\right)}{v(x_k)}\right)
\end{aligned}$$

$$\geq \limsup_{k\to\infty} \mathcal{E}_{x_k+z_2(k)} \left(\exp\left\{ -C_1 \int_0^{\widetilde{\tau}_{x_k+z_1(k)}} \left(v\left(\widetilde{\xi}_s\right)\right)^{\alpha-1} \mathrm{d}s \right\} \right) \frac{v\left(x_k+z_1(k)\right)}{v(x_k)}, \tag{3.5}$$

where in the last inequality, we used the fact that v is decreasing and that $\widetilde{\xi}_{\widetilde{\tau}_{x_k+z_1(k)}} \leq x_k + z_1(k)$. Since $\widetilde{\xi}_s \geq x_k + z_1(k) \geq x_k$ for $s \in (0, \widetilde{\tau}_{x_k+z_1(k)})$ and v is decreasing, by (3.5), we have

$$\begin{aligned} &\phi(y_1) \geq \phi(y_2) \geq \phi(y_1) \limsup_{k\to\infty} \mathcal{E}_{x_k+z_2(k)} \left(\exp\left\{ -C_1 \left(v\left(x_k\right)\right)^{\alpha-1} \widetilde{\tau}_{x_k+z_1(k)} \right\} \right) \\ &= \phi(y_1) \limsup_{k\to\infty} \mathcal{E}_0 \left(\exp\left\{ -C_1 \left(v\left(x_k\right)\right)^{\alpha-1} \widetilde{\tau}_{z_1(k)-z_2(k)} \right\} \right). \end{aligned} \tag{3.6}$$

Set $a := y_2 - y_1 > 0, n_k := \left(v\left(x_k\right)\right)^{-(\alpha-1)}$. Since for $t > 0$,

$$\mathcal{P}_0\left(n_k^{-1} \widetilde{\tau}_{-a n_k^{1/2}} > t \right) = \mathcal{P}_0\left(n_k^{-1/2} \inf_{s \leq t n_k} \widetilde{\xi}_s > -a \right) = \mathcal{P}_0\left(\frac{\inf_{s\leq t} \widetilde{\xi}_{n_k s}}{n_k^{1/2}} > -a \right),$$

it follows from Lemma 2.3 that

$$\begin{aligned} &\lim_{k\to\infty} \mathcal{P}_0\left(n_k^{-1} \widetilde{\tau}_{-a n_k^{1/2}} > t \right) = \lim_{k\to\infty} \mathcal{P}_0\left(\inf_{s\leq t} \frac{\widetilde{\xi}_{n_k s}}{n_k^{1/2}} > -a \right) \\ &= \mathcal{P}_0\left(\eta \inf_{s\leq t} B_s > -a \right) = \mathcal{P}_0\left(\tau^{BM}_{-a\eta^{-1}} > t \right), \end{aligned} \tag{3.7}$$

where τ_b^{BM} is the first time that a standard Brownian motion hits b. Combining (3.6) and (3.7),

$$\phi(y_1) \geq \phi(y_2) \geq \phi(y_1) \mathcal{E}_0\left(\exp\left\{ -C_1 \tau^{BM}_{(y_1-y_2)\eta^{-1}} \right\} \right) = e^{-\sqrt{2C_1}\frac{(y_2-y_1)}{\eta}} \phi(y_1). \tag{3.8}$$

By the definition of ϕ in (3.2), we see that (3.8) holds for all non-negative real numbers $y_1 < y_2$. This implies that ϕ is continuous. Besides, for any $y \geq 0$, we can fix two non-negative rational numbers $y_1 \leq y < y_2$. Then by the monotonicity of v,

$$\begin{aligned} \phi(y_2) &= \lim_{k\to\infty} \frac{v\left(x_k + y_2 v(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)} \leq \liminf_{k\to\infty} \frac{v\left(x_k + y v(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)} \\ &\leq \limsup_{k\to\infty} \frac{v\left(x_k + y v(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)} \leq \lim_{k\to\infty} \frac{v\left(x_k + y_1 v(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)} = \phi(y_1), \end{aligned}$$

which implies (3.3) by letting $y_1 \uparrow y$ and $y_2 \downarrow y$.

Finally we prove the uniform convergence. For any $\epsilon > 0$, we can find $y_0 = 0 < y_1 < \ldots < y_m = K$ such that

$$\sup_{1\leq i\leq m} |\phi(y_i) - \phi(y_{i-1})| < \frac{\epsilon}{2}.$$

Now we can find a common N such that for all $0 \le i \le m$, when $k > N$,

$$\left| \frac{v\left(x_k + y_i v(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)} - \phi(y_i) \right| < \frac{\epsilon}{2}.$$

Therefore, for any $i = 1, \ldots, m$ and $y \in [y_{i-1}, y_i]$, when $k > N$,

$$\begin{aligned} \phi(y) - \epsilon \le \phi(y_{i-1}) - \epsilon < \phi(y_i) - \frac{\epsilon}{2} < \frac{v\left(x_k + y_i v(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)} \le \frac{v\left(x_k + y v(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)} \\ \le \frac{v\left(x_k + y_{i-1} v(x_k)^{-\frac{\alpha-1}{2}}\right)}{v(x_k)} < \frac{\epsilon}{2} + \phi(y_{i-1}) < \epsilon + \phi(y_i) \le \epsilon + \phi(y). \end{aligned} \tag{3.9}$$

Noticing that $\phi(0) = 1$ and $\phi(K) > 0$ which holds by (3.8) with $y_1 = 0, y_2 = K$, by (3.9), we obtain the desired result (3.4). □

Given Lemma 2.3 and Proposition 3.1, the following result seems trivial, but we will give a proof. Recall that $n_k = v(x_k)^{-(\alpha-1)}$ and $\eta = \sqrt{\mathcal{E}_0(\widetilde{\xi}_1^2)}$.

Lemma 3.2. *For any $\theta > 0, y > 0$ and $z \ge y$, it holds that*

$$\begin{aligned} &\lim_{k\to\infty} \mathcal{E}_0 \left(\exp\left\{ -\theta \int_0^{n_k^{-1}\widetilde{\tau}_{-y\sqrt{n_k}}} \left(\frac{v\left(\left(n_k^{-1/2}\widetilde{\xi}_{n_k s} + z\right) v(x_k)^{-\frac{\alpha-1}{2}} + x_k\right)}{v(x_k)} \right)^{\alpha-1} \mathrm{d}s \right\} \right) \\ &= \mathcal{E}_0 \left(\exp\left\{ -\theta \int_0^{\tau_{-y/\eta}^{BM}} (\phi(\eta B_s + z))^{\alpha-1} \,\mathrm{d}s \right\} \right), \end{aligned} \tag{3.10}$$

where $\tau_{-y/\eta}^{BM}$ is the first time that a standard Brownian motion hits $-y/\eta$.

Proof. For simplicity, we set

$$\widetilde{\tau}^{(k)} := n_k^{-1}\widetilde{\tau}_{-y\sqrt{n_k}}, \quad \widetilde{\xi}_s^{(k)} := \frac{\widetilde{\xi}_{n_k s}}{\sqrt{n_k}}.$$

Step 1: In this step, we prove that for any $T, A > 0$,

$$\begin{aligned} &\lim_{k\to\infty} \mathcal{E}_0 \left(\exp\left\{ -\theta \int_0^{\widetilde{\tau}^{(k)}\wedge T} \left(\phi\left(\widetilde{\xi}_s^{(k)} + z\right)\right)^{\alpha-1} \mathrm{d}s \right\} 1_{\{\sup_{s\in[0,T]} \widetilde{\xi}_s^{(k)} < A\}} \right) \\ &= \mathcal{E}_0 \left(\exp\left\{ -\theta \int_0^{\tau_{-y\eta^{-1}}^{BM}\wedge T} (\phi(\eta B_s + z))^{\alpha-1} \,\mathrm{d}s \right\} 1_{\{\eta \sup_{s\in[0,T]} B_s < A\}} \right). \end{aligned} \tag{3.11}$$

For any integer $N > 1$, define $t_i := Ti/N, 1 \le i \le N$. Since ϕ is decreasing, it holds that

$$\int_0^{\widetilde{\tau}^{(k)}\wedge T} \left(\phi\left(\widetilde{\xi}_s^{(k)} + z\right)\right)^{\alpha-1} \mathrm{d}s = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left(\phi\left(\widetilde{\xi}_s^{(k)} + z\right)\right)^{\alpha-1} 1_{\{s<\widetilde{\tau}^{(k)}\}} \mathrm{d}s$$

$$\geq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left(\phi \left(\sup_{s \in [t_{i-1}, t_i]} \widetilde{\xi}_s^{(k)} + z \right) \right)^{\alpha-1} 1_{\{t_i < \widetilde{\tau}^{(k)}\}} \mathrm{d}s$$

$$= \frac{T}{N} \sum_{i=1}^{N} \left(\phi \left(\sup_{s \in [t_{i-1}, t_i]} \widetilde{\xi}_s^{(k)} + z \right) \right)^{\alpha-1} 1_{\{t_i < \widetilde{\tau}^{(k)}\}}. \tag{3.12}$$

Note that

$$\left\{ t_i < \widetilde{\tau}^{(k)} \right\} = \{ \widetilde{\tau}_{-y\sqrt{n_k}} > n_k t_i \} = \left\{ \inf_{s \leq n_k t_i} \widetilde{\xi}_s > -y\sqrt{n_k} \right\} = \left\{ \inf_{s \leq t_i} \widetilde{\xi}_s^{(k)} > -y \right\}$$

by the definition of $\widetilde{\tau}^{(k)}$ and $\widetilde{\xi}_s^{(k)}$. Also, observe that the functionals

$$w \in D[0, T] \mapsto \sup_{s \in [t_{j-1}, t_j]} w(s) \in \mathbb{R}, \quad i = 1, \ldots, N,$$

are continuous with respect to the J_1-topology. Therefore, taking two sequences of bounded continuous functions $h_\ell(x) \uparrow 1_{(-y,+\infty)}(x)$ and $j_\ell(x) \downarrow 1_{(-\infty,A)}(x)$, by Lemma 2.3 and (3.12), using the Lebesgue dominated convergence theorem, we get that

$$\limsup_{k \to \infty} \mathcal{E}_0 \left(\exp \left\{ -\theta \frac{T}{N} \sum_{i=1}^{N} \left(\phi \left(\sup_{s \in [t_{i-1}, t_i]} \widetilde{\xi}_s^{(k)} + z \right) \right)^{\alpha-1} 1_{\{t_i < \widetilde{\tau}^{(k)}\}} \right\} 1_{\{\sup_{s \in [0,T]} \widetilde{\xi}_s^{(k)} < A\}} \right)$$

$$\leq \limsup_{k \to \infty} \mathcal{E}_0 \left(\exp \left\{ -\theta \frac{T}{N} \sum_{i=1}^{N} \left(\phi \left(\sup_{s \in [t_{i-1}, t_i]} \widetilde{\xi}_s^{(k)} + z \right) \right)^{\alpha-1} h_\ell(\inf_{s \leq t_i} \widetilde{\xi}_s^{(k)}) \right\} j_\ell(\sup_{s \in [0,T]} \widetilde{\xi}_s^{(k)}) \right)$$

$$= \mathcal{E}_0 \left(\exp \left\{ -\theta \frac{T}{N} \sum_{i=1}^{N} \left(\phi \left(\eta \sup_{s \in [t_{i-1}, t_i]} B_s + z \right) \right)^{\alpha-1} h_\ell(\eta \inf_{s \leq t_i} B_s) \right\} j_\ell(\eta \sup_{s \in [0,T]} B_s) \right).$$

Then letting $\ell \to +\infty$, by the monotone convergence theorem, we get

$$\limsup_{k \to \infty} \mathcal{E}_0 \left(\exp \left\{ -\theta \int_0^{\widetilde{\tau}^{(k)} \wedge T} \left(\phi \left(\widetilde{\xi}_s^{(k)} + z \right) \right)^{\alpha-1} \mathrm{d}s \right\} 1_{\{\sup_{s \in [0,T]} \widetilde{\xi}_s^{(k)} < A\}} \right)$$

$$\leq \mathcal{E}_0 \left(\exp \left\{ -\theta \frac{T}{N} \sum_{i=1}^{N} \left(\phi \left(\eta \sup_{s \in [t_{i-1}, t_i]} B_s + z \right) \right)^{\alpha-1} 1_{\left\{ t_i < \tau^{BM}_{-y\eta^{-1}} \right\}} \right\} 1_{\{\eta \sup_{s \in [0,T]} B_s < A\}} \right). \tag{3.13}$$

Letting $N \to +\infty$ in (3.13), we get

$$\limsup_{k \to \infty} \mathcal{E}_0 \left(\exp \left\{ -\theta \int_0^{\widetilde{\tau}^{(k)} \wedge T} \left(\phi \left(\widetilde{\xi}_s^{(k)} + z \right) \right)^{\alpha-1} \mathrm{d}s \right\} 1_{\{\sup_{s \in [0,T]} \widetilde{\xi}_s^{(k)} < A\}} \right)$$

$$\leq \mathcal{E}_0 \left(\exp \left\{ -\theta \int_0^{\tau^{BM}_{-y/\eta} \wedge T} (\phi(\eta B_s + z))^{\alpha-1} \mathrm{d}s \right\} 1_{\{\eta \sup_{s \in [0,T]} B_s < A\}} \right).$$

Using a similar argument, we can get

$$\liminf_{k\to\infty}\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\widetilde{\tau}^{(k)}\wedge T}\left(\phi\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\mathrm{d}s\right\}1_{\{\sup_{s\in[0,T]}\widetilde{\xi}_s^{(k)}<A\}}\right)$$
$$\geq \mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\tau_{-y/\eta}^{BM}\wedge T}(\phi(\eta B_s+z))^{\alpha-1}\mathrm{d}s\right\}1_{\{\eta\sup_{s\in[0,T]}B_s<A\}}\right).$$

Combining the two displays above, we get the desired conclusion of this step.

Step 2: In this step, we prove that for any $T, A>0$,

$$\lim_{k\to\infty}\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\widetilde{\tau}^{(k)}\wedge T}\left(\phi^{(k)}\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\mathrm{d}s\right\}1_{\{\sup_{s\in[0,T]}\widetilde{\xi}_s^{(k)}<A\}}\right)$$
$$=\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\tau_{-y/\eta}^{BM}\wedge T}(\phi(\eta B_s+z))^{\alpha-1}\mathrm{d}s\right\}1_{\{\eta\sup_{s\in[0,T]}B_s<A\}}\right), \tag{3.14}$$

where

$$\phi^{(k)}(z):=\frac{v\left(zv(x_k)^{-\frac{\alpha-1}{2}}+x_k\right)}{v(x_k)}.$$

Note that on set $\{\sup_{s\in[0,T]}\widetilde{\xi}_s^{(k)}<A\}$, for any $s<\widetilde{\tau}^{(k)}\wedge T$, it holds that $\widetilde{\xi}_s^{(k)}+z\in(z-y,A+z)\subset[0,A+z]$. It follows from Proposition 3.1 that, for any $\varepsilon>0$, there exists K such that for any $k>K$ and $s\in\widetilde{\tau}^{(k)}\wedge T$,

$$(1-\varepsilon)\left(\phi\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\leq\left(\phi^{(k)}\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\leq(1+\varepsilon)\left(\phi\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}.$$

Therefore, by (3.11),

$$\limsup_{k\to\infty}\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\widetilde{\tau}^{(k)}\wedge T}\left(\phi^{(k)}\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\mathrm{d}s\right\}1_{\{\sup_{s\in[0,T]}\widetilde{\xi}_s^{(k)}<A\}}\right)$$
$$\leq\lim_{k\to\infty}\mathcal{E}_0\left(\exp\left\{-\theta(1-\varepsilon)\int_0^{\widetilde{\tau}^{(k)}\wedge T}\left(\phi\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\mathrm{d}s\right\}1_{\{\sup_{s\in[0,T]}\widetilde{\xi}_s^{(k)}<A\}}\right)$$
$$=\mathcal{E}_0\left(\exp\left\{-\theta(1-\varepsilon)\int_0^{\tau_{-y/\eta}^{BM}\wedge T}(\phi(\eta B_s+z))^{\alpha-1}\mathrm{d}s\right\}1_{\{\eta\sup_{s\in[0,T]}B_s<A\}}\right). \tag{3.15}$$

Letting $\varepsilon\downarrow 0$, we get

$$\limsup_{k\to\infty}\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\widetilde{\tau}^{(k)}\wedge T}\left(\phi^{(k)}\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\mathrm{d}s\right\}1_{\{\sup_{s\in[0,T]}\widetilde{\xi}_s^{(k)}<A\}}\right)$$
$$\leq\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\tau_{-y/\eta}^{BM}\wedge T}(\phi(\eta B_s+z))^{\alpha-1}\mathrm{d}s\right\}1_{\{\eta\sup_{s\in[0,T]}B_s<A\}}\right).$$

Using a similar argument, we can get

$$\liminf_{k\to\infty}\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\widetilde{\tau}^{(k)}\wedge T}\left(\phi^{(k)}\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\mathrm{d}s\right\}1_{\{\sup_{s\in[0,T]}\widetilde{\xi}_s^{(k)}<A\}}\right)$$
$$\geq \mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\tau_{-y/\eta}^{BM}\wedge T}(\phi(\eta B_s+z))^{\alpha-1}\mathrm{d}s\right\}1_{\{\eta\sup_{s\in[0,T]}B_s<A\}}\right).$$

Combining the two displays above, we get the desired conclusion of this step.

Step 3: In this step, we prove (3.10). Noting that

$$\lim_{T\to\infty}\lim_{A\to\infty}\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\tau_{-y/\eta}^{BM}\wedge T}(\phi(\eta B_s+z))^{\alpha-1}\mathrm{d}s\right\}1_{\{\eta\sup_{s\in[0,T]}B_s<A\}}\right)$$
$$=\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\tau_{-y/\eta}^{BM}}(\phi(\eta B_s+z))^{\alpha-1}\mathrm{d}s\right\}\right),$$

it suffices to prove that

$$\lim_{T\to\infty}\limsup_{A\to\infty}\limsup_{k\to\infty}\Bigg|\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\widetilde{\tau}^{(k)}\wedge T}\left(\phi^{(k)}\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\mathrm{d}s\right\}1_{\{\sup_{s\in[0,T]}\widetilde{\xi}_s^{(k)}<A\}}\right)$$
$$-\mathcal{E}_0\left(\exp\left\{-\theta\int_0^{\widetilde{\tau}^{(k)}}\left(\phi^{(k)}\left(\widetilde{\xi}_s^{(k)}+z\right)\right)^{\alpha-1}\mathrm{d}s\right\}\right)\Bigg|=0. \tag{3.16}$$

The proof for (3.16) is standard so we omit the details here. This implies the desired result. □

Proposition 3.2. *The function ϕ defined in (3.1) satisfies the equation*

$$\phi(y)=\mathcal{E}_0\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}\int_0^{\tau_{-y/\eta}^{BM}}(\phi(\eta B_s+y))^{\alpha-1}\mathrm{d}s\right\}\right),\quad y\geq 0.$$

Proof. Fix a constant $\rho>0$ and set $z_k:=x_k+v(x_k)^{-\frac{\alpha-1}{2}+\rho}$. For $y>0$, by Lemma 2.2, we have

$$\frac{v(x_k+yv(x_k)^{-\frac{\alpha-1}{2}}+v(x_k)^{-\frac{\alpha-1}{2}+\rho})}{v(x_k)}=\frac{v(z_k+yv(x_k)^{-\frac{\alpha-1}{2}})}{v(x_k)}$$
$$=\mathcal{E}_{z_k+yv(x_k)^{-\frac{\alpha-1}{2}}}\left(\exp\left\{-\int_0^{\widetilde{\tau}_{z_k}}f\left(v\left(\widetilde{\xi}_s\right)\right)\mathrm{d}s\right\}\frac{v\left(\widetilde{\xi}_{\widetilde{\tau}_{z_k}}\right)}{v(x_k)}\right)$$
$$=\mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(\exp\left\{-\int_0^{\widetilde{\tau}_0}f\left(v\left(\widetilde{\xi}_s+z_k\right)\right)\mathrm{d}s\right\}\frac{v\left(\widetilde{\xi}_{\widetilde{\tau}_0}+z_k\right)}{v(x_k)}\right). \tag{3.17}$$

We first show that

$$\lim_{k\to\infty} \mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(\left|\frac{v\left(\widetilde{\xi}_{\widetilde{\tau}_0}+z_k\right)}{v(x_k)}-1\right|\right)=0. \tag{3.18}$$

Indeed, on the event

$$A := \left\{\widetilde{\xi}_{\widetilde{\tau}_0}+z_k \geq x_k\right\},$$

by the inequality $v(x_k) \geq v\left(\widetilde{\xi}_{\widetilde{\tau}_0}+z_k\right) \geq v(z_k)$, we have

$$\left|\frac{v\left(\widetilde{\xi}_{\widetilde{\tau}_0}+z_k\right)}{v(x_k)}-1\right| = 1-\frac{v\left(\widetilde{\xi}_{\widetilde{\tau}_0}+z_k\right)}{v(x_k)} \leq 1-\frac{v\left(z_k\right)}{v(x_k)},$$

and on A^c, we have

$$\left|\frac{v\left(\widetilde{\xi}_{\widetilde{\tau}_0}+z_k\right)}{v(x_k)}-1\right| \leq \frac{2}{v(x_k)}.$$

Therefore,

$$\mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(\left|\frac{v\left(\widetilde{\xi}_{\widetilde{\tau}_0}+z_k\right)}{v(x_k)}-1\right|\right) \leq \frac{2}{v(x_k)}\mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(A^c\right)+1-\frac{v\left(z_k\right)}{v(x_k)}. \tag{3.19}$$

By Markov's inequality, for any $r > 2$, we have

$$\frac{1}{v(x_k)}\mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(A^c\right) \leq \mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(\left|\widetilde{\xi}_{\widetilde{\tau}_0}\right|^{r-2}\right)\left(v(x_k)\right)^{\left(\frac{\alpha-1}{2}-\rho\right)(r-2)-1}.$$

Since $r > 2\alpha/(\alpha-1)$, we can find a sufficiently small $\rho > 0$ such that $\left(\frac{\alpha-1}{2}-\rho\right)(r-2) > 1$. Therefore, by Lemma 2.1, we have

$$\lim_{k\to\infty}\frac{1}{v(x_k)}\mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(A^c\right)=0. \tag{3.20}$$

Since $\lim_{k\to\infty} v(z_k)/v(x_k) = 1$ by Proposition 3.1, we immediately get (3.18) by combining (3.19) and (3.20).

Letting $k \to \infty$, the left-hand side of (3.17) converges to $\phi(y)$ according to Proposition 3.1. For the right-hand side of (3.17), combining (3.18) and the trivial inequality $|\mathbb{E}(e^{-|X|}Y) - \mathbb{E}(e^{-|X|})| \leq \mathbb{E}(|Y-1|)$, we get that

$$\phi(y) = \lim_{k\to\infty}\mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(\exp\left\{-\int_0^{\widetilde{\tau}_0} f\left(v\left(\widetilde{\xi}_s+z_k\right)\right)\mathrm{d}s\right\}\right). \tag{3.21}$$

Using Lemma 3.1 and the fact that $\sup_{s<\widetilde{\tau}_0} v\left(\widetilde{\xi}_s + z_k\right) \le v(z_k) \to 0$, we get that for any $\varepsilon > 0$, there exists N such that for all $k \ge N$ and $s \in (0, \widetilde{\tau}_0)$,

$$\begin{aligned}&\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}(1-\varepsilon)\left(v\left(\widetilde{\xi}_s + x_k + \varepsilon v(x_k)^{-\frac{\alpha-1}{2}}\right)\right)^{\alpha-1} \le f\left(v\left(\widetilde{\xi}_s + z_k\right)\right)\\ &\le \frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}(1+\varepsilon)\left(v\left(\widetilde{\xi}_s + x_k\right)\right)^{\alpha-1}.\end{aligned}$$

Plugging this into (3.21), we get that

$$\begin{aligned}&\phi(y) \le\\ &\liminf_{k\to\infty} \mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}(1-\varepsilon)\int_0^{\widetilde{\tau}_0}\left(v\left(\widetilde{\xi}_s + x_k + \varepsilon v(x_k)^{-\frac{\alpha-1}{2}}\right)\right)^{\alpha-1} \mathrm{d}s\right\}\right).\end{aligned}$$

Note that for $n_k = v(x_k)^{-(\alpha-1)}$,

$$\begin{aligned}&\mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}(1-\varepsilon)\int_0^{\widetilde{\tau}_0}\left(v\left(\widetilde{\xi}_s + x_k + \varepsilon v(x_k)^{-\frac{\alpha-1}{2}}\right)\right)^{\alpha-1} \mathrm{d}s\right\}\right)\\ &= \mathcal{E}_0\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}(1-\varepsilon)\int_0^{\widetilde{\tau}_{-y\sqrt{n_k}}}\left(v\left(\widetilde{\xi}_s + (y+\varepsilon)v(x_k)^{-\frac{\alpha-1}{2}} + x_k\right)\right)^{\alpha-1} \mathrm{d}s\right\}\right)\\ &= \mathcal{E}_0\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}(1-\varepsilon)\right.\right.\\ &\qquad\left.\left.\int_0^{n_k^{-1}\widetilde{\tau}_{-y\sqrt{n_k}}}\left(v\left(\left(n_k^{-1/2}\widetilde{\xi}_{n_k s} + y + \varepsilon\right)v(x_k)^{-\frac{\alpha-1}{2}} + x_k\right)/v(x_k)\right)^{\alpha-1} \mathrm{d}s\right\}\right).\end{aligned}$$

By Lemma 3.2,

$$\begin{aligned}&\lim_{k\to\infty} \mathcal{E}_{yv(x_k)^{-\frac{\alpha-1}{2}}}\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}(1-\varepsilon)\int_0^{\widetilde{\tau}_0}\left(v\left(\widetilde{\xi}_s + x_k + \varepsilon v(x_k)^{-\frac{\alpha-1}{2}}\right)\right)^{\alpha-1} \mathrm{d}s\right\}\right)\\ &= \mathcal{E}_0\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}(1-\varepsilon)\int_0^{\tau^{BM}_{-y/\eta}}\left(\phi(\eta B_s + y + \varepsilon)\right)^{\alpha-1} \mathrm{d}s\right\}\right).\end{aligned}$$

Therefore, we conclude that

$$\phi(y) \le \mathcal{E}_0\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}(1-\varepsilon)\int_0^{\tau^{BM}_{-y\eta^{-1}}}\left(\phi(\eta B_s + y + \varepsilon)\right)^{\alpha-1} \mathrm{d}s\right\}\right).$$

Let $\varepsilon \downarrow 0$, we obtain that

$$\phi(y) \le \mathcal{E}_0\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}\int_0^{\tau^{BM}_{-y/\eta}}\left(\phi(\eta B_s + y)\right)^{\alpha-1} \mathrm{d}s\right\}\right).$$

Similarly, we also have

$$\phi(y) \geq \mathcal{E}_0\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{\alpha-1}\int_0^{\tau^{BM}_{-y/\eta}}(\phi(\eta B_s+y))^{\alpha-1}\,\mathrm{d}s\right\}\right).$$

Combining the two displays above, we arrive at the desired result. □

Corollary 3.1. *It holds that*

$$\phi(y) = (\theta y+1)^{-\frac{2}{\alpha-1}},$$

where

$$\theta := \left(\frac{\beta\kappa\Gamma(2-\alpha)(\alpha-1)}{\eta^2(\alpha+1)}\right)^{1/2}.$$

Proof. Combining Proposition 3.2 with the scaling property of Brownian motion, we get that, for any $y>0$,

$$\phi(y) = \mathcal{E}_y\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{(\alpha-1)\eta^2}\int_0^{\tau^{BM}_0}(\phi(B_s))^{\alpha-1}\,\mathrm{d}s\right\}\right).$$

By the strong Markov property, for any $n>0$ and $y\in(0,n)$,

$$\phi(y) = \mathcal{E}_y\left(\exp\left\{-\frac{\beta\kappa\Gamma(2-\alpha)}{(\alpha-1)\eta^2}\int_0^{\tau^{BM}_{(0,n)}}(\phi(B_s))^{\alpha-1}\,\mathrm{d}s\right\}\phi\left(B_{\tau^{BM}_{(0,n)}}\right)\right),$$

where $\tau^{BM}_{(0,n)} := \inf\{s\geq 0: B_s\notin(0,n)\}$. By (Chung and Zhao, 1995, Proposition 9.10), ϕ is solution of $\frac{1}{2}\phi''(y) = \frac{\beta\kappa\Gamma(2-\alpha)}{(\alpha-1)\eta^2}(\phi(y))^\alpha$ in $(0,n)$ with boundary condition $\phi(0) = \lim_{y\to 0+}\phi(y) = 1$. By the arbitrariness of n, ϕ is a bounded solution of the following problem:

$$\begin{cases}\frac{1}{2}\phi''(y) = \frac{\beta\kappa\Gamma(2-\alpha)}{(\alpha-1)\eta^2}(\phi(y))^\alpha, & y>0.\\ \phi(0)=1.\end{cases}$$

By (Chung and Zhao, 1995, Proposition 9.19), the bounded solution to the above problem is unique. It is easy to check that $\phi(y) = (\theta y+1)^{-\frac{2}{\alpha-1}}$ solves the above equation. Then the result follows. □

Proof of Theorem 1.1 By Corollary 3.1, the limit ϕ is independent of $\{x_k\}$, which implies that for all $y\geq 0$,

$$(\theta y+1)^{-\frac{2}{\alpha-1}} = \lim_{x\to+\infty}\frac{v\left(x+yv(x)^{-\frac{\alpha-1}{2}}\right)}{v(x)}. \tag{3.22}$$

Set $w(x) = x^{2/(\alpha-1)}v(x)$. Then w is left-continuous and that (3.22) is equivalent to

$$\lim_{x\to+\infty}\frac{w\left(x\left(1+yw(x)^{-\frac{\alpha-1}{2}}\right)\right)}{w(x)}\cdot\frac{(\theta y+1)^{2/(\alpha-1)}}{\left(1+yw(x)^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}} = 1. \tag{3.23}$$

Put $A := \liminf_{x\to\infty} w(x)$, and $B := \limsup_{x\to\infty} w(x)$. Then $0 \le A \le B \le \infty$.

Step 1: In this step, we prove $B > 0$ and $A < \infty$. Assume that $B = 0$. In this case, for $k \in \mathbb{N}$, define $b_k := \sup\{x : w(x) > k^{-1}\}$, then $b_k \to +\infty$ and $w(b_k) \to 0$. Taking $x = b_k$ and $y = 1$ in (3.23), we obtain that

$$\lim_{k\to+\infty} \frac{w\left(b_k\left(1+w(b_k)^{-\frac{\alpha-1}{2}}\right)\right)}{w(b_k)} \cdot \frac{(\theta+1)^{2/(\alpha-1)}}{\left(1+w(b_k)^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}} = 1.$$

Noticing that, by the definition of b_k, $w\left(b_k\left(1+w(b_k)^{-\frac{\alpha-1}{2}}\right)\right) \le k^{-1}$. Also, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $w(b_k - \delta_\varepsilon) > k^{-1}$. Since w is left-continuous and that we can choose δ_ε with $\lim_{\varepsilon\to 0+} \delta_\varepsilon = 0$, we see that $w(b_k) \ge k^{-1}$. Therefore,

$$\frac{w\left(b_k\left(1+w(b_k)^{-\frac{\alpha-1}{2}}\right)\right)}{w(b_k)} \cdot \frac{(\theta+1)^{2/(\alpha-1)}}{\left(1+w(b_k)^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}} \le \frac{(\theta+1)^{2/(\alpha-1)}}{\left(1+w(b_k)^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}} \stackrel{k\to\infty}{\longrightarrow} 0,$$

which is a contradiction. The proof of $A < \infty$ is similar.

Step 2: In this step, we prove $A \le \theta^{-2/(\alpha-1)} \le B$. By the definition of B, there exists $c_k \to +\infty$ such that $w(c_k) \to B$. Taking $x = c_k$ and $y = 1$ in (3.23), we get that

$$\lim_{k\to+\infty} \frac{w\left(c_k\left(1+w(c_k)^{\frac{\alpha-1}{2}}\right)\right)}{B} \cdot \frac{(\theta+1)^{2/(\alpha-1)}}{\left(1+B^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}} = 1. \tag{3.24}$$

Since $\limsup_{k\to\infty} w\left(c_k\left(1+w(c_k)^{\frac{\alpha-1}{2}}\right)\right) \le B$, (3.24) implies that

$$1 \le \frac{(\theta+1)^{2/(\alpha-1)}}{\left(1+B^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}} \quad \Longleftrightarrow \quad B \ge \theta^{-2/(\alpha-1)}.$$

The proof of $A \le \theta^{-2/(\alpha-1)}$ is similar.

Step 3: In this step we show that $A = B$, which leads to the conclusion of the theorem. Otherwise, we can assume $B > \theta^{-2/(\alpha-1)}$ without loss of generality. Let A_1 and B_1 be two fixed constants such that $\theta^{-2/(\alpha-1)} < A_1 < B_1 < B$. Since w is left-continuous and $\liminf_{x\to\infty} w(x) < A_1 < B_1 < \limsup_{x\to\infty} w(x)$, the following sequences are well-defined:

$$\begin{aligned}
&a_1 := \inf\{x > 0 : w(x) < A_1\}, \quad d_1 := \inf\{x > a_1 : w(x) > B_1\},\\
&a_k := \inf\{x > d_{k-1} : w(x) < A_1\}, \quad d_k := \inf\{x > a_k : w(x) > B_1\},\\
&a_k^* := \sup\{x \in [a_k, d_k) : w(x) < A_1\}.
\end{aligned}$$

Note that $a_k \uparrow \infty$ and $d_k \uparrow \infty$. Besides, using the left-continuity of w, we see that for every k, $w(a_k^*) \le A_1$. Taking $x = a_k^*$ in (3.23), by (3.4) and noticing that $\phi(y) = (\theta y+1)^{-\frac{2}{\alpha-1}}$, we get that for any $K > 0$

and any $\varepsilon > 0$ with $(1+\varepsilon)A_1 < B_1$, there exists N such that

$$\sup_{y\in[0,K]}\left|\frac{w\left(a_k^*\left(1+yw(a_k^*)^{-\frac{\alpha-1}{2}}\right)\right)}{w(a_k^*)}\cdot\frac{(\theta y+1)^{2/(\alpha-1)}}{\left(1+yw(a_k^*)^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}}-1\right|<\varepsilon,\quad k>N. \tag{3.25}$$

Since $A_1 > \theta^{-2/(\alpha-1)} \Longleftrightarrow A_1^{-(\alpha-1)/2} < \theta$, by (3.25), we see that when $k > N$, for all $y \in [0, K]$,

$$\begin{aligned}
w\left(a_k^*\left(1+yw(a_k^*)^{-\frac{\alpha-1}{2}}\right)\right) &< (1+\varepsilon)\frac{\left(1+yw(a_k^*)^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}}{(\theta y+1)^{2/(\alpha-1)}}w(a_k^*)\\
&= (1+\varepsilon)\frac{\left(w(a_k^*)^{\frac{\alpha-1}{2}}+y\right)^{2/(\alpha-1)}}{(\theta y+1)^{2/(\alpha-1)}} \le (1+\varepsilon)\frac{\left(A_1^{\frac{\alpha-1}{2}}+y\right)^{2/(\alpha-1)}}{(\theta y+1)^{2/(\alpha-1)}}\\
&= (1+\varepsilon)\frac{\left(1+yA_1^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}}{(\theta y+1)^{2/(\alpha-1)}}A_1 \le (1+\varepsilon)A_1 < B_1,
\end{aligned} \tag{3.26}$$

which implies that for any $k > N$,

$$\left\{a_k^*\left(1+yw(a_k^*)^{-\frac{\alpha-1}{2}}\right) : y\in[0,K]\right\} \subset [a_k^*, d_k) \tag{3.27}$$

by the definition of d_k. On the other hand, for any $K > \delta > 0$, set

$$C_\delta := \sup_{y\in[\delta,K]}\frac{\left(1+yA_1^{-\frac{\alpha-1}{2}}\right)^{2/(\alpha-1)}}{(\theta y+1)^{2/(\alpha-1)}} < 1.$$

Taking ε sufficiently small such that $(1+\varepsilon)C_\delta < 1$, by (3.26), when $k > N$, we have

$$\sup_{y\in[\delta,K]} w\left(a_k^*\left(1+yw(a_k^*)^{-\frac{\alpha-1}{2}}\right)\right) \le (1+\varepsilon)C_\delta A_1 < A_1. \tag{3.28}$$

Therefore, by the left-continuity of w and the definitions of a_k, d_k, a_k^*, for any $k > N$, there exists $m_k > k$ such that

$$\left\{a_k^*\left(1+yw(a_k^*)^{-\frac{\alpha-1}{2}}\right) : y\in[\delta,K]\right\} \subset [a_{m_k}, a_{m_k}^*]. \tag{3.29}$$

Moreover, for $y = K$,

$$a_k^*\left(1+Kw(a_k^*)^{-\frac{\alpha-1}{2}}\right) \ge a_{m_k} \ge a_{k+1} > d_k,$$

which contradicts (3.27). This completes the proof of the theorem. □

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