Three Favorite Edges Occurs Infinitely Often for One-Dimensional Simple Random Walk

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Abstract: For a one-dimensional simple symmetric random walk (S_n), an edge x (between points $x-1$ and x) is called a favorite edge at time n if its local time at n achieves the maximum among all edges. In this paper, we show that with probability 1 three favorite edges occurs infinitely often. Our work is inspired by Tóth and Werner [Combin. Probab. Comput. **6** (1997) 359-369], and Ding and Shen [Ann. Probab. **46** (2018) 2545-2561], disproves a conjecture mentioned in Remark 1 on page 368 of Tóth and Werner [Combin. Probab. Comput. **6** (1997) 359-369].

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1 Introduction

Let $(S_n)_{n\in\mathbb{N}}$ be a one-dimensional simple symmetric random walk with $S_0 = 0$. Following Tóth and Werner [35], we define for any $i \geq 1$,

$$\widetilde{S}_i := \frac{S_i + S_{i-1} + 1}{2},$$

which characterizes the edge of i-th jump (edge x is between points $x-1$ and x), and also define the "local time on the edge x at time n" as follows:

$$L(x,n) := \#\left\{1 \leq j \leq n : \widetilde{S}_j = x\right\}. \tag{1.1}$$

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Hereafter, $\#D$ denotes the cardinality of the set D. An edge x is called a favorite (or most visited) edge of the random walk at time n if

$$L(x,n) = \sup_{y\in\mathbb{Z}} L(y,n).$$

The set of favorite edges of the random walk at time n is denoted by $\mathcal{K}(n)$. $(\mathcal{K}(n))_{n\geq 1}$ is called the favorite edge process of the one-dimensional simple symmetric random walk. We say that *three favorite edges* occurs at time n if $\#\mathcal{K}(n) = 3$.

Theorem 1.1. *For a one-dimensional simple symmetric random walk, with probability* 1*, three favorite edges occurs infinitely often.*

Theorem 1.1 complements the result in [35] which showed that eventually there are no more than three favorite edges, and disproves a conjecture mentioned in [35, Remark 1 on page 368].

For the related problem of the number of favorite sites of one-dimensional simple symmetric random walks, there are many more references (see Shi and Tóth [33] for an overview). This problem was posed by Erdős and Révész [16, 17, 18, 32]. Tóth [34] proved that there are no more than three favorite sites eventually. Ding and Shen [12] proved that with probability 1 three favorite sites occurs infinitely often.

Besides the number of favorite sites, a series of papers focus on the asymptotic behavior of favorite sites, see [3, 7, 8, 28]. In addition, there are a number of papers on favorite sites for other processes including Brownian motion, symmetric stable process, Lévy processes, random walks in random environments and so on, see [2, 4, 13, 14, 15, 20, 21, 24, 26, 27, 29].

For papers on favorite sites of simple random walks in higher dimensions, we refer to [1, 10, 11, 30].

Our proof of Theorem 1.1 is inspired by [12], which in turn was inspired by [34, 35]. Following [34], we define the number of upcrossings and downcrossings of the site x by time n to be

$$\begin{aligned}\xi_U(x,n) &:= \#\{0 < k \leq n : S_k = x, S_{k-1} = x-1\},\\ \xi_D(x,n) &:= \#\{0 < k \leq n : S_k = x, S_{k-1} = x+1\}\end{aligned}$$

respectively. It is easy to check that

$$\xi_U(x,n) - \xi_D(x-1,n) = \mathbf{1}_{\{0<x\leq S_n\}} - \mathbf{1}_{\{S_n<x\leq 0\}}. \tag{1.2}$$

Using (1.1) and (1.2), we can easily get (see [34])

$$\begin{aligned}L(x,n) &= \#\left\{0 < j \leq n : \widetilde{S}_j = x\right\}\\ &= \#\left\{0 < j \leq n : S_j = x, S_{j-1} = x-1\right\} + \#\left\{0 < j \leq n : S_j = x-1, S_{j-1} = x\right\}\\ &= \xi_U(x,n) + \xi_D(x-1,n)\\ &= 2\xi_U(x,n) + \mathbf{1}_{\{S_n<x\leq 0\}} - \mathbf{1}_{\{0<x\leq S_n\}}\\ &= 2\xi_D(x-1,n) + \mathbf{1}_{\{0<x\leq S_n\}} - \mathbf{1}_{\{S_n<x\leq 0\}}.\end{aligned} \tag{1.3}$$

For $r \geq 1$, let $f(r)$ be the (possibly infinite) number of times when there are exactly r favorite edges:

$$f(r) := \# \{n \geq 1 : \#\mathcal{K}(n) = r\} .$$

We will show that

$$f(3) = \infty \quad \text{with probability } 1, \tag{1.4}$$

which implies Theorem 1.1. The idea for the proof comes from [12], i.e., we will show that $f(3) = \infty$ with positive probability and then prove (1.4) by using a 0-1 law. To this end, we need to show that the favorite edge process of a one-dimensional simple symmetric random walk is transient. In fact, by following Bass and Griffin [3], we will prove the following stronger transience result with an escape rate:

Theorem 1.2. *For any $\gamma > 11$, we have*

$$\liminf_{n\to\infty} \frac{\tilde{U}(n)}{\sqrt{n}(\log n)^{-\gamma}} = \infty \qquad a.s., \tag{1.5}$$

where $\tilde{U}(n) := \min\{|x| : x \in \mathcal{K}(n)\}$.

We mention in passing that, similar to [12, 34], we could have defined the following quantity related $f(r)$:

$$\widetilde{f}(r) := \# \left\{ n \geq 1 : \widetilde{S}_n \in \mathcal{K}(n) \text{ and } \#\mathcal{K}(n) = r \right\},$$

which is the number of times at which a new favorite edge appears, tied with $r-1$ other favorite edges. Using the recurrence of one-dimensional simple symmetric random walk, one can easily see that $f(3) = \infty$ a.s. is equivalent to $\widetilde{f}(3) = \infty$ a.s. See for instance, the paragraph below [35, Theorem 1.1].

The rest of the paper is organized as follows: in Section 2, we make some preparations for our proof for Theorem 1.1; in Section 3, we give the proof of Theorem 1.1; in Section 4, we give the proof of Theorem 1.2; in Section 5, we make a remark about a follow-up question; in Section 6, we present the proofs of several lemmas used in the previous sections. We emphasize that the idea for the proof of Theorem 1.1 comes from [12], and our main contribution is to prove Theorem 1.2 by establishing an invariance principle for one-side local times.

2 Preliminaries for the proof of Theorem 1.1

In this section, we make some preparations for the proof of Theorem 1.1. These preparations are modifications of the corresponding materials from [12] and [34].

2.1 Three consecutive favorite edges

We define the inverse edge local times by

$$T_U(x,k) := \min\{n \geq 1 : \xi_U(x,n) = k\} \quad \text{and} \quad T_D(x,k) := \min\{n \geq 1 : \xi_D(x,n) = k\}.$$

For any $x \in \mathbb{Z}$, define

$$\begin{aligned} u(x) &:= \sum_{n=1}^{\infty} \mathbf{1}_{\{S_{n-1}=x-2, S_n=x-1, x\in\mathcal{K}(n), \#\mathcal{K}(n)=3\}} \\ &= \sum_{k=1}^{\infty} \mathbf{1}_{\{x\in\mathcal{K}(T_U(x-1,k)), \#\mathcal{K}(T_U(x-1,k))=3\}} \\ &= \sum_{k=0}^{\infty}\sum_{h=1}^{\infty} \mathbf{1}_{\{x\in\mathcal{K}(T_U(x-1,k+1)), \#\mathcal{K}(T_U(x-1,k+1))=3, L(x,T_U(x-1,k+1))=h\}}. \end{aligned}$$

Thus $f(3) \geq \sum_{x\in\mathbb{Z}} u(x)$.

For $x \in \mathbb{Z}$, $h,k \in \mathbb{N}$, we define

$$A_{x,h}^{(k)} := \{\mathcal{K}(T_U(x-1,k+1)) = \{x,x+1,x+2\}, L(x,T_U(x-1,k+1)) = h\}. \tag{2.1}$$

Here we use $T_U(x-1,k+1)$ instead of $T_U(x,k+1)$, which was used in [12], due to the the following two reasons:

(i) *If $x>0$, then by the definition of local time on the edge x at the time $n = T_U(x,k+1)$, the set of favorite edges $\mathcal{K}(n)$ is not equal to $\{x,x+1,x+2\}$, since $L(x+1,n)$ and $L(x+2,n)$ are even numbers and $L(x,n)$ is an odd number.*

(ii) *$T_U(x-1,k+1)$ is useful to obtain the lower bound on the first moment in Section 4.1.*

Note that the definition of $T_U(x-1,k+1)$ implies $S_{T_U(x-1,k+1)} = x-1$ and $\xi_U(x-1, T_U(k+1,x-1)) = k+1$. Thus by (1.3), we have for $x \geq 1$,

$$L(x-1,T_U(x-1,k+1)) = 2k+2-\mathbf{1}_{\{0<x-1\}}.$$

Hence

$$L(x-1,T_U(x-1,k+1)) = \begin{cases} 2k+1, & \text{if } x>1, \\ 2k+2, & \text{if } x=1. \end{cases}$$

Again using (1.3), we can easily see that, when $x \geq 1$, the h in $A_{x,h}^{(k)}$ has to be even. In the following, we implicitly assume $x>1$, which implies that $L(x-1,T_U(x-1,k+1)) = 2k+1$, unless explicitly mentioned otherwise.

We write the events $A_{x,h}^{(k)}$ in terms of $T_U(x-1,k+1)$ since the events defined this way match the form of the Ray-Knight representation to be discussed later. Let $K_h = \left(\frac{1}{2}(h-2\sqrt{h}), \frac{1}{2}(h-\sqrt{h})\right)$ and define

$$N_H := \sum_{h=8}^{H}\sum_{k\in K_{2h}}\sum_{x=2}^{+\infty} \mathbf{1}_{A_{x,2h}^{(k)}} \quad \text{and} \quad N := \lim_{H\to\infty} N_H = \sum_{h=8}^{+\infty}\sum_{k\in K_{2h}}\sum_{x=2}^{+\infty} \mathbf{1}_{A_{x,2h}^{(k)}}. \tag{2.2}$$

We note that for each h, the events $A^{(k)}_{x,h}, x \in \mathbb{Z}, k \in K_h$ are mutually disjoint. Since $f(3) \geq \sum_{x\in\mathbb{Z}} u(x)$, we have that $f(3) \geq N$, and thus it is enough to show that $N = \infty$ a.s.

2.2 Branching process and the Ray-Knight representation

In the remainder of this paper, we denote by Y_n a critical Galton-Watson branching process with geometric offspring distribution, and by Z_n, R_n two related critical Galton-Watson branching processes with immigration. The precise definitions of these processes are as follows: Let $(X_{n,i})_{n,i}$ be i.i.d. geometric variables with mean 1, that is, for all $k \geq 0, P(X_{n,i} = k) = \frac{1}{2^{k+1}}$. We recursively define

$$Y_{n+1} = \sum_{i=1}^{Y_n} X_{n,i},\ \ Z_{n+1} = \sum_{i=1}^{Z_n+1} X_{n,i},\ \ R_{n+1} = 1 + \sum_{i=1}^{R_n} X_{n,i}. \tag{2.3}$$

One can check that Y_n, Z_n and R_n are Markov chains with state space $\mathbb{N}$ and transition probabilities:

$$P(Y_{n+1} = j|Y_n = i) = \pi(i,j) := \begin{cases} \delta_0(j), & \text{if } i = 0, \\ 2^{-i-j}\dfrac{(i+j-1)!}{(i-1)!j!}, & \text{if } i > 0, \end{cases} \tag{2.4}$$

$$P(Z_{n+1} = j|Z_n = i) = \rho(i,j) := \pi(i+1,j), \tag{2.5}$$
$$P(R_{n+1} = j|R_n = i) = \rho^*(i,j) := \pi(i,j-1). \tag{2.6}$$

Let $k \geq 0$ and x be fixed integers. When $x - 1 \geq 1$, we define the following three processes:

1. $(Z^{(k)}_n)_{n\geq 0}$ is a Markov chain with transition probabilities $\rho(i,j)$ and initial state $Z^{(k)}_0 = k$.
2. $(Y^{(k)}_n)_{n\geq -1}$ is a Markov chain with transition probabilities $\pi(i,j)$ and initial state $Y^{(k)}_{-1} = k$.
3. $(Y'^{(k)}_n)_{n\geq 0}$ is a Markov chain with transition probabilities $\pi(i,j)$ and initial state $Y'^{(k)}_0 = Z^{(k)}_{x-1}$.

We assume that the three processes are independent, except that $Y'^{(k)}$ starts from $Z^{(k)}_{x-1}$. We patch the above three processes together to a single process as follows:

$$\Delta^{(k)}_x(y) := \begin{cases} Z^{(k)}_{x-1-y}, & \text{if } 0 \leq y \leq x-1, \\ Y^{(k)}_{y-x}, & \text{if } x-1 \leq y < \infty, \\ Y'^{(k)}_{-y}, & \text{if } -\infty < y \leq 0. \end{cases} \tag{2.7}$$

By the Ray-Knight theorem on local times of simple random walk on $\mathbb{Z}$ (c.f. [25, Theorem 1.1]), we know that for any integers $x \geq 2$ and $k \geq 0$,

$$(\xi_D(y, T_U(x-1,k+1)), y \in \mathbb{Z}) \overset{\text{law}}{=} (\Delta^{(k)}_{x-1}(y), y \in \mathbb{Z}). \tag{2.8}$$

Similarly, when $x - 1 \leq 0$, we define the processes:

1. $(R_n^{(k)})_{n\ge -1}$ is a Markov chain with transition probabilities $\rho^*(i,j)$ and initial state $R_{-1}^{(k)} = k$.

2. $(Y_n^{(k)})_{n\ge 0}$ is a Markov chain with transition probabilities $\pi(i,j)$ and initial state $Y_0^{(k)} = k$.

3. $(Y_n^{'(k)})_{n\ge -1}$ is a Markov chain with transition probabilities $\pi(i,j)$ and initial state $Y_{-1}^{'(k)} = R_{-1-x}^{(k)}$.

We assume that the three processes are independent, except that $Y'^{(k)}$ starts from $R_{-1-x}^{(k)}$. In this case, we patch the three processes together as follows:

$$\Delta_x^{(k)}(y) \triangleq \begin{cases} Y_y^{'(k)}, & \text{if } -1 \le y < \infty, \\ R_{y-x}^{(k)}, & \text{if } x-1 \le y \le -1, \\ Y_{x-1-y}^{(k)}, & \text{if } -\infty < y \le x-1. \end{cases} \tag{2.9}$$

By the Ray-Knight theorem, we get that for the case $k \ge 0, x \le 1$:

$$(\xi_D(y, T_U(x-1, k+1)), y \in \mathbb{Z}) \stackrel{\text{law}}{=} (\Delta_{x-1}^{(k)}(y), y \in \mathbb{Z}). \tag{2.10}$$

2.3 Three favorite edges under Ray-Knight representation

For $h \in \mathbb{N}$, define the first hitting times of $[h, +\infty)$ for $Y_n^{(k)}$ and $Z_n^{(k)}$ to be $\sigma_h^{(k)}$ and $\tau_h^{(k)}$ respectively, and the extinction time of $Y_n^{(k)}$ to be $\omega^{(k)}$. That is,

$$\sigma_h^{(k)} := \min\{n \ge 0 : Y_n^{(k)} \ge h\}, \tau_h^{(k)} := \min\{n \ge 0 : Z_n^{(k)} \ge h\}, \omega^{(k)} := \min\{n \ge 0 : Y_n^{(k)} = 0\}. \tag{2.11}$$

Using the notation above, we can express $P(A_{x,h}^{(k)})$ in its Ray-Knight representation form. In the remainder of this section, we let $\tilde{n} := T_U(x-1, k+1)$ for simplicity. Now, on $A_{x,h}^{(k)}$, we have that $L(x,\tilde{n}) = L(x+1,\tilde{n}) = L(x+2,\tilde{n}) = h$, $h > L(y,\tilde{n}) = 2\xi_D(y-1,\tilde{n}) + \mathbf{1}_{\{0<y\le x-1\}}$ for $x \ge 2$, $y \ne x, x+1, x+2$. We have the following five cases:

(1) $y \le 0$, $\xi_D(y-1,\tilde{n}) = \frac{L(y,\tilde{n})}{2} < \frac{h}{2}$;

(2) $y \in [1, x-2]$, $\xi_D(y-1,\tilde{n}) = \frac{L(y,\tilde{n})-1}{2} < \frac{h-1}{2}$;

(3) $y = x-1$, $\xi_D(y-1,\tilde{n}) = \frac{L(y,\tilde{n})-1}{2} < \frac{h-1}{2}$;

(4) $y = x, x+1, x+2$, $\xi_D(y-1,\tilde{n}) = \frac{L(y,\tilde{n})}{2} = \frac{h}{2}$;

(5) $y \ge x+3$, $\xi_D(y-1,\tilde{n}) = \frac{L(y,\tilde{n})}{2} < \frac{h}{2}$.

Then by (2.8), we obtain that

$$\begin{aligned} P(A_{x,h}^{(k)}) \;=\; & P\left(Y_0^{(k)} = Y_1^{(k)} = Y_2^{(k)} = \frac{h}{2}, \left\{Y_n^{(k)} < \frac{h}{2}, n \ge 3\right\},\right. \\ & \left.\left\{Z_n^{(k)} < \frac{h-1}{2}, 1 \le n \le x-2\right\}, \left\{Y_n^{'(k)} < \frac{h}{2}, n \ge 1\right\}\right). \end{aligned} \tag{2.12}$$

For all the notation above, when the initial state of a process is obvious, we omit the superscript "(k)" for simplicity. We will also use the notation $P(\cdot|Y_0 = k)$ to indicate the initial state.

2.4 Standard lemmas

In this subsection we recall a few lemmas that will be useful later. In what follows, c_i for $i \geq 1$ and c are all constants.

Lemma 2.1. *([12, Lemma 2.2]) We have that*

(1) *For* $i, j \in \left(\frac{1}{2}(h - 10\sqrt{h}), \frac{1}{2}(h + 10\sqrt{h})\right)$, *there exist positive constants* c *and* C *such that* $ch^{-\frac{1}{2}} \leq \pi(i,j) \leq Ch^{-\frac{1}{2}}$ *for all* $h \geq 100$.

(2) *For* $i + j = h$, $\pi(i,j) \leq O(1)h^{-\frac{1}{2}}$.

(3) *For* $j < i_1 < i_2$, $\pi(i_1, j) > \pi(i_2, j)$.

Lemma 2.2. *([12, Lemma 2.3]) For any* $h \in \mathbb{N}$, *it holds that* $E\tau_h = EZ_{\tau_h} - Z_0$. *In particular, for any* $0 \leq k \leq h$, *we have that* $E[\tau_h | Z_0 = k] \geq h - k$.

Lemma 2.3. *([34, (6.18)]) There exists a constant* $C < \infty$ *such that, for any* $0 \leq k < h$,

$$E(Z_{\tau_h} | Z_0 = k) \leq h + Ch^{\frac{1}{2}}.$$

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by following the idea of [12]. We spell out some details for the reader's convenience and point out some modifications that need to be made.

3.1 Lower bound on the first moment.

The following is the counterpart to [12, Lemma 3.1] and its proof is postponed to Section 6.1.

Lemma 3.1. *Suppose that* $Z_0 = k \in K_h = [\frac{h-2\sqrt{h}}{2}, \frac{h-\sqrt{h}}{2}]$. *Then there exists a constant* $c > 0$ *such that for any* $h > 4$,

$$E\left(\sum_{n=1}^{\tau_{\frac{h-1}{2}}} \frac{h/2 - Z_n}{h/2}\right) \geq c\sqrt{h}.$$

Proposition 3.2. *There exists* $c > 0$ *such that* $EN_H \geq c \log H$ *for all* $H \in [50, \infty)$.

Proof. By the Ray-Knight representation, (2.2) and (2.12), we know that

$$\begin{aligned} EN_H &= \sum_{h=8}^{H}\sum_{k\in K_{2h}} P\left(Y_0^{(k)}=Y_1^{(k)}=Y_2^{(k)}=h, \left\{Y_n^{(k)}<h, \forall n\ge 3\right\}\right) \\ &\quad \cdot \sum_{x=2}^{+\infty} P\left(\left\{Z_n^{(k)}<h-\frac{1}{2}, 1\le n\le x-2\right\}, \left\{Y_n^{\prime(k)}<h, \forall n\ge 1\right\}\right). \end{aligned}$$

It follows that

$$\begin{aligned} & EN_H \\ \ge & \sum_{h=50}^{H}\sum_{k\in K_{2h}} P\left(Y_0^{(k)}=Y_1^{(k)}=Y_2^{(k)}=h, \left\{Y_3^{(k)}\in\left(h-5\sqrt{2h}, h-\frac{\sqrt{2h}}{2}\right), Y_n^{(k)}<h, \forall n\ge 4\right\}\right) \\ & \cdot \sum_{x=2}^{+\infty} P\left(\left\{Z_n^{(k)}<h-\frac{1}{2}, 1\le n\le x-2\right\}, \left\{Y_n^{\prime(k)}<h, n\ge 1\right\}\right) \\ = & \sum_{h=50}^{H}\sum_{k\in K_{2h}} \pi(k,h)\cdot\pi(h,h)\cdot\pi(h,h)\cdot \sum_{m\in(h-5\sqrt{2h}, h-\frac{\sqrt{2h}}{2})} \pi(h,m) P\left(Y_n^{(m)}<h, \forall n\ge 1\right) \\ & \cdot \sum_{x=2}^{+\infty} P\left(\tau_{h-\frac{1}{2}}\ge x-1, \left\{Y_n^{\prime(k)}<h, \forall n\ge 1\right\}\right). \end{aligned} \tag{3.1}$$

By Lemma 2.1, all the $\pi(\cdot,\cdot)$'s in the display above are of the order $h^{-\frac{1}{2}}$. Since Y_n is a martingale, applying the optional stopping theorem at $\sigma_h\wedge\omega$, where σ_h and ω are defined in (2.11), we get for $m\in(h-5\sqrt{2h}, h-\frac{\sqrt{2h}}{2})$,

$$P\left(Y_n^{(m)}<h, \forall n\ge 1\right) = P\left(Y_n^{(m)} \text{ hits 0 before exceeds } h\right) \ge \frac{h-m}{h} \ge c_3 h^{-\frac{1}{2}}. \tag{3.2}$$

The last two inequalities hold since

$$\begin{aligned} m &= EY_{\sigma_h\wedge\omega} = E\left(Y_{\sigma_h}\mathbf{1}_{\{\omega\ge\sigma_h\}}\right) + E\left(Y_\omega \mathbf{1}_{\{\omega<\sigma_h\}}\right) \\ &\ge E\left(Y_{\sigma_h}\mathbf{1}_{\{\omega\ge\sigma_h\}}\right) \\ &\ge h\left(1-P\left(Y_n^{(m)} \text{ hits 0 before exceeds } h\right)\right), \end{aligned}$$

which implies that

$$P\left(Y_n^{(m)} \text{ hits 0 before exceeds } h\right) \ge \frac{h-m}{h} \ge c_3 h^{-\frac{1}{2}}. \tag{3.3}$$

By (3.1) and (3.2), we get

$$EN_H \ge c_4 \sum_{h=50}^{H}\sum_{k\in K_{2h}} h^{-2}\cdot\sum_{x=2}^{+\infty} P\left(\tau_{h-\frac{1}{2}}\ge x-1, \left\{Y_n^{\prime(k)}<h, \forall n\ge 1\right\}\right). \tag{3.4}$$

By the independence of the processes in the Ray-Knight representation, we have

$$\begin{aligned}
&\sum_{x=2}^{+\infty} P\left(\tau_{h-\frac{1}{2}} \geq x-1, \left\{Y_n^{\prime(k)} < h, \forall n \geq 1\right\}\right) \\
&\geq \sum_{x=2}^{+\infty}\sum_{l=0}^{h-1} P\left(Z_n^{(k)} < h - \frac{1}{2} \text{ for } 1 \leq n \leq x-2, Z_{x-1} = l\right) \cdot P\left(Y_n^{(l)} \text{ hits 0 before exceeds } h\right).
\end{aligned}$$

By the definitions of $(Z_n)_{n\geq 0}$ and τ_h, we obtain

$$Z_0 - Z_{\tau_{h-\frac{1}{2}}} \leq h - \frac{\sqrt{2h}}{2} - h + \frac{1}{2} = \frac{1-\sqrt{2h}}{2} \leq 0. \tag{3.5}$$

Similar to (3.3), we have $P(Y_n^{(l)}$ hits 0 before exceeds $h) \geq \frac{h-l}{h}$, which together with (3.5) and Lemma 3.1 implies that

$$\begin{aligned}
&\sum_{x=2}^{+\infty} P\left(\tau_{h-\frac{1}{2}} \geq x-1, \left\{Y_n^{\prime(k)} < h, \forall n \geq 1\right\}\right) \\
&\geq \sum_{x=2}^{+\infty}\sum_{l=0}^{h} P\left(\tau_{h-\frac{1}{2}} \geq x-1, Z_{x-2} = l\right) \cdot \frac{h-l}{h} \\
&= E\left(\sum_{l=0}^{h}\sum_{x=2}^{\tau_{h-\frac{1}{2}}+1} \frac{h-l}{h} \cdot \mathbf{1}_{\{Z_{x-2}=l\}}\right) = E\left(\sum_{n=0}^{\tau_{h-\frac{1}{2}}-1} \frac{h-Z_n}{h}\right) \\
&\geq E\left(\sum_{n=0}^{\tau_{h-\frac{1}{2}}-1} \frac{h-Z_n}{h}\right) + \left(\frac{h-Z_{\tau_{h-\frac{1}{2}}}}{h} - \frac{h-Z_0}{h}\right) \\
&= E\left(\sum_{n=1}^{\tau_{h-\frac{1}{2}}} \frac{h-Z_n}{h}\right) \geq c_5\sqrt{h}.
\end{aligned} \tag{3.6}$$

By (3.4) and (3.6), we obtain

$$EN_H \geq c_4 \sum_{h=50}^{H}\sum_{k\in K_{2h}} h^{-2} \cdot c_5\sqrt{h} \geq c_6 \sum_{h=100}^{H} h^{-1} \geq c\log H.$$

The proof is complete. □

3.2 Upper bound on the second moment

In this subsection, we will give an upper bound on the second moment EN_H^2 following [12]. If we use the N_H defined in (2.2)–(2.1), we could not prove the counterparts of [12, Lemmas 3.3, 3.4]. To overcome this, we give a variant $\widetilde{N}_H$ of N_H with $N_H \leq \widetilde{N}_H$ a.s. In the remainder of this section, we assume $x \geq 2$ unless explicitly mentioned otherwise.

For any positive integer h, we define

$$A_{x,2h} := \{\mathcal{K}(T_D(x-1,h)) = \{x,x+1,x+2\}, L(x,T_D(x-1,h)) = 2h\} \tag{3.7}$$

and

$$\widetilde{N}_H := \sum_{h=50}^{H}\sum_{x=2}^{\infty}\mathbf{1}_{A_{x,2h}}, \quad \widetilde{N} := \lim_{H\to\infty}\widetilde{N}_H = \sum_{h=50}^{\infty}\sum_{x=2}^{\infty}\mathbf{1}_{A_{x,2h}}.$$

We claim that

$$\cup_{k\in K_{2h}}A_{x,2h}^{(k)} \subset A_{x,2h}, \tag{3.8}$$

where $A_{x,2h}^{(k)}$ is defined by (2.1). Suppose that $\omega \in A_{x,2h}^{(k)}$ for some $k \in K_{2h}$. Define

$$\tilde{T}(\omega) := \{m < T_U(x-1,k+1)(\omega) : S_m = x-1, S_{m-1} = x, \mathcal{K}(m) = \{x,x+1,x+2\}, L(x,m) = 2h\}.$$

By the definition of $A_{x,h}^{(k)}$, we know that $\tilde{T}(\omega)$ consists of one unique element $t(\omega)$. Further by (1.3), we get that $t(\omega) = T_D(x-1,h)(\omega)$. Hence $\omega \in A_{x,2h}$ and thus (3.8) holds.

By (3.8), we get that $N_H \le \widetilde{N}_H$ a.s. Now we study the second moment of $\widetilde{N}_H$ following [12, Section 3.2].

Let $D(n) = \big(\xi_D(x,n), x\in\mathbb{Z}\big) \in \mathbb{N}^{\mathbb{Z}}$ be the random vector that records the number of down-crossings of each site by time n. For $l \in \mathbb{N}^{\mathbb{Z}}$, we use $l(i), i\in\mathbb{Z}$ to denote the i-th component of l. For $l\in\mathbb{N}^{\mathbb{Z}}$, define $B_x(l) = \{\exists n < \infty : D(n) = l, S(n-1) = x, S(n) = x-1\}$. Note that if $B_x(l)$ happens, there exists a unique $n \in \mathbb{N}$, such that $D(n) = l, S(n-1) = x$ and $S(n) = x-1$.

Let $\mathcal{P} = \{l : P(B_x(l)) > 0 \text{ for some } x\}$. For $\mathcal{Q} \subset \mathcal{P}$, denote $B_x(\mathcal{Q}) = \bigcup_{l\in\mathcal{Q}} B_x(l)$. Then by virtue of (1.3), we have $A_{x,2h} = B_x(\mathcal{P}_{x,2h})$, where $\mathcal{P}_{x,2h}$ is the set of $l \in \mathcal{P}$ such that

$$l(x-1) = l(x) = l(x+1) = h,\ l(i) < h \text{ for all } i \ne x-1, x, x+1.$$

Let $\mathcal{A}$ be the family of all subsets of $\mathcal{P}$. For any $x\in\mathbb{Z}$, we define a map $\varphi_x : \mathcal{P} \mapsto \mathcal{A}$ by

$$\varphi_x(l) := \{l^* \in \mathcal{P} : l^*(i) < l(i) \text{ for } i = x, x+1, l^*(i) = l(i) \text{ for } i \ne x, x+1\}.$$

Following the argument of [12, Lemma 3.3], we can prove

Lemma 3.3. *Suppose $x_1, x_2 \in \mathbb{Z}$ and h is a positive integer. If $l_i^* \in \varphi_{x_i}(l_i)$ with $l_i \in \mathcal{P}_{x_i,2h}$, $i = 1,2$, we have that $B_{x_1}(l_1^*) \cap B_{x_2}(l_2^*) = \emptyset$, if $(x_1,l_1) \ne (x_2,l_2)$. Further, we have $B_{x_1}(l_1^*) \cap B_{x_2}(l_2^*) = \emptyset$ if $(x_1,l_1) = (x_2,l_2)$ but $l_1^* \ne l_2^*$.*

The following result is the counterpart of [12, Lemma 3.4].

Lemma 3.4. *There exists a constant $c > 0$ such that for any $x \ge 2, h \ge 50, l \in \mathcal{P}_{x,2h}$,*

$$P\big(B_x(\varphi_x(l))\big) \ge chP\big(B_x(l)\big).$$

Proof. We consider $l^* \in \varphi_x(l)$ such that $l^*(x) \in [h - \frac{\sqrt{2h}}{2}, h)$ and $l^*(x+1) \in [h - \frac{\sqrt{2h}}{2}, h)$. According to Lemma 2.1 (1) and (3), there is a constant $c > 0$ such that

$$\frac{P\big(B_x(l^*)\big)}{P\big(B_x(l)\big)} = \frac{\pi\big(h, l^*(x)\big) \cdot \pi\big(l^*(x), l^*(x+1)\big) \cdot \pi\big(l^*(x+1), l(x+2)\big)}{\pi(h,h) \cdot \pi(h,h) \cdot \pi(h, l(x+2))} \geq 4c.$$

Note that there are about $h/2$ of such $l^* \in \varphi_x(l)$ that satisfy the above inequality. Thus by Lemma 3.3, we get that $P\big(B_x(\varphi_x(l))\big) \geq chP\big(B_x(l)\big)$. □

With Lemmas 3.3 and 3.4 in hand, we can follow the proof of [12, Proposition 3.5] to get the following result. We omit the details.

Proposition 3.5. *We have that* $E\widetilde{N}_H^2 = O(\log H) \cdot E\widetilde{N}_H$.

Corollary 3.6. *We have* $EN_H^2 = O(\log^2 H)$.

Proof. By Proposition 3.5 and the Cauchy-Schwarz inequality, we get

$$E\widetilde{N}_H^2 \leq O(\log H) \cdot \left(E\widetilde{N}_H^2\right)^{1/2},$$

which implies that

$$E\tilde{N}_H^2 = O(\log^2 H).$$

Since $N_H \leq \tilde{N}_H$ a.s., the desired result follows immediately. □

3.3 0-1 law

Recall that $N = \lim_{H\to\infty} N_H$. First, we show that $N = \infty$ with positive probability.

Proposition 3.7. *There exists a constant* $\delta > 0$ *such that* $P(N = \infty) \geq \delta$.

Proof. By the Cauchy-Schwarz inequality, we get that

$$\begin{aligned} EN_H &= EN_H \mathbf{1}_{\{N_H > \log\log H\}} + EN_H \mathbf{1}_{\{N_H \leq \log\log H\}} \\ &\leq \sqrt{EN_H^2 \cdot P(N_H > \log\log H)} + \log\log H. \end{aligned} \tag{3.9}$$

Combining this with Proposition 3.2 and Corollary 3.6, we get that there exist constants c, $\delta > 0$ such that

$$P(N_H > \log\log H) \geq \frac{(EN_H - \log\log H)^2}{EN_H^2} \geq c\frac{\log^2 H}{EN_H^2} \geq \delta,$$

for all sufficiently large H. Letting $H \to \infty$, we get that $P(N = \infty) \geq \delta$. □

Recall the definition of $\tilde{U}(n)$ in Theorem 1.2. By Theorem 1.2, we know that

$$\liminf_{n\to\infty} \frac{\tilde{U}(n)}{\sqrt{n}(\log n)^{-\gamma}} = \infty \qquad a.s., \tag{3.10}$$

where $\gamma > 11$.

Using Proposition 3.7 and (3.10), following the argument of [12, Section 3.3] and applying Kolmogorov's 0-1 law, we obtain that

$$P(f(3) = \infty) = 1,$$

which completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. To this end, we will establish an invariance principle for one-side local times of random walks in Section 4.2. The proof of Theorem 1.2 will be given in Section 4.3. In Section 4.1, we give a brief introduction to the invariance principle for (two-side) local times.

4.1 Invariance principle for local times

Recall that the (site) local time process of the random walk $(S_n, n \geq 0)$ is defined by

$$\xi(x,n) := \#\left\{k : 1 \leq k \leq n, S_k = x\right\},\ x \in \mathbb{Z},\ n \geq 1.$$

Define $\xi^*(n) := \sup_{x\in\mathbb{Z}} \xi(x,n)$.

Let $(W(t))_{t\geq 0}$ be a one-dimensional standard Brownian motion (Wiener process). Recall that the local time process $(\eta(x,t))_{t\geq 0, x\in\mathbb{R}}$ of $(W(t))_{t\geq 0}$ is defined by

$$\eta(x,t) := \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{(x-\varepsilon,x+\varepsilon)}(W(s))ds = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \mathbf{1}_{[x,x+\varepsilon)}(W(s))ds. \tag{4.1}$$

Révész [31] established the following strong invariance principle with a rate of convergence: On a rich enough probability space, as $n \to \infty$,

$$\sup_{x\in\mathbb{Z}} |\xi(x,n) - \eta(x,n)| = o(n^{\frac{1}{4}+\varepsilon}) \quad a.s. \tag{4.2}$$

for any $\varepsilon > 0$. Csörgő and Horváth [9, Theorems 1 and 2] showed that the Révész's result is the best possible.

4.2 Invariance principle for one-side local times

Define the one-side local time of the Wiener process by

$$\eta_R(x,t) := \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{[x,x+\varepsilon)}\big(W(s)\big)ds.$$

Then by (4.1), we get that for any $x\in\mathbb{R}$ and $t\geq 0$,

$$\eta_R(x,t) = \frac{1}{2}\eta(x,t). \tag{4.3}$$

The goal in this subsection is to extend (4.2) to the one-side case, that is, prove the following result.

Theorem 4.1. *On a rich enough probability space $(\Omega,\mathcal{F},P)$, one can define a Wiener process $(W(t))_{t\geq 0}$ and a one-dimensional simple symmetric random walk $(S_k)_{k\in\mathbb{N}}$ with $S_0=0$, such that for any $\varepsilon>0$, as $n\to\infty$, we have*

$$\sup_{x\in\mathbb{Z}}|\xi_D(x,n)-\eta_R(x,n)| = o(n^{\frac{1}{4}+\varepsilon}) \qquad a.s.$$

Before giving the proof of Theorem 4.1, we present the following lemma, whose proof is postponed to Section 6.2.

Lemma 4.2. *For any $\varepsilon>0$, we have*

$$\lim_{n\to\infty}\frac{\sup_{x\in\mathbb{Z}}|\xi_D(x+1,n)-\xi_D(x,n)|}{n^{\frac{1}{4}+\varepsilon}} = 0 \quad a.s. \tag{4.4}$$

Proof of Theorem 4.1. This proof is inspired by Révész [31], and Csörgő and Horváth [9]. Let $(W(t))_{t\geq 0}$ be a Wiener process and define $\tau_0 := 0, \tau_1 := \inf\{t : t>0, |W(t)|=1\}, \tau_n := \inf\{t : t>\tau_{n-1}, |W(t)-W(\tau_{n-1})|=1\}, \forall n\geq 2$. Then $X_i = W(\tau_i)-W(\tau_{i-1}) (i\geq 1)$ are i.i.d. r.v.'s with $P\{X_i=1\}=P\{X_i=-1\}=1/2$, and $\tau_i-\tau_{i-1} (i\geq 1)$ are i.i.d. r.v.'s with $E(\tau_i-\tau_{i-1})=1$ and $E(\tau_i-\tau_{i-1})^2<\infty$. Put $\sigma^2 = E(\tau_1-1)^2$. Define $S_k = X_1+\cdots+X_k = W(\tau_k)$.

Let $a_i(x) = \eta(x,\tau_{v(i)+1})-\eta(x,\tau_{v(i)-1}), b_i(x) = \eta_R(x,\tau_{v(i)+1})-\eta_R(x,\tau_{v(i)-1}) (i\in\mathbb{Z}^+)$, where $v(1)=\min\{k\geq 0, W(\tau_k)(=S_k)=x\}, v(n)=\min\{k>v(n-1), W(\tau_k)(=S_k)=x\}, \forall n\geq 2$. Then by (4.3), we have

$$b_i(x) = \frac{a_i(x)}{2}. \tag{4.5}$$

Kesten [22] showed

$$\limsup_{n\to\infty}(2n\log\log n)^{-\frac{1}{2}}\cdot\sup_{x\in\mathbb{Z}}\xi(x,n) = 1 \qquad a.s. \tag{4.6}$$

Csörgő and Horváth [9, (2.7)] says that

$$\max_{-n\le x\le n}\left|\sum_{i=1}^{\xi(x,n)} a_i(x)-\eta(x,\tau_n)\right| = O(\log n) \quad a.s.,$$

which together with (4.3) and (4.5) implies that

$$\max_{-n\le x\le n}\left|\sum_{i=1}^{\xi(x,n)} b_i(x)-\eta_R(x,\tau_n)\right| = O(\log n) \qquad a.s. \tag{4.7}$$

By [9, (2.8)] and (4.5), we have

$$P\left\{\max_{-k\le x\le k}\left|\sum_{i=1}^{k}\left(b_i(x)-\frac{1}{2}\right)\right| > C_1(k\log k)^{\frac{1}{2}}\right\} \le C_2 k^{-2}. \tag{4.8}$$

Combining this with the Borel-Cantelli lemma and then using (4.6), we get that

$$\limsup_{n\to\infty}\frac{\max\limits_{-n\le x\le n}|\sum_{i=1}^{\xi(x,n)} b_i(x)-\frac{\xi(x,n)}{2}|}{n^{\frac{1}{4}}\cdot(\log n)^{\frac{1}{2}}\cdot(\log\log n)^{\frac{1}{4}}} \le C_3 \qquad a.s. \tag{4.9}$$

Since $\tau_i-\tau_{i-1}, i\ge 1$, are i.i.d. r.v.'s with $E\tau_1=1$ and $\sigma^2=E(\tau_1-1)^2<\infty$, by the law of iterated logarithm, we have

$$\limsup_{n\to\infty}\frac{|\tau_n-n|}{\sqrt{2\sigma^2 n\log\log n}} = 1 \qquad a.s. \tag{4.10}$$

By [9, (2.11)] and (4.3), we have

$$\limsup_{n\to\infty}\frac{\sup\limits_x|\eta_R(x,n\pm g(n))-\eta_R(x,n)|}{n^{\frac{1}{4}}\cdot(\log n)^{\frac{1}{2}}\cdot(\log\log n)^{\frac{1}{4}}} \le C_4 \quad a.s.,$$

where $g(n)=(4\sigma^2 n\log\log n)^{\frac{1}{2}}$. (Note that there is a minor typo in the line below [9, (2.11)], the $-\frac{1}{2}$ there should be $\frac{1}{2}$.) Thus by (4.10) we have

$$\limsup_{n\to\infty}\frac{\sup\limits_x|\eta_R(x,\tau_n)-\eta_R(x,n)|}{n^{\frac{1}{4}}\cdot(\log n)^{\frac{1}{2}}\cdot(\log\log n)^{\frac{1}{4}}} \le C_5 \qquad a.s. \tag{4.11}$$

Note that $\xi_D(x,n)=0$ if $|x|>n$ and that $\lim_{n\to\infty}\sup_{|x|>n}\eta_R(x,n)=0$ $a.s.$ Thus

$$\lim_{n\to\infty}\sup_{|x|>n}\frac{\xi_D(x,n)-\eta_R(x,n)}{n^{\frac{1}{4}+\varepsilon}} = 0 \qquad a.s. \tag{4.12}$$

Now by (4.7), (4.9), (4.11), (4.12) and Lemma 4.2, we obtain

$$\limsup_{n\to\infty}\frac{\sup\limits_x|\xi_D(x,n)-\eta_R(x,n)|}{n^{\frac{1}{4}+\varepsilon}}$$

$$
\begin{aligned}
&\le \limsup_{n\to\infty} \frac{\sup_{x\le n} |\xi_D(x,n) - \frac{\xi(x,n)}{2}| + \sup_{x\le n} |\frac{\xi(x,n)}{2} - \sum_{i=1}^{\xi(x,n)} b_i(x)|}{n^{\frac{1}{4}+\varepsilon}} \\
&\quad + \limsup_{n\to\infty} \frac{\sup_{x\le n} |\sum_{i=1}^{\xi(x,n)} b_i(x) - \eta_R(x,\tau_n)| + \sup_{x\le n} |\eta_R(x,\tau_n) - \eta_R(x,n)|}{n^{\frac{1}{4}+\varepsilon}} \\
&\quad + \lim_{n\to\infty} \sup_{|x|>n} \frac{|\xi_D(x,n) - \eta_R(x,n)|}{n^{\frac{1}{4}+\varepsilon}} \\
&\le \limsup_{n\to\infty} \frac{\sup_{x\le n} \frac{|\xi_D(x,n)-\xi_D(x-1,n)|+1}{2} + \sup_{x\le n} |\frac{\xi(x,n)}{2} - \sum_{i=1}^{\xi(x,n)} b_i(x)|}{n^{\frac{1}{4}+\varepsilon}} \\
&\quad + \limsup_{n\to\infty} \frac{\sup_{x\le n} |\sum_{i=1}^{\xi(x,n)} b_i(x) - \eta_R(x,\tau_n)| + \sup_{x\le n} |\eta_R(x,\tau_n) - \eta_R(x,n)|}{n^{\frac{1}{4}+\varepsilon}} \\
&\quad + \lim_{n\to\infty} \sup_{|x|>n} \frac{|\xi_D(x,n) - \eta_R(x,n)|}{n^{\frac{1}{4}+\varepsilon}} \\
&= 0 \ a.s.,
\end{aligned}
$$

where we used the following fact

$$
\left|\xi_D(x,n) - \frac{\xi(x,n)}{2}\right| = \frac{|\xi_D(x,n) - \xi_U(x,n)|}{2} \le \frac{|\xi_D(x,n) - \xi_D(x-1,n)| + 1}{2}.
$$

The proof is complete. □

4.3 Proof of Theorem 1.2

Set

$$
\begin{aligned}
&\eta_R^*(n) := \sup_{x\in\mathbb{R}} \eta_R(x,n), \quad \eta^*(n) := \sup_{x\in\mathbb{R}} \eta(x,n), \\
&\xi_D^*(n) := \sup_{x\in\mathbb{Z}} \xi_D(x,n), \quad T_r := \inf\{n > 0 : \eta_R(0,n) \ge r\}, \\
&I_R(h,n) := \sup_{|x|\le h} \eta_R(x,n),
\end{aligned}
$$

and

$$
\mathcal{V}(n) := \{x \in \mathbb{Z} : \eta_R(x,n) = \eta_R^*(n)\}, \quad \mathcal{U}(n) := \{x \in \mathbb{Z} : \xi_D(x,n) = \xi_D^*(n)\}.
$$

It is easy to see that Theorem 1.2 is a consequence of the following two propositions.

Proposition 4.3. *If $x \in \mathcal{K}(n)$, then $x - 1 \in \mathcal{U}(n)$.*

Proposition 4.4. *For any $\gamma > 11$, we have*

$$
\liminf_{n\to\infty} \frac{U(n)}{\sqrt{n}(\log n)^{-\gamma}} = \infty \qquad a.s., \tag{4.13}
$$

where $U(n) := \min\{|x| : x \in \mathcal{U}(n)\}$.

4.3.1 Proof of Proposition 4.3.

Assume that $x \in \mathcal{K}(n)$, that is, x is a favorite edge at time n. We want to prove that $x-1 \in \mathcal{U}(n)$. Let $\xi_D(x-1,n) = h$ for some nonnegative integer h. Then by (1.3), we know that $L(x,n) \in \{2h-1, 2h, 2h+1\}$. We will prove $x-1 \in \mathcal{U}(n)$ by contradiction. Suppose that $x-1 \notin \mathcal{U}(n)$. Then for any $y \in \mathcal{U}(n)$, we have $\xi_D(y,n) \geq h+1$.

Case 1. $L(x,n) = 2h-1$ or $2h$. By (1.3), we get that for any $y \in \mathcal{U}(n)$,

$$L(y+1,n) \geq 2\xi_D(y,n) - 1 \geq 2(h+1) - 1 = 2h+1.$$

This implies that $x \notin \mathcal{K}(n)$, which is a contradiction.

Case 2. $L(x,n) = 2h+1$. Now by (1.3), we know that $0 < x \leq S_n$. For any $y \in \mathcal{U}(n)$, we have the following two subcases:

Case 2.1. $y \leq 0$ or $y > S_n$. Now by (1.3), we get that

$$L(y+1,n) = 2\xi_D(y,n) \geq 2(h+1) = 2h+2.$$

This implies that $x \notin \mathcal{K}(n)$, which is a contradiction.

Case 2.2. $0 < y \leq S_n$. Now by (1.3), we get that

$$L(y+1,n) = 2\xi_D(y,n) + 1 \geq 2(h+1) + 1 = 2h+3.$$

This implies that $x \notin \mathcal{K}(n)$, which is a contradiction.

Thus we must have $x-1 \in \mathcal{U}(n)$. The proof is complete. □

4.3.2 Proof of Proposition 4.4.

To prove Proposition 4.4, we need several lemmas. By [3, (5.1)] and (4.3), we have

Lemma 4.5. *For any $\alpha > 5$ and $\varepsilon > 0$, there exists n_0 such that, with probability one, we have*

$$\eta_R^*(n) > I_R\left(\frac{\sqrt{n}}{(\log n)^{2\alpha+1+\varepsilon}}, n\right) + \frac{1}{2} n^{\frac{1}{2}-\varepsilon}, \quad n \geq n_0.$$

By [3, Lemma 5.3] and (4.3), we have

Lemma 4.6. *For every $\varepsilon > 0$,*

$$\sup_{k\in\mathbb{Z}} \sup_{t\leq n, x\in[k.k+1]} |\eta_R(x,t) - \eta_R(k,t)| = o(n^{\frac{1}{4}+\varepsilon}) \quad a.s.$$

Lemma 4.7.

$$|\eta_R^*(n) - \xi_D^*(n)| = o(n^{\frac{1}{4}+\varepsilon}) \quad a.s.$$

We postpone the proof of the above lemma to Section 6.3.

Proof of Proposition 4.4. By Theorem 4.1 we can define a simple symmetric random walk S_n and a Wiener process $W(t)$ on a common probability space such that for each $\varepsilon > 0$,

$$\sup_{x\in\mathbb{Z}} |\xi_D(x,n) - \eta_R(x,n)| = o(n^{\frac{1}{4}+\varepsilon}) \qquad a.s. \tag{4.14}$$

For any $\alpha > 5$ and $\varepsilon > 0$, let $K_n = \max_{x\in\mathbb{Z},|x|\le\sqrt{n}(\log n)^{-(2\alpha+1+\varepsilon)}} \xi_D(x,n)$. By (4.14), Lemmas 4.5 and 4.7,

$$\begin{aligned}
\xi_D^*(n) &\ge \eta_R^*(n) - cn^{\frac{1}{4}+\varepsilon} \\
&\ge I_R\left(\frac{\sqrt{n}}{(\log n)^{2\alpha+1+\varepsilon}}, n\right) + \frac{1}{2}n^{\frac{1}{2}-\varepsilon} - cn^{\frac{1}{4}+\varepsilon} \\
&\ge K_n + \frac{1}{2}n^{\frac{1}{2}-\varepsilon} - 2cn^{\frac{1}{4}+\varepsilon} > K_n
\end{aligned}$$

for n sufficiently large. Thus the most visited edges of S_n must be larger in absolute value than $\sqrt{n}(\log n)^{-(2\alpha+1+\varepsilon)}$ for n large. For any $\gamma > 11$, choosing α and ε so that $2\alpha + 1 + \varepsilon < \gamma$, we obtain (4.13). The proof is complete. □

5 Remark

In Section 2, we used the transience of the favorite downcrossing site process to show the transience of the favorite edge process. In fact, from Proposition 4.3, we can see that there is a close relation between favorite edges and favorite downcrossing sites. A natural question arise:

How about the number of favorite downcrossing sites of one-dimensional simple symmetric random walk?

In [19], we will consider this question and prove the following result.

Theorem 5.1. *For a one-dimensional simple symmetric random walk, with probability 1 there are only finitely many times at which there are at least four favorite downcrossing sites and three favorite downcrossing sites occurs infinitely often.*

6 Proofs of several lemmas

6.1 Proof of Lemma 3.1

Let $M_n = \sum_{s=1}^n (Z_s - s) - n(Z_n - n)$, and let $\mathcal{F}_n = \sigma(Z_0, Z_1, ..., Z_n)$. By the proof of [12, Lemma 3.1], we know that (M_n) is a martingale. By the optional stopping theorem, we get that

$$E\left(\sum_{n=1}^{\tau_{\frac{h-1}{2}}} (Z_n - n)\right) = E\tau_{\frac{h-1}{2}}(Z_{\tau_{\frac{h-1}{2}}} - \tau_{\frac{h-1}{2}}),$$

and thus

$$
\begin{aligned}
E\left(\sum_{n=1}^{\tau_{\frac{h-1}{2}}} \frac{h/2 - Z_n}{h/2}\right) &= E\tau_{\frac{h-1}{2}} - \frac{E\left(\sum_{n=1}^{\tau_{\frac{h-1}{2}}} (Z_n - n) + n\right)}{h/2} \\
&= E\tau_{\frac{h-1}{2}} - \frac{E\tau_{\frac{h-1}{2}}(Z_{\tau_{\frac{h-1}{2}}} - \tau_{\frac{h-1}{2}})}{h/2} - \frac{E\frac{(1+\tau_{\frac{h-1}{2}})\tau_{\frac{h-1}{2}}}{2}}{h/2} \\
&= \left(1 - \frac{1}{h}\right) E\tau_{\frac{h-1}{2}} - \frac{2}{h} E\left[\tau_{\frac{h-1}{2}} Z_{\tau_{\frac{h-1}{2}}} - \frac{1}{2}\tau^2_{\frac{h-1}{2}}\right]. \qquad (6.1)
\end{aligned}
$$

Define the process $M'_n = -\frac{1}{4}Z_n^2 + nZ_n - \frac{1}{2}n^2 + \frac{1}{4}n$. By the proof of [12, Lemma 3.1], we know that (M'_n) is a martingale. Applying the optional stopping theorem to (M'_n) at $\tau_{\frac{h-1}{2}}$, we get

$$
E\left[\tau_{\frac{h-1}{2}} Z_{\tau_{\frac{h-1}{2}}} - \frac{1}{2}\tau^2_{\frac{h-1}{2}}\right] = E\left[\frac{1}{4} Z^2_{\tau_{\frac{h-1}{2}}} - \frac{1}{4}\tau_{\frac{h-1}{2}}\right] - \frac{Z_0^2}{4} = \frac{1}{4} E\left[Z^2_{\tau_{\frac{h-1}{2}}} - \tau_{\frac{h-1}{2}}\right] - \frac{Z_0^2}{4}. \qquad (6.2)
$$

Combining (6.1), (6.2) and Lemma 2.2, we get

$$
\begin{aligned}
&E\left(\sum_{n=1}^{\tau_{\frac{h-1}{2}}} \frac{h/2 - Z_n}{h/2}\right) = \left(1 - \frac{1}{2h}\right) E\tau_{\frac{h-1}{2}} - \frac{1}{2h}\left(EZ^2_{\tau_{\frac{h-1}{2}}} - Z_0^2\right) \\
&= \left(1 - \frac{1}{2h}\right) E\left[Z_{\tau_{\frac{h-1}{2}}} - Z_0\right] - \frac{1}{2h} E\left[(Z_{\tau_{\frac{h-1}{2}}} - Z_0)(Z_{\tau_{\frac{h-1}{2}}} + Z_0)\right] \\
&= \frac{1}{2h} E\left[\left(2h - 1 - (Z_{\tau_{\frac{h-1}{2}}} + Z_0)\right)(Z_{\tau_{\frac{h-1}{2}}} - Z_0)\right]. \qquad (6.3)
\end{aligned}
$$

Obviously $Z_{\tau_{\frac{h-1}{2}}} - Z_0 \geq \frac{h-1}{2} - \frac{h-\sqrt{h}}{2} \geq c_1\sqrt{h}$. Then by Lemmas 2.2 and 2.3, we obtain

$$
E\left(\sum_{n=1}^{\tau_{\frac{h-1}{2}}} \frac{h/2 - Z_n}{h/2}\right) \geq \frac{1}{2h} \cdot \left(2h - 1 - \frac{h-1}{2} - c_2\sqrt{h} - \frac{h - \sqrt{h}}{2}\right) \cdot c_1\sqrt{h} \geq c\sqrt{h}. \qquad (6.4)
$$

6.2 Proof of Lemma 4.2

The idea of the proof comes from Csáki and Révész [6, Lemma 5], where the corresponding problem for (two-side) local times was considered in a more general setting. First we show that, for all $m \geq 2$ and all $\delta > 0$, there exists a constant C (which may depends on m but not on n) such that

$$
E(|\xi_D(1, n) - \xi_D(0, n)|^m) \leq C n^{\frac{m}{4} + \delta}. \qquad (6.5)
$$

Define

$$
T_k := \sum_{i=1}^{k} \left(\xi_D(1, \alpha_i) - \xi_D(1, \alpha_{i-1}) - 1\right), \qquad (6.6)
$$

where $\alpha_0 = 0$, $\alpha_i = \min\{j > \alpha_{i-1} : S_j = 0, S_{j-1} = 1\}, \forall i \geq 1$. We claim that

$$E(\xi_D(1, \alpha_1)) = 1, \quad E(\xi_D^m(1, \alpha_1)) < +\infty, \forall m \geq 2. \tag{6.7}$$

To prove this claim, we follow Kesten and Spitzer [23, Lemma 2] and define, for $x, y \in \mathbb{Z}$, $\tau_0(x) := 0, \tau_i(x) := \min\{n > \tau_{i-1}(x) : S_n = x, S_{n-1} = x + 1\}, \forall i \geq 1$. Note that $\alpha_i = \tau_i(0)$. For $j \geq 0$,

$$M_j(x, y) := \sum_{\tau_j(x) < n \leq \tau_{j+1}(x)} \mathbf{1}_{\{S_n = y, S_{n-1} = y+1\}}$$

is the number of downcrossings to y between the j^{th} and $(j+1)^{th}$ downcrossings to x. In particular, $M_0(0, 1) = \xi_D(1, \alpha_1)$. By the strong Markov property, we know that the distribution of $M_j(0, y)$, $j \geq 0$, is independent of j, and the distribution of $M_j(x, y)$, $j \geq 1$, is independent of j. Thus we can define for $j \geq 1$,

$$\begin{aligned} p(x, y) &:= P(M_j(x, y) \neq 0) \\ &= P(S_n = y, S_{n-1} = y + 1 \text{ for some } n \text{ with } \tau_j(x) < n \leq \tau_{j+1}(x)). \end{aligned}$$

We claim that, for any $x, y \in \mathbb{Z}$ with $|x - y| = 1$, it holds that $p(x, y) = p(y, x) = 1/2$. To prove this claim, it suffices to show that $p(0, 1) = p(1, 0) = 1/2$. When $\tau_1(0) < \infty$, there exists $n_0 = \min\{k \geq 0 : S_k = 1\} < \tau_1(0)$. Then $S_{n_0} = 1$, $\{S_{n_0+1} = 0\} \subset \{M_0(0, 1) = 0\}$ and $\{S_{n_0+1} = 2\} \subset \{M_0(0, 1) \neq 0\}$. Since $P(\tau_1(0) < \infty) = 1$, we have

$$\begin{aligned} &p(0, 1) = P(M_0(0, 1) \neq 0, \tau_1(0) < \infty) \\ &= P(S_{n_0+1} = 0, M_0(0, 1) \neq 0, \tau_1(0) < \infty) + P(S_{n_0+1} = 2, M_0(0, 1) \neq 0, \tau_1(0) < \infty) \\ &= P(S_{n_0+1} = 2) = \frac{1}{2}. \end{aligned}$$

Similarly, there exists $n_1 = \min\{k \geq 0 : S_k = 1, S_{k-1} = 2\} < \tau_1(1)$, then

$$\begin{aligned} &p(1, 0) = P(S_{n_1+1} = 2, M_1(1, 0) \neq 0) + P(S_{n_1+1} = 0, M_1(1, 0) \neq 0) \\ &= P(S_{n_1+1} = 0) = \frac{1}{2}. \end{aligned}$$

Combining the claim above with the strong Markov property and the fact $S_{\tau_j(x)} = S_{\tau_{j+1}(x)} = x$, we get

$$P(M_j(0, 1) = k) = \begin{cases} 1 - p(0, 1) = \frac{1}{2}, & \text{if } k = 0, \\ p(0, 1)[1 - p(1, 0)]^{k-1} p(1, 0) = \frac{1}{2^{k+1}}, & \text{if } k \geq 1. \end{cases}$$

It follows that

$$\begin{aligned} E(\xi_D(1, \alpha_1)) &= E(M_j(0, 1)) = \sum_{k=1}^{\infty} k \cdot \frac{1}{2^{k+1}} = 1, \\ E(\xi_D^m(1, \alpha_1)) &= E(M_j^m(0, 1)) = \sum_{k=1}^{\infty} k^m \cdot \frac{1}{2^{k+1}} < +\infty, \end{aligned}$$

which implies that (6.7) holds.

It follows from the strong Markov property that T_k is a sum of i.i.d. r.v.'s with mean 0 and finite moments of all orders. Therefore by Chung [5] and the L^p-maximal inequality for martingales, we have

$$E(|T_k|^m) \le C_1 k^{\frac{m}{2}} \tag{6.8}$$

and for any $\delta_1 > 0$,

$$E\left(\max_{k \le n^{\frac{1}{2}+\delta_1}} |T_k|^m\right) \le C_2 n^{\frac{m}{4}+\frac{m\delta_1}{2}}. \tag{6.9}$$

Note that

$$\xi_D(1, \alpha_{D_n}) - \xi_D(0,n) \le \xi_D(1,n) - \xi_D(0,n) \le \xi_D(1, \alpha_{D_n+1}) - (\xi_D(0,n)+1) + 1,$$

where $D_n = \xi_D(0,n)$. Thus on the event $\{D_n + 1 \le n^{\frac{1}{2}+\delta_1}\}$, we have

$$|\xi_D(1,n) - \xi_D(0,n)| \le \max_{1\le k \le n^{\frac{1}{2}+\delta_1}} |T_k| + 1.$$

Hence

$$E(|\xi_D(1,n) - \xi_D(0,n)|^m) \le CE\left(\max_{1\le k\le n^{\frac{1}{2}+\delta_1}} (|T_k|+1)^m\right) + n^m P\left(D_n + 1 \ge n^{\frac{1}{2}+\delta_1}\right). \tag{6.10}$$

Let $N_n = N(0,n) = \#\{k : 0 < k \le n, S_k = 0\}$, then according to Kesten and Spitzer [23], $E(N_n^m) = O(n^{\frac{m}{2}})$. Combining this with the fact that $0 \le D_n \le N_n$, we get that $E(D_n^m) = O(n^{\frac{m}{2}})$. Then by Markov's inequality, (6.9) and (6.10), letting δ_1 be small enough, we obtain (6.5).

By repeating the argument above and noticing that $p(-1,0) = p(0,-1) = \frac{1}{2}$, we can obtain the inequality (6.5) with $\xi_D(1,n)$ replaced by $\xi_D(-1,n)$.

It is easy to see that for any $x \in \mathbb{Z}$,

$$E(|\xi_D(x+1,n) - \xi_D(x,n)|^m) \le Cn^{\frac{m}{4}+\delta}, \tag{6.11}$$

where the constant C does not depend on x, because $\xi_D(x+1,n) - \xi_D(x,n)$ is stochastically smaller than $\max(|\xi_D(1,n) - \xi_D(0,n)|, |\xi_D(-1,n) - \xi_D(0,n)|)$.

Choosing $m = \frac{2+\delta}{\varepsilon}$, we obtain by Markov's inequality that

$$P\Big(|\xi_D(x+1,n) - \xi_D(x,n)| > n^{\frac{1}{4}+\varepsilon}\Big) \le \frac{C}{n^2}$$

and hence

$$P\left(\sup_{|x|<(n\log n)^{\frac{1}{2}}} |\xi_D(x+1,n) - \xi_D(x,n)| > n^{\frac{1}{4}+\varepsilon}\right) \le \frac{2C(\log n)^{\frac{1}{2}}}{n^{\frac{3}{2}}}.$$

Therefore, by the Borel-Cantelli lemma, we get that

$$\lim_{n\to+\infty} \frac{\sup_{|x|\le (n\log n)^{\frac{1}{2}}} |\xi_D(x+1,n)-\xi_D(x,n)|}{n^{\frac{1}{4}+\varepsilon}} = 0 \quad a.s.$$

The law of the iterated logarithm for S_n implies that $\xi_D(x,n)=0$ $a.s.$ for $|x|\ge (n\log n)^{\frac{1}{2}}$ and n sufficiently large, hence (4.4) holds. □

6.3 Proof of Lemma 4.7

We have

$$\begin{aligned}
\eta_R^*(n)-\xi_D^*(n) &= \sup_{x\in\mathbb{R}} \eta_R(x,n) - \sup_{x\in\mathbb{Z}} \xi_D(x,n) \\
&= \sup_{x\in\mathbb{R},y\in\mathbb{Z},|x-y|\le 1} [\eta_R(x,n)-\xi_D(y,n)+\xi_D(y,n)] - \sup_{x\in\mathbb{Z}} \xi_D(x,n) \\
&\le \sup_{x\in\mathbb{R},y\in\mathbb{Z},|x-y|\le 1} (\eta_R(x,n)-\xi_D(y,n)) + \sup_{y\in\mathbb{Z}} \xi_D(y,n) - \sup_{x\in\mathbb{Z}} \xi_D(x,n) \\
&= \sup_{x\in\mathbb{R},y\in\mathbb{Z},|x-y|\le 1} (\eta_R(x,n)-\xi_D(y,n)) \\
&\le \sup_{x\in\mathbb{R},y\in\mathbb{Z},|x-y|\le 1} |\eta_R(x,n)-\xi_D(y,n)|.
\end{aligned}$$

Similarly, we have

$$\xi_D^*(n)-\eta_R^*(n) \le \sup_{x\in\mathbb{R},y\in\mathbb{Z},|x-y|\le 1} |\eta_R(x,n)-\xi_D(y,n)|.$$

Hence

$$|\eta_R^*(n)-\xi_D^*(n)| \le \sup_{x\in\mathbb{R},y\in\mathbb{Z},|x-y|\le 1} |\eta_R(x,n)-\xi_D(y,n)|.$$

Then by Lemma 4.6 and Theorem 4.1, we get

$$\begin{aligned}
&|\eta_R^*(n)-\xi_D^*(n)| \\
&\le \sup_{x\in\mathbb{R},y\in\mathbb{Z},|x-y|\le 1} |\eta_R(x,n)-\eta_R(y,n)+\eta_R(y,n)-\xi_D(y,n)| \\
&\le \sup_{x\in\mathbb{R},y\in\mathbb{Z},|x-y|\le 1} |\eta_R(x,n)-\eta_R(y,n)| + \sup_{y\in\mathbb{Z}} |\eta_R(y,n)-\xi_D(y,n)| \\
&= o(n^{\frac{1}{4}+\varepsilon}).
\end{aligned}$$

The proof is complete.

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