Lower deviation for the supremum of the support of super-Brownian motion

Yan-Xia Ren* **Renming Song**[†] **Rui Zhang***[‡]

Abstract

We study the asymptotic behavior of the supremum M_t of the support of a supercritical super-Brownian motion. In our recent paper (Stoch. Proc. Appl. **137** (2021), 1–34), we showed that, under some conditions, $M_t - m(t)$ converges in distribution to a randomly shifted Gumbel random variable, where $m(t) = c_0 t - c_1 \log t$. In the same paper, we also studied the upper large deviation of M_t, i.e., the asymptotic behavior of $\mathbb{P}(M_t > \delta c_0 t)$ for $\delta \geq 1$. In this paper, we study the lower large deviation of M_t, i.e., the asymptotic behavior of $\mathbb{P}(M_t \leq \delta c_0 t | \mathcal{S})$ for $\delta < 1$, where $\mathcal{S}$ is the survival event.

AMS Subject Classifications (2020): 60F10, 60J68

Keywords and Phrases: super-Brownian motion, supremum of support, lower large deviation.

1 Introduction

1.1 Super-Brownian motion

Let ψ be a function of the form

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_0^\infty \left(e^{-\lambda y} - 1 + \lambda y\right) n(\mathrm{d}y), \quad \lambda \geq 0,$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and n is a σ-finite measure satisfying

$$\int_0^\infty (y^2 \wedge y) n(\mathrm{d}y) < \infty.$$

ψ is called a branching mechanism. We will always assume that $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$. Let $\{B_t, t \geq 0; \mathrm{P}_x\}$ be a standard Brownian motion starting from $x \in \mathbb{R}$, and let E_x be the

*Y.-X. Ren: LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, P.R. China. Email: yxren@math.pku.edu.cn

[†]R. Song: Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A. Email: rsong@illinois.edu

[‡]R. Zhang*: corresponding author. School of Mathematical Sciences & Academy for Multidisciplinary Studies, Capital Normal University, Beijing, 100048, P.R. China. Email: zhangrui27@cnu.edu.cn

corresponding expectation. We write $\mathrm{P} = \mathrm{P}_0$ and $\mathrm{E} = \mathrm{E}_0$. In this paper we will consider a super-Brownian motion X on $\mathbb{R}$ with branching mechanism ψ.

Let $\mathcal{B}^+(\mathbb{R})$ (resp. $\mathcal{B}_b^+(\mathbb{R})$) be the space of non-negative (resp. bounded non-negative) Borel functions on $\mathbb{R}$, and let $\mathcal{M}_F(\mathbb{R})$ be the space of finite measures on $\mathbb{R}$, equipped with the topology of weak convergence. A super-Brownian motion $X = \{X_t, t \geq 0\}$ with branching mechanism ψ is a Markov process taking values in $\mathcal{M}_F(\mathbb{R})$. For any $\mu \in \mathcal{M}_F(\mathbb{R})$, we denote the law of X with initial configuration μ by $\mathbb{P}_\mu$, and the corresponding expectation by $\mathbb{E}_\mu$. We write $\mathbb{P} = \mathbb{P}_{\delta_0}$ and $\mathbb{E} = \mathbb{E}_{\delta_0}$. As usual, we use the notation $\langle f, \mu \rangle := \int_{\mathbb{R}} f(x)\mu(dx)$ and $\|\mu\| := \langle 1, \mu \rangle$. Then for all $f \in \mathcal{B}_b^+(\mathbb{R})$ and $\mu \in \mathcal{M}_F(\mathbb{R})$,

$$-\log \mathbb{E}_\mu \left(e^{-\langle f, X_t \rangle}\right) = \langle V_f(t, \cdot), \mu \rangle, \qquad t \geq 0, \tag{1.1}$$

where $V_f(t, x)$ is the unique positive solution to the equation

$$V_f(t, x) + \mathrm{E}_x \int_0^t \psi(V_f(t-s, B_s))\mathrm{d}s = \mathrm{E}_x f(B_t), \qquad t \geq 0. \tag{1.2}$$

The existence of such superprocesses is well-known, see, for instance, [8], [12] or [18].

It is well known that $\|X_t\|$ is a continuous state branching process with branching mechanism ψ and that

$$\mathbb{P}(\lim_{t\to\infty} \|X_t\| = 0) = e^{-\lambda^*},$$

where $\lambda^* \in [0, \infty)$ is the largest root of the equation $\psi(\lambda) = 0$. It is known that $\lambda^* > 0$ if and only if $\alpha = -\psi'(0+) > 0$. X is called a supercritical (critical, subcritical) super-Brownian motion if $\alpha > 0$ $(= 0, < 0)$. In this paper, we only deal with the supercritical case, that is, we assume $\alpha > 0$. Let M_t be the supremum of the support of X_t. More precisely, we define the rightmost point $M(\mu)$ of $\mu \in \mathcal{M}_F(\mathbb{R})$ by $M(\mu) := \sup\{x : \mu(x, \infty) > 0\}$. Here we use the convention that $\sup \emptyset = -\infty$. Then M_t is simply $M(X_t)$. Recently, in [19], we studied the asymptotic behavior of M_t under the following two assumptions:

(H1) There exists $\gamma > 0$ such that

$$\int_1^\infty y(\log y)^{2+\gamma} n(\mathrm{d}y) < \infty.$$

(H2) There exist $\vartheta \in (0, 1]$ and $a > 0, b > 0$ such that

$$\psi(\lambda) \geq -a\lambda + b\lambda^{1+\vartheta}, \quad \lambda > 0.$$

It is clear that if $\beta > 0$ or $n(\mathrm{d}y) \geq y^{-1-\vartheta}\,\mathrm{d}y$, then (H2) holds. Condition (H2) implies that the following Grey condition holds:

$$\int^\infty \frac{1}{\psi(\lambda)}\, d\lambda < \infty. \tag{1.3}$$

It is well known that under the above Grey condition, $\lim_{t\to\infty} \mathbb{P}_\mu(\|X_t\| = 0) = e^{-\lambda^*\|\mu\|}$. Denote $\mathcal{S} := \{\forall t \geq 0, \|X_t\| > 0\}$. It is clear that $\mathbb{P}(\mathcal{S}) \in (0, 1)$. Define, for $t \geq 0$,

$$D_t := \langle (\sqrt{2\alpha}t - \cdot)e^{-\sqrt{2\alpha}(\sqrt{2\alpha}t - \cdot)}, X_t \rangle.$$

It has been proven in [17] that $\{D_t, t \geq 0\}$ is a martingale, which is called the derivative martingale of the super-Brownian motion X_t, and that D_t has an almost sure non-negative limit D_∞ as $t \to \infty$. Assumption (H2) also implies that

$$\int^\infty \frac{1}{\sqrt{\int_{\lambda^*}^\xi \psi(u)\, du}} \mathrm{d}\xi < \infty. \tag{1.4}$$

Under (H1) and (1.4), D_∞ is non-degenerate and

$$\frac{M_t}{t} \to \sqrt{2\alpha}, \quad \mathbb{P}\text{-a.s. on } \mathcal{S}, \tag{1.5}$$

see [17, Theorem 2.4 and Corollary 3.2].

For any $f \in \mathcal{B}^+(\mathbb{R})$, put

$$u_f(t,x) := -\log \mathbb{E}\left(e^{-\int_\mathbb{R} f(y-x) X_t(dy)}; M_t \leq x\right), \tag{1.6}$$

Note that u_f only depends on the value of f on $(-\infty, 0]$. Let $\mathcal{H}$ be the space of all the nonnegative bounded Borel functions f on $(-\infty, 0]$ satisfying

$$\int_0^\infty y e^{\sqrt{2\alpha} y} f(-y)\, dy < \infty. \tag{1.7}$$

It has been proved in [19, Theorem 1.3] that under (H1)-(H2), for any $f \in \mathcal{H}$, we have that

$$\lim_{t\to\infty} u_f(t, m_t + x) = w_f(x), \tag{1.8}$$

where

$$m_t = \sqrt{2\alpha} t - \frac{3}{2\sqrt{2\alpha}} \log t, \tag{1.9}$$

and w_f is a traveling wave solution of the F-KPP equation, that is, a solution of

$$\frac{1}{2} w_{xx} + \sqrt{2\alpha} w_x - \psi(w) = 0.$$

Moreover, w_f is given by $w_f(x) = -\log \mathbb{E}\left[\exp\{-\tilde{C}(f) D_\infty e^{-\sqrt{2\alpha} x}\}\right]$, with

$$\tilde{C}(f) := \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty u_f(r, \sqrt{2\alpha} r + y) y e^{\sqrt{2\alpha} y}\, dy \in (0, \infty).$$

In the remainder of this paper, we write $u(t,x)$ and $w(x)$ for $u_f(t,x)$ and $w_f(x)$ respectively when $f \equiv 0$.

1.2 Main results

In [19, Theorem 1.2], we proved the following upper large deviation results for M_t under conditions (H1)-(H2):

(1) For $\delta > 1$,

$$\lim_{t\to\infty} \sqrt{t} e^{\alpha(\delta^2-1)t} \mathbb{P}(M_t > \sqrt{2\alpha}\delta t) \in (0,\infty);$$

(2)

$$\lim_{t\to\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2\alpha}}\log t} \mathbb{P}(M_t > \sqrt{2\alpha} t) \in (0,\infty).$$

However, using the methods in [19], we could not get the asymptotic behavior of the lower large deviation probability $\mathbb{P}(M_t \le \sqrt{2\alpha}\delta t|\mathcal{S})$ for $\delta < 1$. The purpose of this paper is to study the asymptotic behavior of the lower large deviation probability. To accomplish this, we use the skeleton decomposition of super-Brownian motion and adapt some ideas from [7] used in the study of lower larger deviations of the maximum of branching Brownian motion.

For branching Brownian motion, the asymptotic behavior of the maximal position, also denoted by M_t, of the particles alive at time t has been intensively studied. To simplify notation, we consider a standard binary branching Brownian motion in $\mathbb{R}$, i.e., the lifetime of a particle is an exponential random variable of parameter 1 and when it dies, it gives birth to 2 children at the position of its death. Bramson proved in [4] that $P(M_t - m(t) \le x) \to 1 - w(x)$ as $t \to \infty$, where $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$ and $w(x)$ is a traveling wave solution. For the large deviation of M_t, [5, 6] studied the convergence rate of $P(M_t > \sqrt{2}\delta t)$ for $\delta \ge 1$. Recently, Derrida and Shi [9, 10] studied the lower large deviation of M_t, i.e, the asymptotic behavior of $\frac{1}{t}\log P(M_t \le \sqrt{2}\delta t)$ for $\delta < 1$, and found that the rate function has a phase transition at $1-\sqrt{2}$. In [7], Chen, He and Mallein studied the limiting property of $P(M_t \le \sqrt{2}\delta t)$ for $\delta < 1$. For more results on extremal processes of branching Brownian motions, we refer our readers to [1, 2].

To maximize the possibility of $M_t \le \sqrt{2}\delta t$ for $\delta < 1$, a good strategy is to make the first branching time τ as large as possible. It was shown in [7] that, conditioned on $\{M_t \le \sqrt{2}\delta t\}$, $\tau \approx \frac{1-\delta}{\sqrt{2}} t \pm O(1)\sqrt{t}$ when $\delta \in (1-\sqrt{2}, 1)$; $\tau \approx t - O(1)\sqrt{t}$ when $\delta = 1-\sqrt{2}$ and $\tau \approx t - O(1)$ when $\delta < 1-\sqrt{2}$. The asymptotic behaviors of $P(M_t \le \sqrt{2}\delta t)$ are different in these 3 different cases.

The intuition above also works for super-Brownian motion, but we need to use the first branching time of the skeleton process, which is a branching Brownian motion. Put

$$q := \psi'(\lambda^*) > 0, \qquad \rho := \sqrt{1 + \frac{\psi'(\lambda^*)}{\alpha}} = \sqrt{1+\frac{q}{\alpha}}.$$

We also use τ to denote the first branching time of the skeleton process of super-Brownian motion. We will prove that, conditioned on $\{M_t \le \sqrt{2\alpha}\delta t, \mathcal{S}\}$, as $t \to \infty$, $\tau \in [\frac{1-\delta}{\rho}t - (\log t)\sqrt{t}, \frac{1-\delta}{\rho}t + (\log t)\sqrt{t}]$ when $\delta \in (1-\rho, 1)$; $\tau \in t - \sqrt{t}\left[t^{-1/4}, \log t\right]$ when $\delta = 1-\rho$ and $\tau \in [t - O(1), t]$ when $\delta < 1-\rho$. The asymptotic behavior of $\mathbb{P}(M_t \le \sqrt{2\alpha}\delta t|\mathcal{S})$ exhibits a phase transition at $\delta = 1-\rho$.

Now we state our main results.

Theorem 1.1 *Assume that (H1) and (H2) hold. If $\delta \in (1-\rho, 1)$, then for any $f \in \mathcal{H}$,*

$$\lim_{t\to\infty} e^{2\alpha(\rho-1)(1-\delta)t} t^{-3(\rho-1)/2} \mathbb{E}\left(e^{-\int_{\mathbb{R}} f(y-\sqrt{2\alpha}\delta t)X_t(dy)}; M_t \le \sqrt{2\alpha}\delta t|\mathcal{S}\right)$$

$$=\frac{\lambda^*}{e^{\lambda^*}-1}\frac{a_\delta^{3(\rho-1)/2}}{\sqrt{2\alpha}\rho}\int_{-\infty}^{\infty}e^{-\sqrt{2\alpha}(\rho-1)z}A(w_f(z))\mathrm{d}z,$$

where $a_\delta=1-\frac{1-\delta}{\rho}$ *and*

$$A(\lambda)=\frac{1}{\lambda^*}\psi(\lambda)+\psi'(\lambda^*)\left(1-\frac{\lambda}{\lambda^*}\right)\geq 0,\quad \lambda\geq 0.$$

Theorem 1.2 *Assume that (H1) and (H2) hold. Then for any* $f\in\mathcal{H}$,

$$\lim_{t\to\infty}t^{-3(\rho-1)/4}e^{(q+\alpha(\rho-1)^2)t}\mathbb{E}\left(e^{-\int_{\mathbb{R}}f(y-\sqrt{2\alpha}(1-\rho)t)X_t(dy)};M_t\leq\sqrt{2\alpha}(1-\rho)t|\mathcal{S}\right)$$
$$=\frac{\lambda^*}{e^{\lambda^*}-1}\frac{1}{\sqrt{2\pi}}\int_0^\infty s^{3(\rho-1)/2}e^{-\alpha\rho^2s^2}\mathrm{d}s\int_{-\infty}^{\infty}e^{-\sqrt{2\alpha}(\rho-1)z}A(w_f(z))\mathrm{d}z.$$

Theorem 1.3 *Assume that (H1) and (H2) hold. If* $\delta<1-\rho$*, then for any* $f\in\mathcal{B}_b^+(\mathbb{R})$,

$$\lim_{t\to\infty}\sqrt{t}e^{(q+\alpha\delta^2)t}\mathbb{E}\left(e^{-\int_{\mathbb{R}}f(y-\sqrt{2\alpha}\delta t)X_t(dy)};M_t\leq\sqrt{2\alpha}\delta t|\mathcal{S}\right)$$
$$=\frac{\lambda^*}{e^{\lambda^*}-1}\left[\frac{1}{2\sqrt{\pi\alpha}|\delta|}+\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}}e^{\sqrt{2\alpha}\delta z}G_f(s,z)\,\mathrm{d}z\right],$$

where

$$G_f(t,x):=\frac{1}{\lambda^*}\Big[\psi(u_f(t,x))-\psi(\lambda^*+u_f^*(t,x))\Big]+qv_f(t,x),\tag{1.10}$$

with v_f,u_f^* *being defined in* (2.6) *and* (2.7) *below.*

The reason that we assume $f\in\mathcal{H}$ in Theorems 1.1 and 1.2 is that (1.8) plays an important role in the proofs of Lemmas 3.2 and 3.7. Lemma 3.2 is used in the proof of Theorem 1.1 and Lemma 3.7 is used in the proof of Theorem 1.2.

Let $\mathcal{C}_c(\mathbb{R})(\mathcal{C}_c^+(\mathbb{R}))$ be the space of all the (nonnegative) continuous functions with compact support. Let $\mathcal{M}_R(\mathbb{R})$ be the space of all the Radon measures on $\mathbb{R}$ equipped with the vague topology, see [15, p.111]. Recall that for random measures $\mu_t,\mu\in\mathcal{M}_R(\mathbb{R})$, μ_t converges in distribution to μ is equivalent to $\langle f,\mu_t\rangle$ converges in distribution to $\langle f,\mu\rangle$ for any $f\in\mathcal{C}_c^+(\mathbb{R})$. See [15, p.119] for more details.

As a consequence of Theorems 1.1-1.3, we have the following corollary.

Corollary 1.4 *Assume that (H1) and (H2) hold. Conditioned on* $\{M_t\leq\sqrt{2\alpha}\delta t,\mathcal{S}\}$, $X_t-\sqrt{2\alpha}\delta t$ *converges in distribution to a random measure* Ξ_δ*. Moreover, for any* $f\in C_c^+(\mathbb{R})$*, if* $\delta\in[1-\rho,1)$,

$$\mathbb{E}\left(e^{-\int_{\mathbb{R}}f(y)\Xi_\delta(dy)}\right)=\frac{\int_{-\infty}^{\infty}e^{-\sqrt{2\alpha}(\rho-1)z}A(w_f(z))\mathrm{d}z}{\int_{-\infty}^{\infty}e^{-\sqrt{2\alpha}(\rho-1)z}A(w(z))\mathrm{d}z};\tag{1.11}$$

and if $\delta<1-\rho$,

$$\mathbb{E}\left(e^{-\int_{\mathbb{R}}f(y)\Xi_\delta(dy)}\right)=\frac{\frac{1}{\sqrt{2\alpha}|\delta|}+\int_0^\infty e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}}e^{\sqrt{2\alpha}\delta z}G_f(s,z)\,\mathrm{d}z}{\frac{1}{\sqrt{2\alpha}|\delta|}+\int_0^\infty e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}}e^{\sqrt{2\alpha}\delta z}G(s,z)\,\mathrm{d}z},$$

where G_f *is defined in* (1.10) *and* $G(t,x):=G_0(t,x)$.

Proof: We consider the case of $\delta \in [1-\rho, 1)$ first. For any $f \in \mathcal{H}$ and $\theta > 0$, by Theorems 1.1-1.2,

$$\lim_{t\to\infty} \mathbb{E}\left(e^{-\theta \int_{\mathbb{R}} f(y-\sqrt{2\alpha}\delta t)X_t(dy)} | M_t \le \sqrt{2\alpha}\delta t, \mathcal{S}\right) = \frac{\int_{-\infty}^{\infty} e^{-\sqrt{2\alpha}(\rho-1)z} A(w_{\theta f}(z))\mathrm{d}z}{\int_{-\infty}^{\infty} e^{-\sqrt{2\alpha}(\rho-1)z} A(w(z))\mathrm{d}z}.$$

It has been proved in [19, Lemma 3.3] that $\lim_{\theta\to 0} \tilde{C}(\theta f) = \tilde{C}(0)$, which implies that $w_{\theta f}(x) \to w(x)$. Note that $A(\lambda)$ is decreasing on $(0, \lambda^*)$ and $0 \le w_{\theta f}(z) \le \lambda^*$. Thus using the monotone convergence theorem we get that

$$\lim_{\theta\to 0} \frac{\int_{-\infty}^{\infty} e^{-\sqrt{2\alpha}(\rho-1)z} A(w_{\theta f}(z))\mathrm{d}z}{\int_{-\infty}^{\infty} e^{-\sqrt{2\alpha}(\rho-1)z} A(w(z))\mathrm{d}z} = 1.$$

Thus, conditioned on $\{M_t \le \sqrt{2\alpha}\delta t, \mathcal{S}\}$, $\int_{\mathbb{R}} f(y-\sqrt{2\alpha}\delta t)X_t(dy)$ converges in distribution for any $f \in \mathcal{C}_c^+(\mathbb{R})$, which implies that $X_t - \sqrt{2\alpha}\delta t$ converges in distribution to a random measure Ξ_δ with Laplace transform given by (1.11).

Similarly, using Theorem 1.3, we can get the result for $\delta < 1-\rho$. □

Throughout this paper we use C to denote a positive constant whose value may change from one appearance to another. For any two positive functions f and g on $[0,\infty)$, $f \sim g$ as $s \to \infty$ means that $\lim_{s\to\infty} \frac{f(s)}{g(s)} = 1$.

2 Preliminaries

2.1 Skeleton decomposition

Denote by $\mathbb{P}_\mu^*$ the law of X with initial configuration μ conditioned on extinction. It has been proved in [3, Lemma 2] that $(X, \mathbb{P}^*)$ is a super-Brownian motion with branching mechanism $\psi^*(\lambda) = \psi(\lambda + \lambda^*)$. Note that $(\psi^*)'(0+) = \psi'(\lambda^*) = q > 0$. So $(X, \mathbb{P}^*)$ is subcritical.

Let $\mathbb{D}([0,\infty), \mathcal{M}_F(\mathbb{R}))$ be the space of all the right continuous functions $w : [0,\infty) \to \mathcal{M}_F(\mathbb{R})$, and $\mathbb{D}_0^+$ be the space of right continuous functions from $(0,\infty)$ to $\mathcal{M}_F(\mathbb{R})$ having zero as a trap. It has been proved in [13] that there is a family of measures $\{\mathbb{N}_x, x \in \mathbb{R}\}$ on $\mathbb{D}_0^+$ associated with the probability measures $\{\mathbb{P}_{\delta_x}^* : x \in \mathbb{R}\}$ such that

$$\int_{\mathbb{D}_0^+} \left(1 - e^{-\langle f, w_t\rangle}\right) \mathbb{N}_x^*(dw) = -\log \mathbb{P}_{\delta_x}^* \left(e^{-\langle f, X_t\rangle}\right), \tag{2.1}$$

for all $f \in \mathcal{B}_b^+(\mathbb{R})$ and $t > 0$. The branching property of X implies that, under $\mathbb{P}_{\delta_x}^*$, X_t is an infinitely divisible measure, so (2.1) is a Levy-Khinchine formula in which $\mathbb{N}_x^*$ plays the role of Lëvy measure. By the spatial homogeneity of Brownian motion, one can check that

$$\mathbb{P}_{\delta_x}^*\left(e^{-\langle f, X_t\rangle}\right) = \mathbb{P}_{\delta_0}^*\left(e^{-\int f(x+y)X_t(\mathrm{d}y)}\right), \quad \mathbb{N}_x^*\left(1 - e^{-\langle f, w_t\rangle}\right) = \mathbb{N}_0^*\left(1 - e^{-\int f(x+y)w_t(\mathrm{d}y)}\right).$$

It was shown in [3] that the skeleton of the super Brownian motion X_t is a branching Brownian motion Z_t with branching rate $q = \psi'(\lambda^*)$ and an offspring distribution $\{p_n : n \ge 2\}$ such that its generating function φ satisfies

$$q(\varphi(s) - s) = \frac{1}{\lambda^*}\psi(\lambda^*(1-s)).$$

We label the particles in Z using the classical Ulam-Harris notation. Let $\mathcal{T}$ be the set of all the particles. We write $\varnothing$ for the root. For each particle $u \in \mathcal{T}$, we write b_u and σ_u for its birth and death time respectively, N_u for the number of offspring of u, and $\{z_u(r) : r \in [b_u, \sigma_u]$ for its spatial trajectory. $v \preccurlyeq u$ means that v is an ancestor of u. Now we introduce the three kinds of immigrations along the skeleton Z as follows.

1. **Continuous immigration:** The process $I^{\mathbb{N}^*}$ is defined by

$$I_t^{\mathbb{N}^*} := \sum_{u\in\mathcal{T}} \sum_{(r_j,w_j)\in\mathcal{D}_{1,u}} \mathbf{1}_{r_j<t} w_j(t-r_j),$$

   where, given Z, independently for each $u \in \mathcal{T}$, $\mathcal{D}_{1,u} := \{(r_j, w_j) : j \geq 1\}$ are the atoms of a Poisson point process on $(b_u, \sigma_u] \times \mathbb{D}_0^+$ with rate $2\beta \mathrm{d}r \times \mathrm{d}\mathbb{N}^*_{z_u(r)}$.

2. **Discontinuous immigration:** The processes $I^{\mathbb{P}^*}$ is defined by

$$I_t^{\mathbb{P}^*} := \sum_{u\in\mathcal{T}} \sum_{(r_j,w_j)\in\mathcal{D}_{2,u}} \mathbf{1}_{r_j<t} w_j(t-r_j),$$

   where, given Z, independently for each $u \in \mathcal{T}$, $\mathcal{D}_{2,u} := \{(r_j, w_j) : j \geq 1\}$ are the atoms of a Poisson point process on $(b_u, \sigma_u] \times \mathbb{D}([0,\infty), \mathcal{M}_F(\mathbb{R}))$ with rate $\mathrm{d}r \times \int_{y\in(0,\infty)} ye^{-\lambda^* y} n(dy) \mathrm{d}\mathbb{P}^*_{y\delta_{z_u(r)}}$.

3. **Branching point biased immigration:** The process I^η is defined by

$$I_t^\eta := \sum_{u\in\mathcal{T}} \mathbf{1}_{\sigma_u \leq t} X^{(3,u)}_{t-\sigma_u} ,$$

   where, given Z, independently for each $u \in \mathcal{T}$, $X^{(3,u)}_\cdot$ is an independent copy of the canonical process X issued at time σ_u with law $\mathbb{P}^*_{Y_u \delta_{z_u(\sigma_u)}}$ where, given u has $n(\geq 2)$ offspring, Y_u is an independent random variable with distribution

$$\eta_n(\mathrm{d}y) = \frac{1}{p_n \lambda^* q} \left\{ \beta(\lambda^*)^2 \delta_0(\mathrm{d}y) \mathbf{1}_{\{n=2\}} + (\lambda^*)^n \frac{y^n}{n!} e^{-\lambda^* y} n(\mathrm{d}y) \right\}.$$

Now we define another $\mathcal{M}_F(\mathbb{R})$-valued process $I = \{I_t : t \geq 0\}$ by

$$I := I^{\mathbb{N}^*} + I^{\mathbb{P}^*} + I^\eta , \tag{2.2}$$

where $I^{\mathbb{N}^*} = \{I_t^{\mathbb{N}^*} : t \geq 0\}$, $I^{\mathbb{P}^*} = \{I_t^{\mathbb{P}^*} : t \geq 0\}$ and $I^\eta = \{I_t^\eta : t \geq 0\}$, conditioned on Z, are independent of each other. For any integer-valued measure ν, we denote by $\mathbf{P}_\nu$ the law of (Z, I) when the initial configuration of Z is ν. We write $\mathbf{P}$ for $\mathbf{P}_{\delta_0}$.

For any $\mu \in \mathcal{M}_F(\mathbb{R})$, let Z be a branching Brownian motion with Z_0 being a Poisson random measure with intensity measure $\lambda^*\mu$ and I be the immigration process along Z. Let $\widetilde{X}$ be an independent copy of X under $\mathbb{P}^*_\mu$, also independent of I. Then we define a measure-valued process $\Lambda = \{\Lambda_t : t \geq 0\}$ by

$$\Lambda = \widetilde{X} + I. \tag{2.3}$$

We denote the law of Λ by $\mathbf{Q}_\mu$. In particular, under $\mathbf{Q}_{\delta_0}$, $Z_0 = N\delta_0$, where N is a Poisson random variable with parameter λ^*. We write $\mathbf{Q}$ for $\mathbf{Q}_{\delta_0}$. In the rest of the paper, we use $\mathbf{E}$, $\mathbb{E}^*$ and $\mathrm{E}_{\mathbf{Q}}$ to denote the expectations with respect to $\mathbf{P}$, $\mathbb{P}^*$ and $\mathbf{Q}$, respectively. The following result is proved in [3].

Proposition 2.1 *For any* $\mu \in \mathcal{M}_F(\mathbb{R}^d)$*, the process* $(\Lambda, \mathbf{Q}_\mu)$ *is Markovian and has the same law as* $(X, \mathbb{P}_\mu)$*.*

Recall that M_t is the supremum of the support of X_t. Denote the supremum of Λ_t, I_t, Z_t, and $\tilde{X}_t$ by $M_t^\Lambda, M_t^I, M_t^Z$, and $M_t^{\tilde{X}}$, respectively. By (1.1), for any $f \in \mathcal{B}^+(\mathbb{R})$,

$$V_f(t,x) = -\log \mathbb{E}_{\delta_x}\left(e^{-\int_{\mathbb{R}} f(y)X_t(dy)}\right), \quad x \in \mathbb{R}.$$

By the spatial homogeneity of X, we have

$$V_f(t,-x) = -\log \mathbb{E}\left(e^{-\int_{\mathbb{R}} f(y-x)X_t(dy)}\right), \quad x \in \mathbb{R}. \tag{2.4}$$

Setting $f_\theta := f + \theta \mathbf{1}_{(0,\infty)}$, we get

$$u_f(t,x) = \lim_{\theta\to\infty} V_{f_\theta}(t,-x), \quad x \in \mathbb{R}. \tag{2.5}$$

For any $f \in \mathcal{B}^+(\mathbb{R})$, put

$$v_f(t,x) := \mathbf{E}\left(e^{-\int_{\mathbb{R}} f(y-x)I_t(dy)}; M_t^I \le x\right), \tag{2.6}$$

$$u_f^*(t,x) := -\log \mathbb{E}^*\left(e^{-\int_{\mathbb{R}} f(y-x)X_t(dy)}; M_t \le x\right). \tag{2.7}$$

For $f \equiv 0$, we write $v(t,x)$ and $u^*(t,x)$ for $v_f(t,x)$ and $u_f^*(t,x)$, respectively. The relation among u_f, u_f^* and v_f is given by the following lemma.

Lemma 2.2 *For any* $f \in \mathcal{B}^+(\mathbb{R})$*,* $t \ge 0$ *and* $x \in \mathbb{R}$*,*

$$u_f(t,x) = u_f^*(t,x) + \lambda^*(1 - v_f(t,x)).$$

Proof: Recall that under $\mathbf{Q}$, $Z_0 = N\delta_0$, where N is Poisson distributed with parameter λ^*. By the definition of Λ, we get that, for any $t \ge 0, x \in \mathbb{R}$,

$$\begin{aligned}
e^{-u_f(t,x)} &= \mathbb{E}\left(e^{-\int_{\mathbb{R}} f(y-x)X_t(dy)}; M_t \le x\right) = \mathrm{E}_{\mathbf{Q}}\left(e^{-\int_{\mathbb{R}} f(y-x)\Lambda_t(dy)}; M_t^\Lambda \le x\right) \\
&= \mathrm{E}_{\mathbf{Q}}\left(e^{-\int_{\mathbb{R}} f(y-x)\tilde{X}_t(dy)}; M_t^{\tilde{X}} \le x\right) \mathrm{E}_{\mathbf{Q}}\left(e^{-\int_{\mathbb{R}} f(y-x)I_t(dy)}; M_t^I \le x\right) \\
&= \mathbb{E}^*\left(e^{-\int_{\mathbb{R}} f(y-x)X_t(dy)}; M_t \le x\right) \mathrm{E}_{\mathbf{Q}}\left(\left[\mathbf{E}(e^{-\int_{\mathbb{R}} f(y-x)I_t(dy)}; M_t^I \le x\right]^N\right) \\
&= e^{-u_f^*(t,x)} e^{\lambda^*(v_f(t,x)-1)}.
\end{aligned}$$

Thus $u_f(t,x) = u_f^*(t,x) + \lambda^*(1 - v_f(t,x))$. □

Now we give some basic relations among M_t^Z, M_t^Λ , M_t^I and M_t.

Lemma 2.3 *Under* $\mathbf{Q}$*, given* Λ_t*,* Z_t *is a Poisson random measure with intensity* $\lambda^*\Lambda_t$*, which implies that* $M_t^Z \le M_t^\Lambda$*,* $\mathbf{Q}$*-a.s.*

Proof: We refer the readers to the display above [3, (3.14)] for a proof. □

Lemma 2.4 *Under* $\mathbf{P}$*,* $M_t^Z \le M_t^I$*, a.s.*

Proof: First we claim that $\mathbf{Q}(M_t^Z \le M_t^I) = 1$. In fact, for any x, by Lemma 2.3, we have

$$\begin{aligned} 0 = \mathbf{Q}(M_t^Z > x \ge M_t^\Lambda) &= \mathbf{Q}(M_t^Z > x, M_t^I \le x, M_t^{\tilde{X}} \le x) \\ &= \mathbf{Q}(M_t^Z > x, M_t^I \le x)\mathbf{Q}(M_t^{\tilde{X}} \le x). \end{aligned}$$

Using the fact that $\mathbf{Q}(M_t^{\tilde{X}} \le x) > 0$, we get $\mathbf{Q}(M_t^Z > x, M_t^I \le x) = 0$. Since x is arbitrary, the claim is true.

Recall that under $\mathbf{Q}$, $Z_0 = N\delta_0$, where N is Poisson distributed with parameter λ^*. Thus

$$0 = \mathbf{Q}(M_t^Z > M_t^I) \ge \mathbf{Q}[M_t^Z > M_t^I | N = 1]\mathbf{Q}(N = 1) = \mathbf{P}(M_t^Z > M_t^I)e^{-\lambda^*},$$

which implies that $\mathbf{P}(M_t^Z > M_t^I) = 0$. $\square$

The following lemma implies that, to prove our main results, we only need to study the limiting behavior of $v_f(t, \sqrt{2\alpha}\delta t)$.

Lemma 2.5 *For any* $f \in \mathcal{B}^+(\mathbb{R})$ *and* $\delta < 1$*,*

$$\lim_{t\to\infty} \frac{\mathbb{E}\left(e^{-\int_{\mathbb{R}} f(y-\sqrt{2\alpha}\delta t)X_t(dy)}; M_t \le \sqrt{2\alpha}\delta t | \mathcal{S}\right)}{v_f(t, \sqrt{2\alpha}\delta t)} = \frac{\lambda^*}{e^{\lambda^*} - 1}. \tag{2.8}$$

Proof: We also use $\mathcal{S}$ to denote the survival event of Λ. It is clear that, under $\mathbf{Q}$, $\mathcal{S} \subset \{N \ge 1\}$ and $\mathbf{Q}(\mathcal{S}) = \mathbf{Q}(N \ge 1) = 1 - e^{-\lambda^*}$. It follows that $\mathcal{S} = \{N \ge 1\}$, $\mathbf{Q}$-a.s. Then, by Proposition 2.1,

$$\begin{aligned} &\mathbb{E}\left(e^{-\int_{\mathbb{R}} f(y-x)X_t(dy)}; M_t \le x|\mathcal{S}\right) = \mathrm{E}_{\mathbf{Q}}\left(e^{-\int_{\mathbb{R}} f(y-x)\Lambda_t(dy)}; M_t^\Lambda \le x|N \ge 1\right) \\ =&\mathrm{E}_{\mathbf{Q}}\left(e^{-\int_{\mathbb{R}} f(y-x)\tilde{X}_t(dy)}; M_t^{\tilde{X}} \le x\right)\mathrm{E}_{\mathbf{Q}}\left(e^{-\int_{\mathbb{R}} f(y-x)I_t(dy)}; M_t^I \le x|N \ge 1\right) \\ =&e^{-u_f^*(t,x)}\mathrm{E}_{\mathbf{Q}}(v_f(t,x)^N|N \ge 1) = e^{-u_f^*(t,x)}\frac{e^{\lambda^* v_f(t,x)} - 1}{e^{\lambda^*} - 1}. \end{aligned} \tag{2.9}$$

Since $(X, \mathbb{P}^*)$ is subcritical, we have, for any δ,

$$e^{-u_f^*(t,\sqrt{2\alpha}\delta t)} \ge \mathbb{P}^*(\|X_t\| = 0) \to 1, \quad t \to \infty,$$

which implies that $e^{-u_f^*(t,\sqrt{2\alpha}\delta t)} \to 1$, as $t \to \infty$. By (1.5), we have for any $\delta < 1$,

$$\mathbb{E}\left(e^{-\int_{\mathbb{R}} f(y-\sqrt{2\alpha}\delta t)X_t(dy)}; M_t \le \sqrt{2\alpha}\delta t|\mathcal{S}\right) \le \mathbb{P}(M_t \le \sqrt{2\alpha}\delta t|\mathcal{S}) \to 0.$$

Thus by (2.9), $v_f(t, \sqrt{2\alpha}\delta t) \to 0$ for any $\delta < 1$. The desired result follows immediately. $\square$

To study the behavior of $v_f(t, \sqrt{2\alpha}\delta t)$ as $t \to \infty$, the following decomposition of v_f plays a fundamental role.

Proposition 2.6 *For any $f \in \mathcal{B}^+(\mathbb{R})$, $t > 0$ and $x \in \mathbb{R}$,*

$$v_f(t,x) = U_{1,f}(t,x) + U_{2,f}(t,x), \tag{2.10}$$

where

$$U_{1,f}(t,x) = E\Big[e^{-\int_0^t \psi'(\lambda^*+u_f^*(t-r,x-B_r))\,dr}, B_t \le x\Big], \tag{2.11}$$

$$U_{2,f}(t,x) = E\int_0^t e^{-\int_0^s \psi'(\lambda^*+u_f^*(t-r,x-B_r))\,dr}\hat{G}_f(t-s, x-B_s)\,\mathrm{d}s, \tag{2.12}$$

with $\hat{G}_f(t,x)$ being defined by

$$\begin{aligned}\hat{G}_f(t,x) &= \frac{1}{\lambda^*}\left[\beta(\lambda^*)^2 v_f(t,x)^2 + \int_0^\infty \left(e^{\lambda^* v_f(t,x)y} - 1 - \lambda^* v_f(t,x)y\right)e^{-(\lambda^*+u_f^*(t,x))y}\,n(\mathrm{d}y)\right] \\ &= \frac{1}{\lambda^*}\Big[\psi(u_f(t,x)) - \psi(\lambda^*+u_f^*(t,x)) + \psi'(\lambda^*+u_f^*(t,x))\lambda^* v_f(t,x)\Big]. \end{aligned} \tag{2.13}$$

Proof: Let τ be the first splitting time of Z, that is $\tau = \sigma_\varnothing$. By considering the cases $\tau > t$ and $\tau \le t$ separately, we get

$$\begin{aligned} v_f(t,x) &= \mathbf{E}\left(e^{-\int_{\mathbb{R}} f(y-x)I_t(dy)}; M_t^I \le x\right) \\ &= \mathbf{E}\left(e^{-\int_{\mathbb{R}} f(y-x)I_t(dy)}; M_t^I \le x, \tau > t\right) + \mathbf{E}\left(e^{-\int_{\mathbb{R}} f(y-x)I_t(dy)}; M_t^I \le x, \tau \le t\right) \\ &=: U_{1,f}(t,x) + U_{2,f}(t,x). \end{aligned} \tag{2.14}$$

By Lemma 2.4, $U_{1,f}(t,x) = \mathbf{E}\left(e^{-\int_{\mathbb{R}} f(y-x)I_t(dy)}; M_t^I \le x, M_t^Z \le x, \tau > t\right)$. By the decomposition of I in (2.2), on the event $\{\tau > t\}$, we have that $I_t = I_t^{\mathbb{N}^*} + I_t^{\mathbb{P}^*}$. Thus using [3, Lemma 3], we have that, on the event $\{\tau > t\}$, for any $x \in \mathbb{R}$,

$$\begin{aligned} \mathbf{E}\left(e^{-\int_{\mathbb{R}} f(y-x)I_t(dy)}; M_t^I \le x | \mathcal{F}_t^Z\right) &= \lim_{\theta\to\infty} \mathbf{E}\left(e^{-\int_{\mathbb{R}}[f(y-x)+\theta\mathbf{1}_{(0,\infty)}(y-x)]I_t(dy)} | \mathcal{F}_t^Z\right) \\ &= \exp\left\{-\int_0^t \langle \phi(u_f^*(t-s, x-\cdot)), Z_s\rangle\,\mathrm{d}s\right\}, \end{aligned}$$

where $\{\mathcal{F}_t^Z, t \ge 0\}$ is the natural filtration of Z and

$$\phi(\lambda) := \psi'(\lambda+\lambda^*) - \psi'(\lambda^*) = 2\beta\lambda + \int_0^\infty (1-e^{-\lambda x})xe^{-\lambda^* x}n(dx). \tag{2.15}$$

Note that, on the event $\{\tau > t\}$, $Z_s = \delta_{z_\varnothing(s)}$ and $\{z_\varnothing(s), s \le t\} \overset{d}{=} \{B_s, s \le t\}$. Thus

$$\begin{aligned} U_{1,f}(t,x) &= e^{-qt}\mathrm{E}\Big[\exp\Big\{-\int_0^t \phi(u_f^*(t-r, x-B_r))\,dr\Big\}; B_t \le x\Big] \\ &= \mathrm{E}\Big[\exp\Big\{-\int_0^t \psi'(\lambda^*+u_f^*(t-r,x-B_r))\,dr\Big\}; B_t \le x\Big]. \end{aligned} \tag{2.16}$$

On the event $\{\tau \le t\}$, the immigration process I has the following expression:

$$I_t = \sum_{(r_j,w_j)\in\mathcal{D}_{1,\varnothing}} w_j(t-r_j) + \sum_{(r_j,w_j)\in\mathcal{D}_{2,\varnothing}} w_j(t-r_j) + X_{t-\tau}^{(3,\varnothing)} + \sum_{i=1}^{N_\varnothing} I_{t-\tau}^i$$

$$=: \mathcal{J}_{1,t} + \mathcal{J}_{2,t} + \mathcal{J}_{3,t} + \mathcal{J}_{4,t}, \tag{2.17}$$

where, given Z_τ, $I^i, i = 1, \cdots, N_\varnothing$, are i.i.d copies of I under $\mathbf{P}_{z_\varnothing(\tau)}$. Since, given $\mathcal{F}_t^Z$, $\mathcal{J}_{i,t}, i = 1, 2, 3, 4$, are independent, we have

$$\begin{aligned} U_{2,f}(t,x) &= \mathbf{E}\left[\mathbf{E}\left(e^{-\int_{\mathbb{R}} f(y-x) I_t(dy)}; M_t^I \le x | \mathcal{F}_t^Z\right); \tau \le t\right] \\ &= \mathbf{E}\left[H_{1,t} H_{2,t} H_{3,t} H_{4,t}; \tau \le t\right], \end{aligned} \tag{2.18}$$

where

$$H_{i,t} = \mathbf{E}(e^{-\int_{\mathbb{R}} f(y-x)\mathcal{J}_{i,t}(dy)}; \mathcal{J}_{i,t}(x,\infty) = 0 | \mathcal{F}_t^Z), \quad i = 1, 2, 3, 4.$$

Put $f_\theta = f + \theta \mathbf{1}_{(0,\infty)}$. By the bounded convergence theorem, we have

$$H_{i,t} = \lim_{\theta\to\infty} \mathbf{E}\left(e^{-\int_{\mathbb{R}} f_\theta(y-x)\mathcal{J}_{i,t}(dy)} | \mathcal{F}_t^Z\right). \tag{2.19}$$

By the definition of $\mathcal{D}_{1,\varnothing}$ and (2.19), we have that, on the event $\{\tau \le t\}$,

$$H_{1,t} = \lim_{\theta\to\infty} \exp\left\{-2\beta \int_0^\tau \int_{\mathbb{D}_0^+} \left(1 - e^{-\int_{\mathbb{R}} f_\theta(y-x) w_{t-r}(dy)}\right) \mathbb{N}^*_{z_\varnothing(r)}(dw)\, dr\right\}. \tag{2.20}$$

Using (2.1), we get that

$$\begin{aligned} &\lim_{\theta\to\infty} \int_{\mathbb{D}_0^+} \left(1 - e^{-\int_{\mathbb{R}} f_\theta(y-x) w_{t-r}(dy)}\right) \mathbb{N}_z^*(dw) = \lim_{\theta\to\infty} -\log \mathbb{E}^*_{\delta_z}\left[e^{-\int_{\mathbb{R}} f_\theta(y-x) X_{t-r}(dy)}\right] \\ &= -\log \mathbb{E}^*_{\delta_z}\left[e^{-\int_{\mathbb{R}} f(y-x) X_{t-r}(dy)}; M_{t-r} \le x\right] = u_f^*(t-r, x-z). \end{aligned}$$

Thus we have that

$$H_{1,t} = \exp\left\{-2\beta \int_0^\tau u_f^*(t-r, x - z_\varnothing(r))\, dr\right\}. \tag{2.21}$$

For $H_{2,t}$, on the event $\{\tau \le t\}$, we have that

$$H_{2,t} = \lim_{\theta\to\infty} \exp\left\{-\int_0^\tau dr \int_0^\infty y e^{-\lambda^* y} n(\mathrm{d}y) \mathbb{E}^*_{y\delta_{z_\varnothing(r)}}\left(1 - e^{-\int_{\mathbb{R}} f_\theta(y-x) X_{t-r}(dy)}\right) dr\right\}. \tag{2.22}$$

It follows from the branching property of X that

$$\lim_{\theta\to\infty} \mathbb{P}^*_{y\delta_z}(e^{-\int_{\mathbb{R}} f_\theta(y-x) X_{t-r}(dy)}) = \lim_{\theta\to\infty} \left[\mathbb{P}^*_{\delta_z}(e^{-\int_{\mathbb{R}} f_\theta(y-x) X_{t-r}(dy)})\right]^y = e^{-u_f^*(t-r,x-z)y},$$

which implies that

$$H_{2,t} = \exp\left\{-\int_0^\tau \int_0^\infty y[1 - e^{-u_f^*(t-r,x-z_\varnothing(r))y}] e^{-\lambda^* y} n(\mathrm{d}y)\, dr\right\}. \tag{2.23}$$

Combining the definition of ϕ in (2.15) with (2.21) and (2.23), we get that

$$H_{1,t} H_{2,t} = \exp\left\{-\int_0^\tau \phi(u_f^*(t-r, x - z_\varnothing(r)))\, dr\right\}. \tag{2.24}$$

By the definition of $X^{(3,\varnothing)}$, on the event $\{\tau \le t\}$, we have that

$$\begin{aligned} H_{3,t} =& \lim_{\theta\to\infty} \mathbf{E}\left(\mathbb{P}^*_{Y_\varnothing \delta_y}\left(e^{-\int_{\mathbb{R}} f_\theta(y-x)X_{t-s}(dy)}\right)|\mathcal{F}^Z_t\right)|_{s=\tau, y=z_\varnothing(\tau)} \\ =&\mathbf{E}\left(e^{-u^*_f(t-\tau, x-z_\varnothing(\tau))Y_\varnothing}|\mathcal{F}^Z_t\right) \\ =&\frac{1}{p_{N_\varnothing}\lambda^* q}\left(\beta(\lambda^*)^2\mathbf{1}_{N_\varnothing=2} + \int_0^\infty \frac{(\lambda^* y)^{N_\varnothing}}{N_\varnothing!}e^{-u^*_f(t-\tau,x-z_\varnothing(\tau))y}e^{-\lambda^* y}\, n(\mathrm{d}y)\right). \end{aligned} \tag{2.25}$$

It follows from the branching property that on the event $\{\tau \le t\}$,

$$H_{4,t} = \left[\mathbf{P}_{\delta_{z_\varnothing(\tau)}}\left(e^{-\int_{\mathbb{R}} f(y-x)X_{t-s}(dy)}; M^I_{t-s} \le x\right)\right]^{N_\varnothing}_{s=\tau} = v_f(t-\tau, x - z_\varnothing(\tau))^{N_\varnothing}. \tag{2.26}$$

Note that

$$\begin{aligned} &\mathbf{E}(H_{3,t}H_{4,t}; \tau \le t|\tau, \{z_\varnothing(r), 0 \le r \le \tau\}) = \sum_{n=2}^\infty \mathbf{E}(H_{3,t}H_{4,t}; \tau \le t, N_\varnothing = n|\tau, z_\varnothing(\tau)) \\ =&\mathbf{1}_{\tau\le t}\sum_{n=2}^\infty p_n \frac{1}{p_n\lambda^* q}\left(\beta(\lambda^*)^2\mathbf{1}_{n=2} + \int_0^\infty \frac{(\lambda^* y)^n}{n!}e^{-u^*_f(t-\tau,x-z_\varnothing(\tau))y}e^{-\lambda^* y}\, n(\mathrm{d}y)\right) v_f(t-\tau, x-z_\varnothing(\tau))^n \\ =&\mathbf{1}_{\tau\le t}\frac{1}{\lambda^* q}\Big[(\beta(\lambda^*)^2 v_f(t-\tau, x-z_\varnothing(\tau))^2 \\ &+ \int_0^\infty \left(e^{\lambda^* v_f(t-\tau,x-z_\varnothing(\tau))y} - 1 - \lambda^* v_f(t-\tau, x-z_\varnothing(\tau))y\right) e^{-(\lambda^*+u^*_f(t-\tau,x-z_\varnothing(\tau)))y} n(\mathrm{d}y)\Big] \\ =&q^{-1}\hat{G}_f(t-\tau, x-z_\varnothing(\tau))\mathbf{1}_{\tau\le t}. \end{aligned} \tag{2.27}$$

Combining (2.18),(2.24) and (2.27), we get that

$$\begin{aligned} U_{2,f}(t,x) &= \mathbf{E}\left[H_{1,t}H_{2,t}\mathbf{E}(H_{3,t}H_{4,t}; \tau \le t|\tau, \{z_\varnothing(r), 0 \le r \le \tau\})\right] \\ &= q^{-1}\mathbf{P}\left(\exp\left\{-\int_0^\tau \phi(u^*_f(t-r, x-z_\varnothing(r)))\, dr\right\}\hat{G}_f(t-\tau, x-z_\varnothing(\tau)); \tau \le t\right) \\ &= \mathrm{E}\int_0^t \exp\left\{-\int_0^s \left(q + \phi(u^*_f(t-r, x-B_r))\right)\, dr\right\}\hat{G}_f(t-s, x-B_s)\,\mathrm{d}s, \end{aligned}$$

where in the last equality, we use the fact that τ is exponentially distributed with parameter q and $\{z_\varnothing(r) : r \ge 0\}$ is a standard Brownian motion. Note that $q + \phi(\lambda) = \psi'(\lambda^* + \lambda)$. The proof is now compete. □

Note that $\frac{e^x-1-x}{x^2} = \sum_{k=2}^\infty \frac{x^{k-2}}{k!}$ is increasing in x on $(0,\infty)$. So $e^{\lambda^* v_f(t,x)y} - 1 - \lambda^* v_f(t,x)y \le v_f(t,x)^2(e^{\lambda^* y} - 1 - \lambda^* y)$, which implies that

$$\begin{aligned} \hat{G}_f(t,x) &\le \frac{1}{\lambda^*}\left[(\beta(\lambda^*)^2 + \int_0^\infty (e^{\lambda^* y} - 1 - \lambda^* y)e^{-\lambda^* y}n(\mathrm{d}y))\right] v_f(t,x)^2 \\ &= (\psi'(\lambda^*) - \psi(\lambda^*)/\lambda^*)\, v_f(t,x)^2 = qv_f(t,x)^2 \le qv(t,x). \end{aligned} \tag{2.28}$$

Here in the last inequality, we use the fact that $v_f(t,x) \le v(t,x)$.

2.2 Some useful estimates

In this subsection we give some useful estimates for $u_f^*(t,x)$ and $v_f(t,x)$. Recall that $q = \psi'(\lambda^*)$ and $\rho = \sqrt{1+q/\alpha}$.

Lemma 2.7 **(1)** *For any $f \in \mathcal{B}^+(\mathbb{R})$ and $t>0, x\in\mathbb{R}$,*

$$u_f^*(t,x) \le k(t) := -\log \mathbb{P}^*(X_t = 0),$$

and $t \mapsto e^{qt}k(t)$ is decreasing on $(0,\infty)$.

(2) *If (H2) holds, then there exists a positive constant c_2 such that*

$$k(t) \le \left[\frac{1}{e^{c_2\vartheta t}-1}\right]^{1/\vartheta}, \quad t>0, \tag{2.29}$$

and for any $f \in \mathcal{B}_b^+(\mathbb{R})$, there exists a positive constant c_3 such that

$$u_f^*(t,x) \le c_3(1+x^{-2/\vartheta})e^{(a+\alpha)t}, \quad t,x>0. \tag{2.30}$$

Proof: Since $\mathbb{E}^*\left(e^{-\int_R f(y-x)X_t(dy)}; M_t \le x\right) \ge \mathbb{P}^*(X_t = 0)$ for any $t>0, x\in\mathbb{R}$, we have $u_f^*(t,x) \le k(t)$. By the branching property and Markov property, we get that

$$\mathbb{P}^*(\|X_t\|=0) = \mathbb{E}^*\left(\mathbb{P}^*_{X_{t-s}}(\|X_s\|=0)\right) = \mathbb{E}^*\left(e^{-k(s)\|X_{t-s}\|}\right).$$

Put $u_\theta^*(t) := -\log \mathbb{E}^*\left(e^{-\theta\|X\|_t}\right)$. Then $k(t) = u^*_{k(s)}(t-s)$. Under $\mathbb{P}^*$, $\|X_t\|$ is a continuous state branching process with branching mechanism $\psi(\lambda^*+\lambda)$. Then according to [16, Theorem 10.1], we have

$$k'(t) = -\psi\left(\lambda^* + u^*_{k(s)}(t-s)\right) = -\psi(\lambda^*+k(t)). \tag{2.31}$$

Since $\psi(\lambda^*)=0$ and ψ' is increasing on $(0,\infty)$, $\psi(\lambda^*+\lambda) \ge \psi'(\lambda^*)\lambda = q\lambda$. Thus $k'(t) \le -qk(t)$. Using this one can check that $(e^{qt}k(t))' \le 0$. The proof of (1) is complete.

Assume that (H2) holds. Then there exists $c_2>0$ such that $\psi(\lambda^*+\lambda) \ge c_2(\lambda+\lambda^{1+\vartheta})$. Thus, by (2.31), we have that

$$k'(t) \le -c_2(k(t)+k(t)^{1+\vartheta}),$$

which implies that

$$-c_2 t \ge \int_0^t \frac{k'(s)}{k(s)+k(s)^{1+\vartheta}}\,ds = \frac{-1}{\vartheta}\log(1+k(s)^{-\vartheta})|_0^t = \frac{-1}{\vartheta}\log(1+k(t)^{-\vartheta}).$$

Hence (2.29) follows immediately.

Since $u_f^*(t,x) \le u_f(t,x)$, it suffices to show that (2.30) is true for $u_f(t,x)$. By [19, Lemma 2.3(2)], we have that

$$V_{f_1+f_2}(t,x) \le V_{f_1}(t,x) + V_{f_2}(t,x).$$

By (2.5),

$$u_f(t,x) = \lim_{\theta\to\infty} V_{f_\theta}(t,-x) \le V_f(t,-x) + \lim_{\theta\to\infty} V_{\theta\mathbf{1}_{(0,\infty)}}(t,-x) = V_f(t,-x) + u(t,x),$$

where $f_\theta = f + \theta \mathbf{1}_{(0,\infty)}$. By (2.4) and Jensen's inequality, we have that

$$V_f(t,-x) = -\log \mathbb{E}\left(e^{-\int_{\mathbb{R}} f(y-x)X_t(dy)}\right) \le \mathbb{E}\left(\int_{\mathbb{R}} f(y-x)X_t(dy)\right) = e^{\alpha t}\mathrm{E}(f(B_t - x)) \le e^{\alpha t}\|f\|.$$

By [19, Lemma 4.2 and 4.3] (with A being replaced by x, and x there replaced by 0), we get that there exists a positive constant C such that

$$u(t,x) \le C(1 + x^{-2/\vartheta})e^{at}, \quad t, x > 0.$$

Combining the two displays above, we get that

$$u_f(t,x) \le e^{\alpha t}\|f\| + C(1 + x^{-2/\vartheta})e^{at} \le (C + \|f\|)(1 + x^{-2/\vartheta})e^{(a+\alpha)t}.$$

Now (2.30) follows immediately. □

Lemma 2.8 *Assume that (H1) and (H2) hold. For any $A > 0$ and $\epsilon > 0$,*

$$\int_0^A \phi(k(s))s^\epsilon \, \mathrm{d}s < \infty.$$

Proof: Note that, by (2.31),

$$k'(s) = -\psi(k(s) + \lambda^*), \quad k''(s) = -\psi'(k(s) + \lambda^*)k'(s).$$

Thus, using (2.15), we have

$$0 \le \phi(k(s)) = \psi'(k(s) + \lambda^*) - q = \frac{k''(s)}{-k'(s)} - q \le \frac{k''(s)}{-k'(s)}.$$

It follows that

$$\int_0^A \phi(k(s))s^\epsilon \, \mathrm{d}s \le \int_0^A \frac{k''(s)}{-k'(s)} s^\epsilon \, \mathrm{d}s = \int_0^A s^\epsilon \, \mathrm{d}(-\log(-k'(s)))$$
$$= -\log(-k'(A))A^\epsilon + \lim_{s \to 0} s^\epsilon \log(-k'(s)) + \epsilon \int_0^A \log(-k'(s))s^{\epsilon - 1} \, \mathrm{d}s. \quad (2.32)$$

Note that for $\lambda > 0$, $\psi''(\lambda + \lambda^*)$ exists and is decreasing. By Taylor's expansion, since $\psi(\lambda^*) = 0$, we have that

$$\psi(\lambda + \lambda^*) \le \psi'(\lambda^*)\lambda + \psi''(\lambda^*)\lambda^2, \quad \lambda > 0.$$

By (2.29), we have that $k(s) \le Cs^{-1/\vartheta}$. Thus we get that

$$-k'(s) = \psi(k(s) + \lambda^*) \le \psi'(\lambda^*)k(s) + \psi''(\lambda^*)k(s)^2$$
$$\le C(s^{-1/\vartheta} + s^{-2/\vartheta}) \le Cs^{-2/\vartheta}, \quad s \in [0, A].$$

Now the desired result follows immediately from (2.32). □

Now we give some upper estimates of $v(t,x)$.

Lemma 2.9 **(1)** *For any $t>0$,*

$$v(t,x)\le \mathrm{P}(B_t\le x),\qquad x\in\mathbb{R}, \tag{2.33}$$

and

$$v(t,x)\le \mathrm{P}(B_t\le x)\le \frac{\sqrt{t}}{\sqrt{2\pi}|x|}e^{-\frac{x^2}{2t}},\qquad x<0. \tag{2.34}$$

(2) *There exist $t_0>1$ and $c>0$ such that for any $t>t_0$,*

$$\begin{aligned} v(t,\sqrt{2\alpha}\theta t-\sqrt{t}) &\le \mathbf{P}(M_t^Z\le \sqrt{2\alpha}\theta t-\sqrt{t})\\ &\le ct\begin{cases} e^{-(q+\alpha\theta^2)t}, & \theta<1-\rho;\\ e^{-2\alpha(\rho-1)(1-\theta)t}, & 1-\rho\le\theta<1. \end{cases} \end{aligned} \tag{2.35}$$

Proof: (1) By Proposition 2.6, we have

$$\begin{aligned} v(t,x)=&\mathrm{E}\Big[e^{-\int_0^t\psi'(\lambda^*+u^*(t-r,x-B_r))\,dr},B_t\le x\Big]\\ &+\mathrm{E}\int_0^t e^{-\int_0^s\psi'(\lambda^*+u^*(t-r,x-B_r))\,dr}\hat{G}(t-s,x-B_s)\,\mathrm{d}s, \end{aligned}$$

which is equivalent to

$$\begin{aligned} v(t,x)=&\mathrm{E}_x\Big[e^{-\int_0^t\psi'(\lambda^*+u^*(t-r,B_r))\,dr},B_t\ge 0\Big]\\ &+\mathrm{E}_x\int_0^t e^{-\int_0^s\psi'(\lambda^*+u^*(t-r,B_r))\,dr}\hat{G}(t-s,B_s)\,\mathrm{d}s, \end{aligned} \tag{2.36}$$

where $\hat{G}$ is the $\hat{G}_f$ defined in Proposition 2.6 with $f\equiv 0$. Thus, by [12, Lemma 1.5, page 1211], the integral equation (2.36) implies that

$$v(t,x)+\mathrm{E}_x\int_0^t\psi'(\lambda^*+u^*(t-s,B_s))v(t-s,B_s)\mathrm{d}s=\mathrm{P}_x\Big[B_t\ge 0\Big]+\mathrm{E}_x\int_0^t\hat{G}(t-s,B_s)\mathrm{d}s. \tag{2.37}$$

Here we remark that since $\psi'(\lambda^*+u^*(t-s,z))$ may not be bounded as a function of (s,z), we can not use [12, Lemma 1.5, page 1211] directly. However, since $\psi'(\lambda^*+u^*(t-s,z))\ge 0$, the argument of [12, Lemma 1.5, page 1211] still works in the present case. Note that the right hand side of (2.37) is finite, and thus we have $\mathrm{E}_x\int_0^t\psi'(\lambda^*+u^*(t-s,B_s))v(t-s,B_s)\mathrm{d}s<\infty$. Combining (2.37) with the definition of $\hat{G}$, we get

$$v(t,x)=\mathrm{P}_x\Big[B_t\ge 0\Big]+\frac{1}{\lambda^*}\mathrm{E}_x\int_0^t\left[\psi(u(s,B_{t-s}))-\psi(\lambda^*+u^*(s,B_{t-s}))\right]\mathrm{d}s.$$

Note that $\psi(\lambda)<0$ for $\lambda\in(0,\lambda^*)$, $\psi(\lambda)>0$ for $\lambda>\lambda^*$ and ψ is increasing on (λ^*,∞). Thus for any $\lambda_0>\lambda^*$,

$$\sup_{0\le\lambda\le\lambda_0}\psi(\lambda)=\sup_{\lambda^*\le\lambda\le\lambda_0}\psi(\lambda)=\psi(\lambda_0).$$

Since $u(s,z) \le \lambda^* + u^*(s,z)$, using the above property with $\lambda_0 = \lambda^* + u^*(s, B_{t-s})$, we get that $\psi(u(s,B_{t-s})) - \psi(\lambda^* + u^*(s,B_{t-s})) \le 0$. Therefore we have that

$$v(t,x) \le \mathrm{P}_x\Big[B_t \ge 0\Big] = \mathrm{P}\Big[B_t \le x\Big], \qquad x \in \mathbb{R}.$$

For $x < 0$,

$$\begin{aligned}\mathrm{P}\Big[B_t \le x\Big] = \mathrm{P}\Big[B_1 \ge |x|t^{-1/2}\Big] &= \frac{1}{2\pi}\int_{|x|t^{-1/2}}^{\infty} e^{-y^2/2}\mathrm{d}y \\ &\le \frac{1}{2\pi}\int_{|x|t^{-1/2}}^{\infty} \frac{y}{|x|t^{-1/2}} e^{-y^2/2}\mathrm{d}y \le \frac{\sqrt{t}}{\sqrt{2\pi}|x|}e^{-x^2/(2t)}. \end{aligned} \tag{2.38}$$

Thus (2.34) follows.

(2) We claim that there exists $t_0 > 0$ such that for any $t > t_0$ and x,

$$\mathbf{P}(M_t^Z \le z) \le (2qt+1)\sup_{0\le s\le t} e^{-qs}\mathrm{P}\left(B_1 \le (z - \sqrt{2\alpha}(t-s) + \sqrt{t})/\sqrt{s}\right). \tag{2.39}$$

It is shown in [10] (see the discussion below [10, Lemma 3]) that the claim is true when $p_2 = 1$ and $q = 1$. Using similar arguments we see that it is also true for the general case. We omit the proof here.

Put $a(t) := \sqrt{2\alpha}(1-\theta)t$. By (2.39), for $t > t_0$,

$$\mathbf{P}(M_t^Z \le \sqrt{2\alpha}\theta t - \sqrt{t}) \le (2qt+1)\sup_{0\le s\le t} e^{-qs}\mathrm{P}(B_1 \le (\sqrt{2\alpha}s - a(t))/\sqrt{s}).$$

Note that by (2.38), $P(B_1 \le -y) \le \frac{1}{\sqrt{2\pi}}y^{-1}e^{-y^2/2}$ for all $y > 0$. Thus, if $\sqrt{2\alpha}s < a(t)$, we have

$$\begin{aligned} e^{-qs}\mathrm{P}\left(B_1 \le (\sqrt{2\alpha}s - a(t))/\sqrt{s}\right) &\le \frac{\sqrt{s}}{\sqrt{2\pi}}\frac{1}{a(t)-\sqrt{2\alpha}s}e^{-qs}e^{-(\sqrt{2\alpha}s - a(t))^2/2s} \\ &= \frac{\sqrt{s}}{\sqrt{2\pi}}\frac{1}{a(t)-\sqrt{2\alpha}s}e^{\sqrt{2\alpha}a(t)}e^{-\alpha\rho^2 s - \frac{a(t)^2}{2s}}. \end{aligned} \tag{2.40}$$

It is clear that

$$\alpha\rho^2 s + \frac{a(t)^2}{2s} \ge \sqrt{2\alpha}\rho a(t), \tag{2.41}$$

and is decreasing on $(0, \frac{a(t)}{\sqrt{2\alpha}\rho})$. We now prove the desired result in four cases.

(i) If $a(t) > \sqrt{2\alpha}\rho t$ (that is, $\theta < 1-\rho$), then $\sqrt{2\alpha}s < a(t)$ for $s \in [0,t]$ and thus by (2.40) we have that

$$\begin{aligned}&\sup_{0\le s\le t} e^{-qs}\mathrm{P}\left(B_1 \le (\sqrt{2\alpha}s - a(t))/\sqrt{s}\right) \le \frac{\sqrt{t}}{\sqrt{2\pi}}\frac{1}{a(t)-\sqrt{2\alpha}t}e^{\sqrt{2\alpha}a(t)}e^{-\alpha\rho^2 t - \frac{a(t)^2}{2t}} \\ &= \frac{\sqrt{t}}{\sqrt{2\pi}}\frac{1}{a(t)-\sqrt{2\alpha}t}e^{-(\alpha\rho^2 + \alpha(1-\theta)^2 - 2\alpha(1-\theta))t} \le \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\alpha}(\rho-1)\sqrt{t}}e^{-(q+\alpha\theta^2)t}.\end{aligned}$$

(ii) If $\sqrt{2\alpha}\frac{\rho+1}{2}t \le a(t) \le \sqrt{2\alpha}\rho t$ (that is, $1-\rho \le \theta \le (1-\rho)/2$), then $\sqrt{2\alpha}s < a(t)$ for $s\in[0,t]$, and thus by (2.40) and (2.41) we have that

$$\begin{aligned}\sup_{0\le s\le t} e^{-qs}\mathrm{P}\left(B_1 \le (\sqrt{2\alpha}s - a(t))/\sqrt{s}\right) &\le \frac{1}{\sqrt{2\pi}}\frac{2}{\sqrt{2\alpha}(\rho-1)\sqrt{t}}e^{-\sqrt{2\alpha}(\rho-1)a(t)}\\ &= \frac{1}{\sqrt{2\pi}}\frac{2}{\sqrt{2\alpha}(\rho-1)\sqrt{t}}e^{-2\alpha(\rho-1)(1-\theta)t}.\end{aligned}$$

(iii) If $1 < a(t) < \sqrt{2\alpha}\frac{\rho+1}{2}t$ (that is, $(1-\rho)/2 < \theta < 1 - \frac{1}{\sqrt{2\alpha}t}$), then

$$\begin{aligned}&\sup_{0\le s\le t} e^{-qs}\mathrm{P}\left(B_1 \le (\sqrt{2\alpha}s - a(t))/\sqrt{s}\right)\\ \le& \sup_{0\le s\le \frac{2}{\sqrt{2\alpha}(\rho+1)}a(t)} e^{-qs}\mathrm{P}\left(B_1 \le (\sqrt{2\alpha}s - a(t))/\sqrt{s}\right) + e^{-q\frac{2}{\sqrt{2\alpha}(\rho+1)}a(t)}\\ \le& \sup_{0\le s\le \frac{2}{\sqrt{2\alpha}(\rho+1)}a(t)} \frac{1}{\sqrt{2\pi}}\frac{\sqrt{s}}{a(t)-\sqrt{2\alpha}s}e^{-\sqrt{2\alpha}(\rho-1)a(t)} + e^{-\sqrt{2\alpha}(\rho-1)a(t)}\\ \le& \sqrt{\frac{1}{\sqrt{2\alpha}\pi(\rho+1)}}\frac{1}{\frac{\rho-1}{(\rho+1)}\sqrt{a(t)}}e^{-\sqrt{2\alpha}(\rho-1)a(t)} + e^{-\sqrt{2\alpha}(\rho-1)a(t)}\\ \le& \left(\sqrt{\frac{\rho+1}{\sqrt{2\alpha}\pi}}\frac{1}{\rho-1}+1\right)e^{-2\alpha(\rho-1)(1-\theta)t}.\end{aligned}$$

Here in the second inequality we used (2.40), (2.41) and the fact that

$$q\frac{2}{\sqrt{2\alpha}(\rho+1)} = \frac{2\alpha(\rho^2-1)}{\sqrt{2\alpha}(\rho+1)} = \sqrt{2\alpha}(\rho-1).$$

(iv) Finally, if $0 < a(t) \le 1$ (that is, $1 - \frac{1}{\sqrt{2\alpha}t} \le \theta < 1$), then

$$\mathbf{P}\left(M_t^Z \le \sqrt{2\alpha}\theta t - \sqrt{t}\right) \le 1 \le e^{\sqrt{2\alpha}(\rho-1)}e^{-\sqrt{2\alpha}(\rho-1)a(t)} = e^{\sqrt{2\alpha}(\rho-1)}e^{-2\alpha(\rho-1)(1-\theta)t}.$$

The proof is now complete. □

Recall that $m_t = \sqrt{2\alpha}t - \frac{3}{2\sqrt{2\alpha}}\log t$. The next lemma gives another estimate of $v(t,z)$. The proof will be given in Appendix.

Lemma 2.10 *For any $\epsilon \in (0, \sqrt{2\alpha}(\rho-1))$, there exist $c_\epsilon > 1$ and $T_\epsilon \ge 1$ such that*

$$v(t, m_t - z) \le \mathbf{P}\left(M_t^Z \le m_t - z\right) \le c_\epsilon e^{-\sqrt{2\alpha}(\rho-1)z}e^{\epsilon z}, \quad t \ge T_\epsilon, z > 0.$$

3 Proofs of the main results

Put $\zeta_f(t,x) := \psi'(\lambda^* + u_f^*(t,x))$. It is clear that $\zeta_f(t,x) \ge \psi'(\lambda^*) = q$.

Lemma 3.1 *For any $f \in \mathcal{B}^+(\mathbb{R})$,*

$$U_{1,f}(t, \sqrt{2\alpha}\delta t) \le \begin{cases} e^{-qt}, & \delta \ge 0; \\ \frac{1}{2\sqrt{\pi\alpha}|\delta|}t^{-1/2}e^{-(q+\alpha\delta^2)t}, & \delta < 0. \end{cases}$$

Proof: Since $\zeta_f(t,x) \geq \psi'(\lambda^*) = q$, by (2.11), we have that

$$U_{1,f}(t, \sqrt{2\alpha}\delta t) = \mathrm{E}\left(e^{-\int_0^t \zeta_f(t-r,\sqrt{2\alpha}\delta t - B_r)\,\mathrm{d}s}; B_t \leq \sqrt{2\alpha}\delta t\right) \leq e^{-qt}\mathrm{P}\Big[B_t \leq \sqrt{2\alpha}\delta t\Big].$$

Thus, the desired result follows easily from (2.38) with $x = \sqrt{2\alpha}\delta t$. □

Note that by the change of variables $s \to t-s$, we have

$$U_{2,f}(t,x) = \mathrm{E}\int_0^t e^{-\int_s^t \zeta_f(r,x-B_{t-r})\,dr}\hat{G}_f(s, x - B_{t-s})\,\mathrm{d}s.$$

3.1 Proof of Theorem 1.1: $\delta \in (1-\rho, 1)$

It follows from Lemma 2.5 that, to prove Theorem 1.1, we only need to consider the limiting property of $v_f(t, \sqrt{2\alpha}\delta t)$. Note that

$$q + \alpha\delta^2 - 2\alpha(\rho-1)(1-\delta) = \alpha(\rho - 1 + \delta)^2, \tag{3.1}$$

and

$$2\alpha(\rho-1)(1-\delta) \leq 2\alpha(\rho-1) < \alpha(\rho^2-1) = q, \quad \delta \in [0,1). \tag{3.2}$$

It follows from Lemma 3.1 that for any $\delta \in (1-\rho, 1)$,

$$\lim_{t\to\infty} \frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}} U_{1,f}(t, \sqrt{2\alpha}\delta t) = 0.$$

Thus, by the decomposition (2.10), to prove the desired result, it suffices to show that

$$\lim_{t\to\infty} \frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}} U_{2,f}(t, \sqrt{2\alpha}\delta t) = \frac{a_\delta^{3(\rho-1)/2}}{\sqrt{2\alpha}\rho}\int_{-\infty}^{\infty} e^{-\sqrt{2\alpha}(\rho-1)z} A(w_f(z))\mathrm{d}z,$$

where $a_\delta = 1 - \frac{1-\delta}{\rho}$ and $A(\lambda) = \frac{1}{\lambda^*}\psi(\lambda) + \psi'(\lambda^*)(1 - \lambda/\lambda^*)$.

The result above follows from Lemmas 3.2 and 3.3 below. In Lemma 3.3, we will show that for $\delta \in (1-\rho, 1)$,

$$\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}\mathbf{P}\Big(M_t^I \leq \sqrt{2\alpha}\delta t, \tau \notin \Big[\frac{1-\delta}{\rho}t - (\log t)\sqrt{t}, \frac{1-\delta}{\rho}t + (\log t)\sqrt{t}\Big]\Big) \to 0.$$

Thus, on the event $\big\{M_t^I \leq \sqrt{2\alpha}\delta t\big\}$, with large probability, the first branching time of the skeleton happens in the interval $\Big[\frac{1-\delta}{\rho}t - (\log t)\sqrt{t}, \frac{1-\delta}{\rho}t + (\log t)\sqrt{t}\Big]$.

Lemma 3.2 *Let $\delta \in (1-\rho, 1)$ and $\mathcal{I}_t = [a_\delta t - (\log t)\sqrt{t}, a_\delta t + (\log t)\sqrt{t}] \cap [0,t]$. Then for any $f \in \mathcal{H}$,*

$$\begin{aligned}&\lim_{t\to\infty} \frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}\mathrm{E}\int_{\mathcal{I}_t} e^{-\int_s^t \zeta_f(r,\sqrt{2\alpha}\delta t - B_{t-r})\,dr}\hat{G}_f(s, \sqrt{2\alpha}\delta t - B_{t-s})\,\mathrm{d}s\\ =&\frac{a_\delta^{3(\rho-1)/2}}{\sqrt{2\alpha}\rho}\int_{-\infty}^{\infty} e^{-\sqrt{2\alpha}(\rho-1)z}A(w_f(z))\mathrm{d}z.\end{aligned}$$

Proof: In this proof, we always assume that $t \geq 1$ is large enough such that $a_\delta t/2 \leq a_\delta t - (\log t)\sqrt{t} \leq a_\delta t + (\log t)\sqrt{t} \leq (1 + a_\delta)t/2$. Since ψ' is increasing and ψ'' is decreasing, it follows that, for any $\lambda \geq 0$

$$q = \psi'(\lambda^*) \leq \psi'(\lambda^* + \lambda) \leq q + \psi''(\lambda^*)\lambda.$$

Thus we have, for any $s \in \mathcal{I}_t$,

$$\begin{aligned} q(t-s) \leq \int_s^t \zeta_f(r, \sqrt{2\alpha}\delta t - B_{t-r})\, dr &\leq q(t-s) + \psi''(\lambda^*) \int_s^t u_f^*(r, \sqrt{2\alpha}\delta t - B_{t-r})\, dr \\ &\leq q(t-s) + \psi''(\lambda^*) t k(a_\delta t - (\log t)\sqrt{t}). \end{aligned}$$

Here the last inequality follows from Lemma 2.7(1) and the fact that the function k is decreasing. By Lemma 2.7(1), $\sup_{t>1} e^{qt} k(t) < \infty$, which implies that $tk(a_\delta t - (\log t)\sqrt{t}) \to 0$ as $t \to \infty$. Thus as $t \to \infty$,

$$\mathrm{E} \int_{\mathcal{I}_t} e^{-\int_s^t \zeta_f(r, \sqrt{2\alpha}\delta t - B_{t-r})\, dr} \hat{G}_f(s, \sqrt{2\alpha}\delta t - B_{t-s})\, \mathrm{d}s \sim \int_{\mathcal{I}_t} e^{-q(t-s)} \mathrm{E}[\hat{G}_f(s, \sqrt{2\alpha}\delta t - B_{t-s})]\, \mathrm{d}s. \tag{3.3}$$

Recall the definition of m_t in (1.9). By the change of variables $s = s(u) := a_\delta t + u\sqrt{t}$, we get that

$$\begin{aligned} &\int_{\mathcal{I}_t} e^{-q(t-s)} \mathrm{E}[\hat{G}_f(s, \sqrt{2\alpha}\delta t - B_{t-s})]\, \mathrm{d}s \\ =&\int_{\mathcal{I}_t} e^{-q(t-s)} \mathrm{E}[\hat{G}_f(s, m_s + (\sqrt{2\alpha}\delta t - m_s - B_{t-s}))]\, \mathrm{d}s \\ =&\int_{\mathcal{I}_t} e^{-q(t-s)} \mathrm{d}s \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(z+m_s-\sqrt{2\alpha}\delta t)^2}{2(t-s)}} \hat{G}_f(s, m_s + z)\, \mathrm{d}z \\ =&\sqrt{t} \int_{-\log t}^{\log t} \frac{e^{-q(1-a_\delta)t} e^{q\sqrt{t}u}}{\sqrt{2\pi(t - s(u))}} du \int_{-\infty}^{\infty} e^{-\frac{(m_{s(u)}+z-\sqrt{2\alpha}\delta t)^2}{2(t-s(u))}} \hat{G}_f(s(u), m_{s(u)} + z)\, \mathrm{d}z. \end{aligned} \tag{3.4}$$

For $u \in (-\log t, \log t)$, we have that

$$\begin{aligned} (m_{s(u)} + z - \sqrt{2\alpha}\delta t)^2 &= \left(\sqrt{2\alpha}(a_\delta - \delta)t + \sqrt{2\alpha}u\sqrt{t} - \frac{3}{2\sqrt{2\alpha}} \log(a_\delta t + u\sqrt{t}) + z\right)^2 \\ &= 2\alpha(a_\delta - \delta)^2 t^2 + 2\alpha u^2 t + 4\alpha(a_\delta - \delta)ut\sqrt{t} - 3(a_\delta - \delta)t\log(a_\delta t) \\ &\quad + 2\sqrt{2\alpha}(a_\delta - \delta)zt + R_1(t, u, z) \\ &= 2\alpha(a_\delta - \delta)^2 t^2 + 4\alpha(a_\delta - \delta)ut^{3/2} - 3(a_\delta - \delta)t\log t \\ &\quad + [2\alpha u^2 + 2\sqrt{2\alpha}(a_\delta - \delta)z - 3(a_\delta - \delta)\log a_\delta]t + R_1(t, u, z), \end{aligned}$$

where $R_1(t, u, z) = \left(-\frac{3}{2\sqrt{2\alpha}} \log(a_\delta t + u\sqrt{t}) + z\right)^2 - 3u\sqrt{t}\log(a_\delta t + u\sqrt{t}) + 2\sqrt{2\alpha}u\sqrt{t}z - 3(a_\delta - \delta)t\log(1 + u/(a_\delta\sqrt{t}))$. Using this one can check that for $|u| \leq \log t$,

$$R_1(t, u, z) \geq -3|u|\sqrt{t}\log(a_\delta t + |u|\sqrt{t}) - 2\sqrt{2\alpha}|u|\sqrt{t}|z| - 3(a_\delta - \delta)t\frac{|u|}{a_\delta\sqrt{t}}$$

$$\geq -3(\log t)^2\sqrt{t} - 2\sqrt{2\alpha}(\log t)\sqrt{t}|z| - \frac{3(a_\delta - \delta)}{a_\delta}\sqrt{t}\log t. \tag{3.5}$$

Using the Taylor expansion of $(1-x)^{-1}$, we obtain that

$$\begin{aligned}\frac{1}{2(t-s(u))} &= \frac{1}{2(1-a_\delta)t}\frac{1}{1-u/[(1-a_\delta)\sqrt{t}]} \\ &= \frac{1}{2(1-a_\delta)t}\left(1+\frac{u}{(1-a_\delta)\sqrt{t}}+\frac{u^2}{(1-a_\delta)^2 t}+R_2(t,u)\right),\end{aligned}$$

where

$$|R_2(t,u)| = \left|\sum_{n=3}^{\infty}\left[\frac{u}{(1-a_\delta)\sqrt{t}}\right]^n\right| \leq \sum_{n=3}^{\infty}\left[\frac{\log t}{(1-a_\delta)\sqrt{t}}\right]^n \leq \frac{2}{(1-a_\delta)^3}(\log t)^3 t^{-3/2}, \tag{3.6}$$

here we used the fact that $\log t/[(1-a_\delta)\sqrt{t}] \leq 1/2$, and for $0 \leq x \leq 1/2$, $\sum_{n=3}^\infty x^n = \frac{x^3}{1-x} \leq 2x^3$. Using the above estimates, we get that for $u \in (-\log t, \log t)$,

$$\begin{aligned}&\frac{(m_{s(u)}+z-\sqrt{2\alpha}\delta t)^2}{2(t-s(u))} \\ =&\frac{\alpha(a_\delta-\delta)^2}{1-a_\delta}\left(t+\frac{u}{(1-a_\delta)}\sqrt{t}+\frac{u^2}{(1-a_\delta)^2}\right)+\frac{4\alpha(a_\delta-\delta)u}{2(1-a_\delta)}\left(\sqrt{t}+\frac{u}{(1-a_\delta)}\right) \\ &-\frac{3(a_\delta-\delta)}{2(1-a_\delta)}\log t+\frac{2\alpha u^2-3(a_\delta-\delta)\log(a_\delta)+2\sqrt{2\alpha}(a_\delta-\delta)z}{2(1-a_\delta)} \\ &+R_3(t,u,z) \\ =&\frac{\alpha(\rho-1)^2(1-\delta)}{\rho}t+qu\sqrt{t}-\frac{3(\rho-1)}{2}\log t+\frac{\alpha\rho^3}{1-\delta}u^2+\sqrt{2\alpha}(\rho-1)z \\ &-\frac{3}{2}(\rho-1)\log(a_\delta)+R_3(t,u,z),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}R_3(t,u,z) =&\frac{4\alpha(a_\delta-\delta)u}{2(1-a_\delta)}\frac{u^2}{(1-a_\delta)^2\sqrt{t}}-\frac{3(a_\delta-\delta)}{2(1-a_\delta)}\log t\left(\frac{u}{(1-a_\delta)\sqrt{t}}+\frac{u^2}{(1-a_\delta)^2 t}\right) \\ &+\frac{2\alpha u^2-3(a_\delta-\delta)\log(a_\delta)+2\sqrt{2\alpha}(a_\delta-\delta)z}{2(1-a_\delta)}\left(\frac{u}{(1-a_\delta)\sqrt{t}}+\frac{u^2}{(1-a_\delta)^2 t}\right) \\ &+\Big(2\alpha(a_\delta-\delta)^2t^2+4\alpha(a_\delta-\delta)ut^{3/2}-3(a_\delta-\delta)t\log t \\ &\quad+[2\alpha u^2+2\sqrt{2\alpha}(a_\delta-\delta)z-3(a_\delta-\delta)\log a_\delta]t\Big)\frac{R_2(t,u)}{2(1-a_\delta)t} \\ &+\frac{R_1(t,u,z)}{2(t-s(u))}.\end{aligned}$$

Then it follows from (3.5) and (3.6) that $\lim_{t\to\infty} R_3(t,u,z) = 0$ and for any $u \in (-\log t, \log t)$,

$$-R_3(t,u,z) \leq \frac{2\alpha(a_\delta-\delta)}{(1-a_\delta)^3}(\log t)^3 t^{-1/2}+\frac{3(a_\delta-\delta)}{2(1-a_\delta)}\log t\left(\frac{\log t}{(1-a_\delta)\sqrt{t}}+\frac{(\log t)^2}{(1-a_\delta)^2 t}\right)$$

$$
\begin{aligned}
&+\frac{2\alpha(\log t)^2+3(a_\delta-\delta)\log(a_\delta)+2\sqrt{2\alpha}(a_\delta-\delta)|z|}{2(1-a_\delta)}\left(\frac{\log t}{(1-a_\delta)\sqrt{t}}+\frac{(\log t)^2}{(1-a_\delta)^2 t}\right)\\
&+\Big(2\alpha(a_\delta-\delta)^2t^2+4\alpha(a_\delta-\delta)(\log t)t^{3/2}+3(a_\delta-\delta)t\log t\\
&\quad+[2\alpha(\log t)^2+2\sqrt{2\alpha}(a_\delta-\delta)|z|+3(a_\delta-\delta)\log a_\delta]t\Big)\frac{1}{(1-a_\delta)^4}(\log t)^3t^{-5/2}\\
&+\left(3(\log t)^2\sqrt{t}+2\sqrt{2\alpha}(\log t)\sqrt{t}|z|+\frac{3(a_\delta-\delta)}{a_\delta}\sqrt{t}\log t\right)\frac{1}{2((1-a_\delta)t-(\log t)\sqrt{t})}.
\end{aligned}
$$

Thus there exists a positive function $r(\cdot)$ with $\lim_{t\to\infty} r(t)=0$ such that for any $u\in(-\log t,\log t)$,

$$
-R_3(t,u,z)\le r(t)(1+|z|). \tag{3.8}
$$

For any $\epsilon>0$, choose t_ϵ such that $r(t)\le\epsilon$ for any $t>t_\epsilon$. Noticing that $q(1-a_\delta)+\frac{\alpha(\rho-1)^2(1-\delta)}{\rho}=2\alpha(\rho-1)(1-\delta)$, by (3.3), (3.4) and (3.7), we get that

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}\mathrm{E}\int_{I_t} e^{-\int_s^t\zeta(r,\sqrt{2\alpha}\delta t-B_{t-r})\,dr}\hat G_f(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s\\
=&a_\delta^{3(\rho-1)/2}\lim_{t\to\infty}\int_{-\log t}^{\log t}\frac{\sqrt{t}}{\sqrt{2\pi(t-s(u))}}e^{-\frac{\alpha\rho^3}{1-\delta}u^2}du\int_{-\infty}^{\infty}e^{-\sqrt{2\alpha}(\rho-1)z}e^{-R_3(t,u,z)}\hat G_f(s(u),m_{s(u)}+z)\mathrm{d}z.
\end{aligned}\tag{3.9}
$$

Note that $u_f^*(t,m_t+x)\le-\log\mathbb{P}^*(X_t=0)\to0$, as $t\to\infty$. Then it follows from (1.8) and Lemma 2.2 that

$$
\lim_{t\to\infty}v_f(t,m_t+z)=1-\frac{w_f(z)}{\lambda^*}.
$$

Recall the definition of $\hat G$ in (2.13). It follows that

$$
\begin{aligned}
&\lim_{t\to\infty}\hat G_f(a_\delta t+u\sqrt{t},m_{a_\delta t+u\sqrt{t}}+z)=\lim_{t\to\infty}\hat G_f(t,m_t+z)\\
=&\frac{1}{\lambda^*}\left(\psi(w_f(z))+q(\lambda^*-w_f(z))\right)=A(w_f(z)).
\end{aligned}\tag{3.10}
$$

Thus, as $t\to\infty$, the limit of the integrand in (3.9) is

$$
\frac{a_\delta^{3(\rho-1)/2}}{\sqrt{2\pi(1-a_\delta)}}e^{-\frac{\alpha\rho^3}{1-\delta}u^2-\sqrt{2\alpha}(\rho-1)z}A(w_f(z)).
$$

By (3.8), (2.28) and Lemma 2.10, we have that, for η small enough, there exist $T_\eta>1$ and $c_\eta>0$ such that, for $t>T_\eta+t_\epsilon$, the integrand in (3.9) is smaller than

$$
q\frac{a_\delta^{3(\rho-1)/2}}{\sqrt{\pi(1-a_\delta)}}e^{-\frac{\alpha\rho^3}{1-\delta}u^2-\sqrt{2\alpha}(\rho-1)z}e^{\epsilon(1+|z|)}\times\begin{cases}c_\eta^2e^{2[\sqrt{2\alpha}(\rho-1)-\eta]z}, & z<0;\\ 1, & z>0,\end{cases}
$$

which is integrable over $\mathbb{R}\times\mathbb{R}$ if we choose $\epsilon<\sqrt{2\alpha}(\rho-1)$ and $-2\eta+\sqrt{2\alpha}(\rho-1)-\epsilon>0$. Thus using the dominated convergence theorem in (3.9), we have that

$$
\lim_{t\to\infty}\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}\mathrm{E}\int_{\mathcal{I}_t}e^{-\int_s^t\zeta_f(r,\sqrt{2\alpha}\delta t-B_{t-r})\,dr}\hat G_f(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s
$$

$$
\begin{aligned}
=&\frac{a_\delta^{3(\rho-1)/2}}{\sqrt{2\pi(1-a_\delta)}}\int_{-\infty}^{\infty} e^{-\frac{\alpha\rho^3}{1-\delta}u^2}du\int_{-\infty}^{\infty} e^{-\sqrt{2\alpha}(\rho-1)z}A(w_f(z))\mathrm{d}z\\
=&\frac{a_\delta^{3(\rho-1)/2}}{\sqrt{2\alpha}\rho}\int_{-\infty}^{\infty} e^{-\sqrt{2\alpha}(\rho-1)z}A(w_f(z))\mathrm{d}z.
\end{aligned}
$$

□

Lemma 3.3 *For $\delta\in(1-\rho,1)$, it holds that for $\delta\in(1-\rho,1)$, it holds that for any $f\in\mathcal{B}^+(\mathbb{R})$,*

$$
\lim_{t\to\infty}\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}\mathrm{E}\int_{[0,t]\setminus\mathcal{I}_t} e^{-\int_s^t\zeta_f(r,\sqrt{2\alpha}\delta t-B_{t-r})\,dr}\hat{G}_f(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s=0.
$$

Since $\zeta_f(t,x)\ge q$, using (2.28) and the fact that $v_f(t,x)\le v(t,x)$, to prove Lemma 3.3, we only need to show that

$$
\lim_{t\to\infty}\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}\mathrm{E}\int_{[0,t]\setminus\mathcal{I}_t} e^{-q(t-s)}v^2(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s=0. \tag{3.11}
$$

Note that

$$
\begin{aligned}
[0,t]\setminus\mathcal{I}_t\subset&[0,\epsilon t]\cup\Big([(a_\delta-\epsilon)t,a_\delta t-(\log t)\sqrt{t}])\cup[a_\delta t+(\log t)\sqrt{t},(a_\delta+\epsilon)t]\Big)\\
&\cup\left([\epsilon t,(a_\delta-\epsilon)t]\cup[(a_\delta+\epsilon)t,t]\right).
\end{aligned}
$$

In the following three lemmas we handle the integral in (3.11) over $[0,\epsilon t]$, $[(a_\delta-\epsilon)t,a_\delta t-(\log t)\sqrt{t}]\cup[a_\delta t+(\log t)\sqrt{t},(a_\delta+\epsilon)t]$ and $[\epsilon t,(a_\delta-\epsilon)t]\cup[(a_\delta+\epsilon)t,t]$ separately.

Lemma 3.4 *Let $\delta\in(1-\rho,1)$. For $\epsilon>0$ small enough,*

$$
\lim_{t\to\infty}\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}\mathrm{E}\int_0^{\epsilon t} e^{-q(t-s)}v^2(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s=0.
$$

Proof: By (2.33), we have that

$$
v(s,\sqrt{2\alpha}\delta t-B_{t-s})\le \mathrm{P}_{B_{t-s}}(B_s\le\sqrt{2\alpha}\delta t)=\mathrm{P}[B_t\le\sqrt{2\alpha}\delta t|\sigma(B_r:r\le t-s)].
$$

Thus it follows that

$$
\mathrm{E}(v^2\left(s,\sqrt{2\alpha}\delta t-B_{t-s})\right)\le\mathrm{E}\left(v(s,\sqrt{2\alpha}\delta t-B_{t-s})\right)\le\mathrm{P}(B_t\le\sqrt{2\alpha}\delta t). \tag{3.12}
$$

Hence, for any $\epsilon>0$,

$$
\begin{aligned}
&\mathrm{E}\int_0^{\epsilon t} e^{-q(t-s)}v^2(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s\le q^{-1}e^{q\epsilon t}e^{-qt}\mathrm{P}(B_t\le\sqrt{2\alpha}\delta t)\\
&\le q^{-1}e^{q\epsilon t}\times\left\{\begin{array}{ll} e^{-qt}, & \delta\ge0;\\ \frac{1}{2\sqrt{\pi\alpha}|\delta|}t^{-1/2}e^{-(q+\alpha\delta^2)t}, & \delta<0,\end{array}\right.
\end{aligned}\tag{3.13}
$$

where in the last inequality we used (2.38). Using (3.1) and (3.2), we can choose ϵ small enough so that

$$
2\alpha(\rho-1)(1-\delta)+q\epsilon<\left\{\begin{array}{ll} q, & \delta\ge0;\\ q+\alpha\delta^2, & \delta\in(1-\rho,0),\end{array}\right.
$$

which implies the desired result. □

Lemma 3.5 *Let $\delta \in (1-\rho, 1)$. For $\epsilon > 0$ small enough,*

$$\lim_{t\to\infty} \frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}} \mathrm{E}\left(\int_{(a_\delta-\epsilon)t}^{a_\delta t-(\log t)\sqrt{t}} + \int_{a_\delta t+(\log t)\sqrt{t}}^{(a_\delta+\epsilon)t}\right) e^{-q(t-s)} v^2(s, \sqrt{2\alpha}\delta t - B_{t-s})\,\mathrm{d}s = 0.$$

Proof: Put $S_t := (a_\delta - \epsilon, a_\delta - (\log t)/\sqrt{t}) \cup (a_\delta + (\log t)/\sqrt{t}, a_\delta + \epsilon)$. Recall the definition of m_t given by (1.9). By the change of variables $s = rt$, applying Lemma 2.10 for $z > 0$ and the fact $v \le 1$ for $z \le 0$, we get that, for η small enough, there exists $c_\eta \ge 1$ such that for t large enough,

$$\begin{aligned}
&\mathrm{E}\left(\int_{(a_\delta-\epsilon)t}^{a_\delta t-(\log t)\sqrt{t}} + \int_{a_\delta t+\log t\sqrt{t}}^{(a_\delta+\epsilon)t}\right) e^{-q(t-s)} v^2(s, \sqrt{2\alpha}\delta t - B_{t-s})\,\mathrm{d}s \\
=&\mathrm{E}\left(\int_{(a_\delta-\epsilon)t}^{a_\delta t-(\log t)\sqrt{t}} + \int_{a_\delta t+\log t\sqrt{t}}^{(a_\delta+\epsilon)t}\right) e^{-q(t-s)} v^2\left(s, m_s - (m_s - \sqrt{2\alpha}\delta t + B_{t-s})\right)\,\mathrm{d}s \\
\le& c_\eta^2 t \int_{S_t} e^{-q(1-r)t} \mathrm{E}\left[e^{-2(\sqrt{2\alpha}(\rho-1)-\eta)(m_{rt}-\sqrt{2\alpha}\delta t + B_{(1-r)t})} \wedge 1\right]\,dr.
\end{aligned}$$

We claim that for any $b_1 > b_2 > 0$,

$$\mathrm{E}\left(e^{-b_1(b_2+B_1)} \wedge 1\right) \le \frac{1}{\sqrt{2\pi}}\left(\frac{1}{b_1-b_2} + \frac{1}{b_2}\right) e^{-b_2^2/2}. \tag{3.14}$$

Indeed, the left-hand side of (3.14) can be written as

$$\mathrm{E}\left(e^{-b_1(b_2+B_1)}; B_1 + b_2 > 0\right) + \mathrm{E}(B_1 + b_2 \le 0).$$

By (2.38), we have that

$$\mathrm{E}(B_1 + b_2 \le 0) = \mathrm{E}(B_1 > b_2) \le \frac{1}{\sqrt{2\pi}}\frac{1}{b_2} e^{-b_2^2/2}.$$

By the Girsanov theorem, we have

$$\begin{aligned}
&\mathrm{E}\left(e^{-b_1(b_2+B_1)}; B_1 + b_2 > 0\right) = e^{-b_1 b_2} e^{b_1^2/2} \mathrm{E}(B_1 - b_1 + b_2 > 0) \\
\le& \frac{1}{\sqrt{2\pi}}\frac{1}{b_1-b_2} e^{-b_1 b_2} e^{b_1^2/2} e^{-(b_1-b_2)^2/2} = \frac{1}{\sqrt{2\pi}}\frac{1}{b_1-b_2} e^{-b_2^2/2}.
\end{aligned}$$

Now (3.14) follows immediately.

We will use (3.14) with $b_1 = 2(\sqrt{2\alpha}(\rho-1)-\eta)\sqrt{(1-r)t}$ and $b_2 = \frac{m_{rt}-\sqrt{2\alpha}\delta t}{\sqrt{(1-r)t}}$. For $\epsilon \in \left(0, \frac{a_\delta-\delta}{2\rho-1} \wedge (1-a_\delta)\right)$, we have for any $r \in S_t \subset (a_\delta - \epsilon, a_\delta + \epsilon)$,

$$\frac{\sqrt{2\alpha}(a_\delta + \epsilon - \delta)}{\sqrt{(1-a_\delta-\epsilon)}}\sqrt{t} \ge b_2 \ge \frac{\sqrt{2\alpha}(a_\delta - \epsilon - \delta)}{\sqrt{(1-a_\delta+\epsilon)}}\sqrt{t} - \frac{\frac{3}{2\sqrt{2\alpha}}}{\sqrt{(1-a_\delta-\epsilon)}}\frac{\log t}{\sqrt{t}},$$

and

$$b_1 - b_2 \ge 2(\sqrt{2\alpha}(\rho-1)-\eta)\sqrt{(1-a_\delta-\epsilon)t} - \frac{\sqrt{2\alpha}(a_\delta+\epsilon-\delta)}{\sqrt{(1-a_\delta-\epsilon)}}\sqrt{t}$$

$$= \frac{\sqrt{2\alpha}}{\sqrt{1-a_\delta-\epsilon}}\left[2(\rho-1)(1-a_\delta)-(a_\delta-\delta)-(2\rho-1)\epsilon-\frac{2\eta}{\sqrt{2\alpha}}(1-a_\delta-\epsilon)\right]\sqrt{t}$$
$$\geq \frac{\sqrt{2\alpha}}{\sqrt{1-a_\delta-\epsilon}}\left[a_\delta-\delta-(2\rho-1)\epsilon-\frac{2\eta}{\sqrt{2\alpha}}\right]\sqrt{t},$$

where in the final inequality, we used $(\rho-1)(1-a_\delta)=(a_\delta-\delta)$. So if we choose $\eta \in \big(0, \sqrt{2\alpha}[a_\delta-\delta-(2\rho-1)\epsilon]/2\big)$, and then for t large enough, $b_1 > b_2 > 0$. Thus, using (3.14), we have that, for t large enough and $r \in S_t$,

$$\mathrm{E}\left[e^{-2(\sqrt{2\alpha}(\rho-1)-\epsilon)(m_{rt}-\sqrt{2\alpha}\delta t+B_{(1-r)t})}\wedge 1\right] \leq Ct^{-1/2}e^{-\frac{(m_{rt}-\sqrt{2\alpha}\delta t)^2}{2(1-r)t}}$$
$$\leq Ct^{-1/2}t^{\frac{3(1-\delta)}{2(1-a_\delta-\epsilon)}}e^{-\frac{\alpha(r-\delta)^2}{(1-r)}t}. \tag{3.15}$$

Here in the last inequality we used the following facts: $r \leq a_\delta+\epsilon < 1$ and

$$e^{-\frac{(m_{rt}-\sqrt{2\alpha}\delta t)^2}{2(1-r)t}} \leq (rt)^{\frac{3(r-\delta)}{2(1-r)}}e^{-\frac{\alpha(r-\delta)^2}{(1-r)}t} \leq t^{\frac{3(1-\delta)}{2(1-a_\delta-\epsilon)}}e^{-\frac{\alpha(r-\delta)^2}{(1-r)}t}.$$

For any $x \in (0,1)$ and $c \in \mathbb{R}$, one can prove that

$$q(1-x)+\frac{\alpha(x-c)^2}{1-x} \geq 2\alpha(\rho-1)(1-c)+\alpha\rho^2\Big(1-\frac{1-c}{\rho}-x\Big)^2. \tag{3.16}$$

For a proof of the above inequality, see Lemma A.2 in the Appendix. Using the inequality above, we get that, for $r \in S_t$,

$$q(1-r)+\frac{\alpha(r-\delta)^2}{(1-r)} \geq 2\alpha(\rho-1)(1-\delta)+\alpha\rho^2(a_\delta-r)^2 \geq 2\alpha(\rho-1)(1-\delta)+\alpha\rho^2\frac{(\log t)^2}{t}.$$

Thus, there exists θ such that

$$\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}\mathrm{E}\left(\int_{(a_\delta-\epsilon)t}^{a_\delta t-\log t\sqrt{t}}+\int_{a_\delta t+\log t\sqrt{t}}^{(a_\delta+\epsilon)t}\right)e^{-q(t-s)}v^2(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s$$
$$\leq Ct^\theta e^{-\alpha\rho^2(\log t)^2} \to 0, \qquad \text{as } t\to\infty.$$

□

Lemma 3.6 *Let $\delta \in (1-\rho,1)$. For $\epsilon > 0$ small enough,*

$$\limsup_{t\to\infty}\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}\mathrm{E}\left(\int_{\epsilon t}^{(a_\delta-\epsilon)t}+\int_{(a_\delta+\epsilon)t}^{t}\right)e^{-q(t-s)}v^2(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s = 0.$$

Proof: Set $\mathcal{I}=(\epsilon, a_\delta-\epsilon)\cup(a_\delta+\epsilon,1)$. By the change of variables $r=s/t$, we get that

$$\mathrm{E}\left(\int_{\epsilon t}^{(a_\delta-\epsilon)t}+\int_{(a_\delta+\epsilon)t}^{t}\right)e^{-q(t-s)}v^2(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s$$
$$=t\mathrm{E}\int_{\mathcal{I}}e^{-q(1-r)t}v^2(rt,\sqrt{2\alpha}\delta t-B_{t-rt})\,\mathrm{d}r$$

$$
\begin{aligned}
=&t\mathrm{E}\int_{\mathcal{I}} e^{-q(1-r)t}\,\mathrm{d}r\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(1-r)t}}e^{-\frac{(z-\sqrt{2\alpha}\delta t)^2}{2(1-r)t}}v^2(rt,z)\,\mathrm{d}z\\
=&\sqrt{2\alpha}t^2\int_{\mathcal{I}} dr\int_{\mathbb{R}}\frac{r}{\sqrt{2\pi(1-r)t}}e^{-q(1-r)t}e^{-\frac{(\sqrt{2\alpha}\alpha rt-\sqrt{rt}-\sqrt{2\alpha}\delta t)^2}{2(1-r)t}}v^2(rt,\sqrt{2\alpha}\theta rt-\sqrt{rt})\,\mathrm{d}\theta\\
=&\frac{\sqrt{\alpha}}{\sqrt{\pi}}t^{3/2}\int_{\mathcal{I}}\frac{r\,\mathrm{d}r}{\sqrt{1-r}}\left(\int_{-\infty}^{1-\rho}+\int_{1-\rho}^{1}+\int_{1}^{\infty}\right)e^{-\frac{\alpha\left(\theta r-\frac{\sqrt{r}}{\sqrt{2\alpha t}}-\delta\right)^2 t}{(1-r)}-q(1-r)t}v^2(rt,\sqrt{2\alpha}\theta rt-\sqrt{rt})\,\mathrm{d}\theta\\
=:&\frac{\sqrt{\alpha}}{\sqrt{\pi}}(I_1(t)+I_2(t)+I_3(t)).
\end{aligned}
$$

For $I_1(t)$, by Lemma 2.9(2) with t replaced by rt, we have that for $\epsilon t>t_0$ and $\theta<1-\rho$,

$$
v(rt,\sqrt{2\alpha}\theta rt-\sqrt{rt})\le crte^{-\alpha\theta^2 rt}e^{-qrt}.
$$

Then by the change of variables $\theta\to-\theta$ in $I_1(t)$, we get that for $t>t_0/\epsilon$,

$$
\begin{aligned}
I_1(t)\le&\ c^2t^{7/2}\int_{\mathcal{I}}\frac{r^3\,dr}{\sqrt{1-r}}\int_{\rho-1}^{\infty}\exp\Big\{-\Big[q(1+r)+\frac{\alpha\left(\theta r+\frac{\sqrt{r}}{\sqrt{2\alpha t}}+\delta\right)^2}{(1-r)}+2\alpha\theta^2 r\Big]t\Big\}\,\mathrm{d}\theta\\
\le&\ c^2t^{7/2}e^{-q(1+\epsilon)t}e^{-\alpha\delta^2 t}\int_{\mathcal{I}}\frac{r^3\,\mathrm{d}r}{\sqrt{1-r}}\int_{-\infty}^{\infty}e^{-2\alpha\theta^2 rt}\,\mathrm{d}\theta\\
=&\ c^2t^{7/2}e^{-q(1+\epsilon)t}e^{-\alpha\delta^2 t}\int_{\mathcal{I}}\sqrt{\frac{\pi}{2\alpha rt}}\frac{r^3\,dr}{\sqrt{1-r}}\,\mathrm{d}r\le Ct^3e^{-q\epsilon t}e^{-(q+\alpha\delta^2)t}.
\end{aligned}
$$

Since $q+\alpha\delta^2>2\alpha(\rho-1)(1-\delta)$, it holds that

$$
\lim_{t\to\infty}\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}I_1(t)=0.
$$

For $I_2(t)$, by Lemma 2.9(2) and the change of variables $\theta-\frac{1}{\sqrt{2\alpha rt}}\to\theta$, we get that for $\epsilon t>t_0$, $I_2(t)$ is less than or equal to

$$
\begin{aligned}
&c^2t^{7/2}\int_{\mathcal{I}}\frac{r^3}{\sqrt{1-r}}\,\mathrm{d}r\int_{1-\rho}^{1}\exp\Big\{-\Big[q(1-r)+\frac{\alpha\left(\theta r-\frac{\sqrt{r}}{\sqrt{2\alpha t}}-\delta\right)^2}{(1-r)}+4\alpha(\rho-1)(1-\theta)r\Big]t\Big\}\,\mathrm{d}\theta\\
=&c^2t^{7/2}\int_{\mathcal{I}}\frac{r^3}{\sqrt{1-r}}\,\mathrm{d}r\int_{1-\rho-\frac{1}{\sqrt{2\alpha rt}}}^{1-\frac{1}{\sqrt{2\alpha rt}}}e^{-\Big[q(1-r)+\frac{\alpha(\theta r-\delta)^2}{(1-r)}+4\alpha(\rho-1)(1-\theta)r\Big]t}e^{2\sqrt{2\alpha}(\rho-1)\sqrt{rt}}\,\mathrm{d}\theta\\
\le&Ct^{7/2}e^{2\sqrt{2\alpha}(\rho-1)\sqrt{t}}e^{-\inf_{r\in\mathcal{I},\theta<1}H(\theta,r)t},
\end{aligned}
$$

where $H(\theta,r):=q(1-r)+\frac{\alpha(\theta r-\delta)^2}{(1-r)}+4\alpha(\rho-1)(1-\theta)r$. We claim that

$$
\inf_{r\in\mathcal{I},\theta<1}H(\theta,r)>2\alpha(\rho-1)(1-\delta). \tag{3.17}
$$

Then it follows that

$$
\lim_{t\to\infty}\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}I_2(t)=0.
$$

Now we prove (3.17). Note that

$$H(\theta,r) = \frac{\alpha r^2}{1-r}\left(\theta - \frac{\delta+2(\rho-1)(1-r)}{r}\right)^2 - \alpha(\rho-1)(3\rho-1)(1-r) + 4\alpha(\rho-1)(1-\delta).$$

For $r^* := \frac{\delta+2(\rho-1)}{2\rho-1} \le r < 1$ (that is $\frac{\delta+2(\rho-1)(1-r)}{r} \le 1$) and $\theta < 1$,

$$H(\theta,r) \ge -\alpha(\rho-1)(3\rho-1)(1-r^*) + 4\alpha(\rho-1)(1-\delta) = 2\alpha(\rho-1)(1-\delta) + \frac{\alpha(\rho-1)^2(1-\delta)}{2\rho-1}.$$

For $r \in [0, r^*] \cap \mathcal{I}$ and $\theta < 1$, since $\frac{\delta+2(\rho-1)(1-r)}{r} \ge 1$, we have that

$$\begin{aligned} H(\theta,r) \ge H(1,r) &= q(1-r) + \frac{\alpha(r-\delta)^2}{(1-r)} \\ &\ge 2\alpha(\rho-1)(1-\delta) + \alpha\rho^2(a_\delta - r)^2 \\ &\ge 2\alpha(\rho-1)(1-\delta) + \alpha\rho^2\epsilon^2, \end{aligned}$$

where in the second inequality we used (3.16). Thus (3.17) is valid.

Finally, we deal with $I_3(t)$. Since $v(t,x) \le 1$, we have

$$\begin{aligned} I_3(t) &\le t^{3/2}\int_{\mathcal{I}} \frac{r\,\mathrm{d}r}{\sqrt{1-r}} \int_1^\infty e^{-\frac{\alpha\left(\theta r - \frac{\sqrt{r}}{\sqrt{2\alpha t}} - \delta\right)^2 t}{(1-r)} - q(1-r)t}\,\mathrm{d}\theta \\ &= \frac{1}{\sqrt{2\alpha}} t \int_{\mathcal{I}} dr \int_{\frac{\sqrt{2\alpha t}(r-\delta)-\sqrt{r}}{\sqrt{1-r}}}^\infty e^{-q(1-r)t} e^{-z^2/2}\,\mathrm{d}z \\ &\le \frac{\sqrt{\pi}}{\sqrt{\alpha}} t \int_{\mathcal{I}} e^{-q(1-r)t}\mathrm{P}\left(B_1 \ge \frac{\sqrt{2\alpha t}(r-\delta)-1}{\sqrt{1-r}}\right)\mathrm{d}r. \end{aligned} \tag{3.18}$$

If $r \le \delta + \frac{2}{\sqrt{2\alpha t}}$, then

$$\begin{aligned} &e^{-q(1-r)t}\mathrm{P}\left(B_1 \ge \frac{\sqrt{2\alpha t}(r-\delta)-1}{\sqrt{1-r}}\right) \le e^{-q(1-r)t} \le e^{-q(1-\delta)t} e^{\frac{2q}{\sqrt{2\alpha}}\sqrt{t}} \\ =&e^{-2\alpha(\rho-1)(1-\delta)t} e^{-\alpha(\rho-1)^2(1-\delta)t} e^{\frac{2q}{\sqrt{2\alpha}}\sqrt{t}}. \end{aligned} \tag{3.19}$$

If $\delta + \frac{2}{\sqrt{2\alpha t}} < r < 1$, then $\frac{\sqrt{2\alpha t}(r-\delta)-1}{\sqrt{1-r}} > 1$, and thus by (2.38),

$$\begin{aligned} e^{-q(1-r)t}\mathrm{P}\left(B_1 \ge \frac{\sqrt{2\alpha t}(r-\delta)-1}{\sqrt{1-r}}\right) &\le \frac{1}{\sqrt{2\pi}} \frac{\sqrt{1-r}}{\sqrt{2\alpha t}(r-\delta)-1} e^{-q(1-r)t} e^{-\frac{(\sqrt{2\alpha t}(r-\delta)-1)^2}{2(1-r)}} \\ &\le e^{-q(1-r)t} e^{-\frac{\alpha\left(r-\delta-\frac{1}{\sqrt{2\alpha t}}\right)^2 t}{(1-r)}}. \end{aligned} \tag{3.20}$$

It follows from (3.16) that for $r \in \mathcal{I}$,

$$q(1-r) + \frac{\alpha(r-\delta-\frac{1}{\sqrt{2\alpha t}})^2}{(1-r)}$$

$$\geq 2\alpha(\rho-1)\left(1-\delta-\frac{1}{\sqrt{2\alpha t}}\right)+\alpha\rho^2\left(a_\delta-r+\frac{1}{\rho\sqrt{2\alpha t}}\right)^2$$
$$\geq 2\alpha(\rho-1)(1-\delta)+\alpha\rho^2\left(\epsilon-\frac{1}{\sqrt{2\alpha t}\rho}\right)^2-\sqrt{2\alpha}(\rho-1)t^{-1/2}.$$

Then we continue the estimates in (3.20) to get that, if $\delta+\frac{2}{\sqrt{2\alpha t}}<r<1$, then

$$e^{-q(1-r)t}\mathrm{P}\left(B_1\geq\frac{\sqrt{2\alpha t}(r-\delta)-1}{\sqrt{1-r}}\right)\leq e^{-2\alpha(\rho-1)(1-\delta)t}e^{-\alpha\rho^2\left(\epsilon-\frac{1}{\sqrt{2\alpha t}\rho}\right)^2t+\sqrt{2\alpha}(\rho-1)\sqrt{t}}. \tag{3.21}$$

Combining (3.18), (3.19) and (3.21), we get

$$\limsup_{t\to\infty}\frac{e^{2\alpha(\rho-1)(1-\delta)t}}{t^{3(\rho-1)/2}}I_3(t)=0.$$

The proof is now complete. □

Proof of Lemma 3.3: By Lemmas 3.4-3.6, we have (3.11) holds. Hence, by the paragraph above Lemma 3.4, the assertion of Lemma 3.3 follows. □

3.2 Proof of Theorem 1.2: $\delta=1-\rho$

It follows from Lemma 2.5 that, to prove Theorem 1.2, we only need to consider the limiting property of $v_f(t,\sqrt{2\alpha}\delta t)$. It follows from Lemma 3.1 that for $\delta=1-\rho<0$,

$$\lim_{t\to\infty}t^{-3(\rho-1)/4}e^{(q+\alpha(1-\rho)^2)t}U_{1,f}(t,\sqrt{2\alpha}(1-\rho)t)=0.$$

Thus, by the decomposition (2.10), to prove the desired result, it suffices to show that

$$\lim_{t\to\infty}t^{-3(\rho-1)/4}e^{(q+\alpha(\rho-1)^2)t}U_{2,f}(t,\sqrt{2\alpha}(\rho-1)t)$$
$$=\frac{1}{\sqrt{2\pi}}\int_0^\infty s^{3(\rho-1)/2}e^{-\alpha\rho^2s^2}\mathrm{d}s\int_{-\infty}^\infty e^{-\sqrt{2\alpha}(\rho-1)z}A(w_f(z))\mathrm{d}z.$$

The display above follows from Lemmas 3.7 and 3.8 below. In Lemma 3.8, we will show that

$$t^{-3(\rho-1)/4}e^{(q+\alpha(1-\rho)^2)t}\mathbf{P}\Big(M_t^I\leq\sqrt{2\alpha}(1-\rho)t,\tau\notin\Big[t-(\log t)\sqrt{t},t-t^{1/4}\Big]\Big)\to 0.$$

Thus, on the event $\left\{M_t^I\leq\sqrt{2\alpha}(1-\rho)t\right\}$, with large probability, the first branching time of the skeleton should happens in the interval $\Big[t-(\log t)\sqrt{t},t-t^{1/4}\Big]$.

Lemma 3.7 *It holds that for any $f\in\mathcal{H}$,*

$$\lim_{t\to\infty}t^{-3(\rho-1)/4}e^{(q+\alpha(1-\rho)^2)t}\mathrm{E}\int_{t^{1/4}}^{(\log t)\sqrt{t}}e^{-\int_s^t\zeta_f(r,\sqrt{2\alpha}(1-\rho)t-B_{t-r})\,dr}\hat{G}_f(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s})\,\mathrm{d}s$$
$$=\frac{1}{\sqrt{2\pi}}\int_0^\infty s^{3(\rho-1)/2}e^{-\alpha\rho^2s^2}\mathrm{d}s\int_{-\infty}^\infty e^{-\sqrt{2\alpha}(\rho-1)z}A(w_f(z))\mathrm{d}z.$$

Proof: In this proof, we always assume that $t \geq 1$ is large enough such that $\log t \leq \sqrt{t}$. Using an argument similar to that in the first paragraph of the proof of Lemma 3.2, we get that, as $t \to \infty$,

$$\begin{aligned}
&\mathrm{E}\int_{t^{1/4}}^{(\log t)\sqrt{t}} e^{-\int_s^t \zeta_f(r,\sqrt{2\alpha}(1-\rho)t-B_{t-r})\,dr}\hat{G}_f\left(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s}\right)\,\mathrm{d}s\\
\sim&\mathrm{E}\int_{t^{1/4}}^{(\log t)\sqrt{t}} e^{-q(t-s)}\hat{G}_f\left(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s}\right)\,\mathrm{d}s\\
=&\sqrt{t}\int_{t^{-1/4}}^{\log t}\frac{e^{-q(t-u\sqrt{t})}}{\sqrt{2\pi(t-u\sqrt{t})}}\mathrm{d}u\int_{\mathbb{R}} e^{-\frac{\left(m_{u\sqrt{t}}+z+\sqrt{2\alpha}(\rho-1)t\right)^2}{2(t-u\sqrt{t})}}\hat{G}_f\left(u\sqrt{t},m_{u\sqrt{t}}+z\right)\mathrm{d}z. \qquad (3.22)
\end{aligned}$$

For $u \in (t^{-1/4}, \log t)$, we have that

$$\begin{aligned}
(m_{u\sqrt{t}}+z+\sqrt{2\alpha}(\rho-1)t)^2 =& \left(\sqrt{2\alpha}(\rho-1)t+\sqrt{2\alpha}u\sqrt{t}-\frac{3}{2\sqrt{2\alpha}}\log(u\sqrt{t})+z\right)^2\\
=&2\alpha(\rho-1)^2t^2+2\alpha u^2t+4\alpha(\rho-1)ut\sqrt{t}-3(\rho-1)t\log(u\sqrt{t})\\
&+2\sqrt{2\alpha}(\rho-1)zt+R_4(t,u,z),
\end{aligned}$$

where

$$\begin{aligned}
R_4(t,u,z) =& \left(-\frac{3}{2\sqrt{2\alpha}}\log(u\sqrt{t})+z\right)^2-3u\sqrt{t}\log(u\sqrt{t})+2\sqrt{2\alpha}u\sqrt{t}z\\
\geq& -3(\log t)^2\sqrt{t}-2\sqrt{2\alpha}(\log t)\sqrt{t}|z|.
\end{aligned}$$

Using the Taylor expansion of $(1-x)^{-1}$, we obtain that, for $u \in (t^{-1/4}, \log t)$,

$$\frac{1}{2(t-u\sqrt{t})}=\frac{1}{2t}\frac{1}{1-u/\sqrt{t}}=\frac{1}{2t}\left(1+\frac{u}{\sqrt{t}}+\frac{u^2}{t}+R_5(t,u)\right),$$

where $|R_5(t,u)| \leq 2(\log t)^3 t^{-3/2}$. Thus

$$\begin{aligned}
\frac{(m_{u\sqrt{t}}+z+\sqrt{2\alpha}(\rho-1)t)^2}{2(t-u\sqrt{t})} =& \alpha(\rho-1)^2t+qu\sqrt{t}-\frac{3(\rho-1)}{4}\log t\\
&+\alpha\rho^2u^2+\sqrt{2\alpha}(\rho-1)z-\frac{3}{2}(\rho-1)\log(u)+R_6(t,u,z).
\end{aligned}$$

Here $\lim_{t\to\infty} R_6(t,u,z) = 0$ and there is a positive function $r^*(\cdot)$ with $\lim_{t\to\infty} r^*(t) = 0$ such that $-R_6(t,u,z) \leq r^*(t)(1+|z|)$ for all $u \in (t^{-1/4}, \log t)$. Now, using (3.22), we get that

$$\begin{aligned}
&\lim_{t\to\infty} t^{-3(\rho-1)/4}e^{(q+\alpha(1-\rho)^2)t}\mathrm{E}\int_{t^{1/4}}^{(\log t)\sqrt{t}} e^{-\int_s^t \zeta_f(r,\sqrt{2\alpha}(1-\rho)t-B_{t-r})\,dr}\hat{G}_f\left(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s}\right)\,\mathrm{d}s\\
=&\lim_{t\to\infty}\int_{t^{-1/4}}^{\log t}\frac{\sqrt{t}}{\sqrt{2\pi(t-u\sqrt{t})}}u^{3(\rho-1)/2}e^{-\alpha\rho^2u^2}\mathrm{d}u\int_{\mathbb{R}} e^{-\sqrt{2\alpha}(\rho-1)z}e^{-R_6(t,u,z)}\hat{G}_f\left(u\sqrt{t},m_{u\sqrt{t}}+z\right)\mathrm{d}z.
\end{aligned}$$

Using an arguments similar to those in the proof of Lemma 3.2, the desired result follows from the the dominated convergence theorem. □

Lemma 3.8 *It holds that for any $f \in \mathcal{B}^+(\mathbb{R})$,*

$$\lim_{t\to\infty} t^{-3(\rho-1)/4} e^{(q+\alpha(1-\rho)^2)t} \mathrm{E} \int_{[0,t]\setminus(t^{1/4},(\log t)\sqrt{t})} e^{-\int_s^t \zeta_f(r,\sqrt{2\alpha}(1-\rho)t-B_{t-r})\,dr} \hat{G}_f(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s})\,\mathrm{d}s$$
$$= 0.$$

Proof: We only need to show that

$$\lim_{t\to\infty} \frac{e^{(q+\alpha(1-\rho)^2)t}}{t^{3(\rho-1)/4}} \mathrm{E} \int_{(0,t)\setminus(t^{1/4},(\log t)\sqrt{t})} e^{-q(t-s)} v^2(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s})\,\mathrm{d}s = 0.$$

We prove the above result in three steps.

Step 1: By (3.12), we have that

$$\mathrm{E}\left(v^2(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s})\right) \le \mathrm{P}\left(B_t \le \sqrt{2\alpha}(1-\rho)t\right) \le \frac{1}{2\sqrt{\pi\alpha}(\rho-1)\sqrt{t}} e^{-\alpha(\rho-1)^2 t}.$$

Thus, for any $T > 0$,

$$\begin{aligned} &\frac{e^{(q+\alpha(\rho-1)^2)t}}{t^{3(\rho-1)/4}} \mathrm{E} \int_0^T e^{-q(t-s)} v^2(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s})\,\mathrm{d}s \\ \le &\int_0^T e^{qs}\,\mathrm{d}s \frac{1}{2\sqrt{\pi\alpha}(\rho-1)\sqrt{t}} \frac{1}{t^{3(\rho-1)/4}} \to 0, \quad \text{as } t\to\infty. \end{aligned} \tag{3.23}$$

Step 2: Using arguments similar to those in the proofs of Lemmas 3.5 and 3.6, we get that,

$$\frac{e^{(q+\alpha(\rho-1)^2)t}}{t^{3(\rho-1)/4}} \mathrm{E} \int_{\sqrt{t}\log t}^t e^{-q(t-s)} v^2(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s})\,\mathrm{d}s \to 0, \quad \text{as } t\to\infty.$$

Step 3: Note that there exists T_0 such that $m_s > 0$ for all $s > T_0$. Using Lemma 2.10, we get that, for η small enough, there exist $c_\eta > 1$ and $T_\eta > 1$ such that for $T > T_\eta + T_0$,

$$\begin{aligned} &\mathrm{E} \int_T^{t^{1/4}} e^{-q(t-s)} v^2(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s})\,\mathrm{d}s \\ =&\mathrm{E} \int_T^{t^{1/4}} e^{-q(t-s)} v^2(s, m(s)-(m(s)+\sqrt{2\alpha}(\rho-1)t+B_{t-s}))\,\mathrm{d}s \\ \le& c_\eta^2 \int_T^{t^{1/4}} e^{-q(t-s)} \mathrm{E}[e^{-2(\sqrt{2\alpha}(\rho-1)-\eta)(m(s)+\sqrt{2\alpha}(\rho-1)t+B_{t-s})} \wedge 1]\,\mathrm{d}s. \end{aligned} \tag{3.24}$$

Similar to (3.15), we have that, for $T < s < t^{1/4}$,

$$\begin{aligned} &\mathrm{E}[e^{-2(\sqrt{2\alpha}(\rho-1)-\eta)(m(s)+\sqrt{2\alpha}(\rho-1)t+B_{t-s})} \wedge 1] \\ \le& Ct^{-1/2} e^{-\frac{(m(s)+\sqrt{2\alpha}(\rho-1)t)^2}{2(t-s)}} \\ \le& Ct^{-1/2} t^{\frac{3(\rho-1)}{8}} e^{-\alpha(\rho-1)^2 t - qs} \end{aligned} \tag{3.25}$$

with C being a positive constant. Here in the last inequality, we used the fact that

$$\begin{aligned}\frac{(m(s)+\sqrt{2\alpha}(\rho-1)t)^2}{2(t-s)} &= \frac{(\sqrt{2\alpha}\rho s-\frac{3}{2\sqrt{2\alpha}}\log s+\sqrt{2\alpha}(\rho-1)(t-s))^2}{2(t-s)}\\ &\geq \alpha(\rho-1)^2(t-s)+\sqrt{2\alpha}(\rho-1)\left(\sqrt{2\alpha}\rho s-\frac{3}{2\sqrt{2\alpha}}\log s\right)\\ &= \alpha(\rho-1)^2 t+qs-\frac{3}{2}(\rho-1)\log s.\end{aligned}$$

Putting (3.25) back to (3.24), we get that

$$\frac{e^{(q+\alpha(\rho-1)^2)t}}{t^{3(\rho-1)/4}}\mathrm{E}\int_T^{t^{1/4}} e^{-q(t-s)}v^2(s,\sqrt{2\alpha}(1-\rho)t-B_{t-s})\,\mathrm{d}s \leq Ct^{-1/4}t^{\frac{-3(\rho-1)}{8}} \to 0, \quad \text{as } t\to\infty.$$

Now the proof is complete. □

3.3 Proof of Theorem 1.3 : $\delta < 1-\rho$

Note that, by Proposition 2.6, we have

$$\begin{aligned}v_f(t,x) =&\mathrm{E}\Big[e^{-\int_0^t \psi'(\lambda^*+u_f^*(t-r,x-B_r))\,dr}, B_t\leq x\Big]\\ &+\mathrm{E}\int_0^t e^{-\int_0^s \psi'(\lambda^*+u_f^*(t-r,x-B_r))\,dr}\hat{G}_f(t-s,x-B_s)\,\mathrm{d}s.\end{aligned}$$

Using the same arguments as those in [12, Lemma 1.5, page 1211] or [18, Proposition 2.9], the above integral equation implies that

$$v_f(t,x) = e^{-qt}\mathrm{E}\Big[B_t\leq x\Big]+\mathrm{E}\int_0^t e^{-q(t-s)}G_f(s,x-B_{t-s})\,\mathrm{d}s, \tag{3.26}$$

where

$$\begin{aligned}G_f(t,x) :&= \hat{G}_f(t,x)-(\psi'(\lambda^*+u_f^*(t,x))-q)v_f(t,x)\\ &= \frac{1}{\lambda^*}\Big[\psi(\lambda^*+u_f^*(t,x)-\lambda^* v_f(t,x))-\psi(\lambda^*+u_f^*(t,x))\Big]+qv_f(t,x).\end{aligned} \tag{3.27}$$

It follows from Lemma 2.5 that, to prove Theorem 1.3, we only need to consider the limiting property of $v_f(t,\sqrt{2\alpha}\delta t)$. Using L'Hospital's rule, one has that

$$\lim_{x\to\infty}\frac{\mathrm{P}(B_1>x)}{x^{-1}e^{-x^2/2}} = \frac{1}{\sqrt{2\pi}}\lim_{x\to\infty}\frac{\int_x^\infty e^{-y^2/2}\,\mathrm{d}y}{x^{-1}e^{-x^2/2}} = \frac{1}{\sqrt{2\pi}}. \tag{3.28}$$

It follows that

$$\lim_{t\to\infty}\sqrt{t}e^{(q+\alpha\delta^2)t}e^{-qt}\mathrm{E}\Big[B_t\leq\sqrt{2\alpha}\delta t\Big] = \lim_{t\to\infty}\sqrt{t}e^{(q+\alpha\delta^2)t}e^{-qt}\mathrm{E}\Big[B_1\geq\sqrt{2\alpha}|\delta|\sqrt{t}\Big] = \frac{1}{2\sqrt{\pi\alpha}|\delta|}. \tag{3.29}$$

Hence, by (3.26), to prove the desired result, we only need to prove that

$$\begin{aligned}&\lim_{t\to\infty}\sqrt{t}e^{(q+\alpha\delta^2)t}\mathrm{E}\int_0^t e^{-q(t-s)}G_f(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s\\=&\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}}e^{\sqrt{2\alpha}\delta z}G_f(s,z)\,\mathrm{d}z,\end{aligned}$$

which will follow from Lemmas 3.9 and 3.10 below. In Lemma 3.10, we will show that, for any $T>0$,

$$\sqrt{t}e^{(q+\alpha\delta^2)t}\mathbf{P}\Big(M_t^I\le\sqrt{2\alpha}\delta t,\tau\in[0,t-T]\Big)\to 0.$$

Thus, on the event $\{M_t^I\le\sqrt{2\alpha}\delta t\}$, with large probability, the first branching of the skeleton happens in the interval $[t-T,t]$.

Lemma 3.9 *If $\delta<1-\rho$, then for any $f\in\mathcal{B}_b^+(\mathbb{R})$ and any $T>0$, it holds that*

$$\begin{aligned}&\lim_{t\to\infty}\sqrt{t}e^{(q+\alpha\delta^2)t}\mathrm{E}\int_0^{t-T} e^{-q(t-s)}G_f(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s\\=&\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}}e^{\sqrt{2\alpha}\delta z}G_f(s,z)\,\mathrm{d}z.\end{aligned}$$

Proof: Note that

$$\begin{aligned}&\sqrt{t}e^{(q+\alpha\delta^2)t}\mathrm{E}\int_0^{t-T} e^{-q(t-s)}G_f(s,\sqrt{2\alpha}\delta t-B_{t-s})\,\mathrm{d}s\\=&\int_0^{t-T}\frac{\sqrt{t}}{\sqrt{2\pi(t-s)}}e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}}e^{\sqrt{2\alpha}\delta z}e^{-\frac{(z-\sqrt{2\alpha}\delta s)^2}{2(t-s)}}G_f(s,z)\,\mathrm{d}z.\end{aligned}$$

The absolute value of the integrand above is less than $\frac{1}{\sqrt{2\pi}}\sqrt{1+s/T}e^{(q-\alpha\delta^2)s}e^{\sqrt{2\alpha}\delta z}|G_f(s,z)|$, thus by the dominated convergence theorem, it suffices to show that

$$\int_0^\infty\sqrt{s+T}e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}}e^{\sqrt{2\alpha}\delta z}|G_f(s,z)|\,\mathrm{d}z<\infty.\tag{3.30}$$

By (3.27), (2.28) and the fact that $v_f(t,x)\le v(t,x)$, we have that

$$|G_f(s,z)|\le\phi(u_f^*(s,z))v_f(s,z)+\hat{G}_f(s,z)\le\phi(u_f^*(s,z))v(s,z)+qv(s,z)^2.\tag{3.31}$$

We will prove (3.30) in two steps. Recall that $k(t)=-\log\mathbb{P}^*(\|X_t\|=0)$.

Step 1: First we consider the integral over $s\in(0,A)$, where $A>0$ is a constant. Since ϕ is increasing, by Lemma 2.7(1), $\phi(u_f^*(s,z))\le\phi(k(s))$. By lemma 2.9(1), $v(s,z)\le\mathrm{P}(B_s\le z)=\mathrm{P}(B_1\le z/\sqrt{s})$. Thus we have for $0<s<A$,

$$\begin{aligned}&\int_{-\infty}^0 e^{\sqrt{2\alpha}\delta z}\phi(u_f^*(s,z))v(s,z)\,\mathrm{d}z\le\phi(k(s))\int_{-\infty}^0 e^{\sqrt{2\alpha}\delta z}\mathrm{P}(B_1\le z/\sqrt{s})\mathrm{d}z\\=&\sqrt{s}\phi(k(s))\int_0^\infty e^{\sqrt{2\alpha}|\delta|\sqrt{s}z}\mathrm{P}(B_1\ge z)\mathrm{d}z\le\sqrt{s}\phi(k(s))\int_0^\infty e^{\sqrt{2\alpha}|\delta|\sqrt{A}z}\mathrm{P}(B_1\ge z)\mathrm{d}z.\end{aligned}$$

Since $\mathrm{P}(B_1 \geq z) \sim \frac{1}{\sqrt{2\pi}} z^{-1} e^{-z^2/2}$ as $z \to \infty$, we have $\int_0^\infty e^{\sqrt{2\alpha}|\delta|\sqrt{A}z}\mathrm{P}(B_1 \geq z)\mathrm{d}z < \infty$. Thus

$$\int_{-\infty}^{0} e^{\sqrt{2\alpha}\delta z}\phi(u_f^*(s,z))v(s,z)\,\mathrm{d}z \leq C\sqrt{s}\phi(k(s)). \tag{3.32}$$

For any $\epsilon > 0$, since $v(s,z) \leq 1$, we have

$$\int_{0}^{s^\epsilon} e^{\sqrt{2\alpha}\delta z}\phi(u_f^*(s,z))v(s,z)\,\mathrm{d}z \leq s^\epsilon\phi(k(s)). \tag{3.33}$$

By (3.32), (3.33) and Lemma 2.8, for any $\epsilon > 0$,

$$\int_0^A \sqrt{s+T}e^{(q-\alpha\delta^2)s}\int_{-\infty}^{s^\epsilon} e^{\sqrt{2\alpha}\delta z}\phi(u_f^*(s,z))v(s,z)\,\mathrm{d}z\mathrm{d}s < \infty. \tag{3.34}$$

Since $\phi'(\lambda) = \psi''(\lambda^* + \lambda)$ is decreasing and $\phi(0) = 0$, we have

$$\phi(\lambda) \leq \phi'(0)\lambda. \tag{3.35}$$

Thus, by (2.30),

$$\phi(u_f^*(s,z)) \leq \phi'(0)u_f^*(s,z) \leq C(1 + z^{-2/\vartheta})e^{(a+\alpha)s}, \quad z > 0.$$

Since $v(s,z) \leq 1$, we have for $0 < s < A$,

$$\begin{aligned}&\int_{s^\epsilon}^\infty e^{\sqrt{2\alpha}\delta z}\phi(u_f^*(s,z))v(s,z)\,\mathrm{d}z \leq Ce^{(a+\alpha)s}\int_{s^\epsilon}^\infty e^{-\sqrt{2\alpha}|\delta|z}(1 + z^{-2/\vartheta})\mathrm{d}z\\ &\leq Ce^{(a+\alpha)A}\Big[\int_{s^\epsilon}^{A^\epsilon}(1 + z^{-2/\vartheta})\mathrm{d}z + \int_{A^\epsilon}^\infty e^{-\sqrt{2\alpha}|\delta|z}(1 + z^{-2/\vartheta})\mathrm{d}z\Big] \leq C(1 + s^{\epsilon(1-2/\vartheta)}).\end{aligned}$$

Now we choose ϵ small enough such that $\epsilon(2/\vartheta - 1) < 1$. Thus

$$\int_0^A \sqrt{s+T}e^{(q-\alpha\delta^2)s}\int_{s^\epsilon}^{\infty} e^{\sqrt{2\alpha}\delta z}\phi(u_f^*(s,z))v(s,z)\,\mathrm{d}z\mathrm{d}s < \infty. \tag{3.36}$$

Combining (3.34) and (3.36), we obtain that

$$\int_0^A \sqrt{s+T}e^{(q-\alpha\delta^2)s}\int_{-\infty}^{\infty} e^{\sqrt{2\alpha}\delta z}\phi(u_f^*(s,z))v(s,z)\,\mathrm{d}z\mathrm{d}s < \infty. \tag{3.37}$$

Step 2: By Lemma 2.7(1), $\sup_{s>A} e^{qs}k(s) = e^{qA}k(A) < \infty$. Hence we have for $s > A$,

$$\phi(u_f^*(s,z)) \leq \phi'(0)u_f^*(s,z) \leq \phi'(0)k(s) \leq \phi'(0)e^{qA}k(A)e^{-qs}.$$

Thus we get that, for $s > A$,

$$\begin{aligned}&\int_{\mathbb{R}} e^{\sqrt{2\alpha}\delta z}\phi(u_f^*(s,z))v(s,z)\,\mathrm{d}z \leq Ce^{-qs}\int_{\mathbb{R}} e^{\sqrt{2\alpha}\delta z}v(s,z)\,\mathrm{d}z\\ =&C\sqrt{2\alpha}se^{-qs}e^{-\sqrt{2\alpha}\delta\sqrt{s}}\int_{\mathbb{R}} e^{2\alpha\delta s\theta}v(s,\sqrt{2\alpha}\theta s - \sqrt{s})\,\mathrm{d}\theta.\end{aligned} \tag{3.38}$$

We will divide the above integral into three parts: $\int_1^\infty + \int_{1-\rho}^1 + \int_{-\infty}^{1-\rho}$. We deal with them one by one. Using Lemma 2.9(2), we have that for $A > t_0$ and $s > A$,

$$\int_1^\infty e^{2\alpha\delta s\theta} v(s, \sqrt{2\alpha}\theta s - \sqrt{s})\,\mathrm{d}\theta \le \int_1^\infty e^{-2\alpha|\delta|s\theta}\,\mathrm{d}\theta = \frac{1}{\sqrt{2\alpha}|\delta|s} e^{-2\alpha|\delta|s},$$

$$\begin{aligned}\int_{1-\rho}^1 e^{2\alpha\delta s\theta} v(s, \sqrt{2\alpha}\theta s - \sqrt{s})\,\mathrm{d}\theta &\le cs\int_{1-\rho}^1 e^{2\alpha\delta s\theta} e^{-2\alpha(\rho-1)(1-\theta)s}\,\mathrm{d}\theta \\ &\le cs\rho e^{-2\alpha(\rho-1)(\rho+\delta)s},\end{aligned}$$

and

$$\begin{aligned}&\int_{-\infty}^{1-\rho} e^{2\alpha\delta s\theta} v(s, \sqrt{2\alpha}\theta s - \sqrt{s})\,\mathrm{d}\theta \le cs\int_{-\infty}^{1-\rho} e^{2\alpha\delta s\theta} e^{-(q+\alpha\theta^2)s}\,\mathrm{d}\theta \\ =& cse^{(-q+\alpha\delta^2)s}\int_{-\infty}^{1-\rho} e^{-\alpha s(\theta-\delta)^2}\mathrm{d}\theta \le Cs^{1/2}e^{(-q+\alpha\delta^2)s}.\end{aligned}$$

For $\delta < 1-\rho$, one can check that

$$2\alpha\delta \le -2\alpha(\rho-1)(\rho+\delta) \le -q+\alpha\delta^2.$$

Thus for $s > A$,

$$\int_{-\infty}^\infty e^{2\alpha\delta s\theta} v(s, \sqrt{2\alpha}\theta s - \sqrt{s})\,\mathrm{d}\theta \le Cse^{(-q+\alpha\delta^2)s}. \tag{3.39}$$

It follows from (3.38) and (3.39) that

$$\begin{aligned}&\int_A^\infty \sqrt{s+T}e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}} e^{\sqrt{2\alpha}\delta z}\phi(u^*(s,z))v(s,z)\,\mathrm{d}z \\ \le& C\int_A^\infty \sqrt{s+T}s^2e^{-qs}e^{-\sqrt{2\alpha}\delta\sqrt{s}}\,\mathrm{d}s < \infty.\end{aligned}$$

Combining the two steps above, we get

$$\int_0^\infty \sqrt{s+T}e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}} e^{\sqrt{2\alpha}\delta z}\phi(u_f^*(s,z))v(s,z)\,\mathrm{d}z < \infty.$$

Similarly, one can prove that

$$\int_0^\infty \sqrt{s+T}e^{(q-\alpha\delta^2)s}\,\mathrm{d}s\int_{\mathbb{R}} e^{\sqrt{2\alpha}\delta z}v(s,z)^2\,\mathrm{d}z < \infty.$$

Hence (3.30) holds and the desired result follows immediately. □

Lemma 3.10 *If $\delta < 1-\rho$, then for any $f \in \mathcal{B}_b^+(\mathbb{R})$ and $T > 0$,*

$$\lim_{t\to\infty}\sqrt{t}e^{(q+\alpha\delta^2)t}\mathrm{E}\int_{t-T}^t e^{-q(t-s)}G_f(s, \sqrt{2\alpha}\delta t - B_{t-s})\,\mathrm{d}s = 0.$$

Proof: Note that

$$\mathrm{E}\int_{t-T}^{t} e^{-q(t-s)}|G_f(s,\sqrt{2\alpha}\delta t-B_{t-s})|\,\mathrm{d}s=\int_0^T e^{-qs}\mathrm{E}|G_f(t-s,\sqrt{2\alpha}\delta t-B_s)|\,\mathrm{d}s$$
$$=\int_0^T e^{-qs}\mathrm{E}[|G_f(t-s,\sqrt{2\alpha}\delta t-B_s)|;B_s<-(\epsilon t-\sqrt{t})]\,\mathrm{d}s$$
$$+\int_0^T e^{-qs}\mathrm{E}[|G_f(t-s,\sqrt{2\alpha}\delta t-B_s)|;B_s\geq-(\epsilon t-\sqrt{t})]\,\mathrm{d}s,$$

where $\epsilon<1-\rho-\delta$ is a small constant.

By (3.35) and Lemma 2.7(1), $\sup_{t>1}\phi(u_f^*(t,x))\leq\phi'(0)\sup_{t>1}u_f^*(t,x)\leq\phi'(0)k(1)<\infty$. Since $v(t,x)\leq 1$, we have $\sup_{t>1}\sup_x|G_f(t,x)|<+\infty$. Hence we have, for $t>1$ large enough, and $s\in(0,T)$,

$$\mathrm{E}\left[|G_f(t-s,\sqrt{2\alpha}\delta t-B_s)|;B_s\leq-(\epsilon t-\sqrt{t})\right]\leq C\mathrm{P}\left(B_s\geq(\epsilon t-\sqrt{t})\right)$$
$$\leq C\frac{\sqrt{s}}{\epsilon t-\sqrt{t}}e^{-(\epsilon t-\sqrt{t})^2/(2s)}\leq C\frac{\sqrt{T}}{\epsilon t-\sqrt{t}}e^{-(\epsilon t-\sqrt{t})^2/(2T)},$$

where in the second inequality, we used (2.38).

Thus for any $\epsilon>0$, as $t\to\infty$,

$$\sqrt{t}e^{(q+\alpha\delta^2)t}\int_0^T e^{-qs}\mathrm{E}\left[|G_f(t-s,\sqrt{2\alpha}\delta t-B_s)|;B_s<-(\epsilon t-\sqrt{t})\right]\,\mathrm{d}s\to 0.$$

Note that if $B_s\geq-(\epsilon t-\sqrt{t})$, then

$$\sqrt{2\alpha}\delta t-B_s\leq\sqrt{2\alpha}(\delta+\epsilon)t-\sqrt{t}\leq\sqrt{2\alpha}(\delta+\epsilon)(t-s)-\sqrt{t-s}.$$

Using Lemma 2.9(2) with $\theta=\delta+\epsilon<1-\rho$, for $t>t_0+T$ and $s\in(0,T)$,

$$v(t-s,\sqrt{2\alpha}\delta t-B_s)\leq v(t-s,\sqrt{2\alpha}(\delta+\epsilon)(t-s)-\sqrt{t-s})\leq cte^{-q(t-s)}e^{-\alpha(\delta+\epsilon)^2(t-s)}.$$

By Lemma 2.7(1), we have that for $t\geq t_0+T$ and $s\in(0,T)$,

$$\phi(u_f^*(t-s,\sqrt{2\alpha}\delta t+z))\leq\phi'(0)u_f^*(t-s,\sqrt{2\alpha}\delta t+z)$$
$$\leq\phi'(0)k(t-s)\leq\phi'(0)e^{qt_0}k(t_0)e^{-q(t-s)}.$$

Thus, by (3.31), we get that, if $B_s\geq-(\epsilon t-\sqrt{t})$,

$$|G_f(t-s,\sqrt{2\alpha}\delta t-B_s)|\leq Ct^2e^{-2q(t-s)}e^{-\alpha(\delta+\epsilon)^2(t-s)}$$
$$\leq Ce^{2qs}e^{\alpha(\delta+\epsilon)^2s}t^2e^{-2qt}e^{-\alpha\delta^2t}e^{-2\alpha\delta\epsilon t}. \tag{3.40}$$

It follows that, as $t\to\infty$,

$$\sqrt{t}e^{(q+\alpha\delta^2)t}\int_0^T e^{-qs}\mathrm{E}[|G_f(t-s,\sqrt{2\alpha}\delta t-B_s)|;B_s\geq-(\epsilon t-\sqrt{t})]\,\mathrm{d}s$$
$$\leq Ct^{5/2}e^{-(q+2\alpha\delta\epsilon)t}\int_0^T e^{qs}e^{\alpha(\delta+\epsilon)^2s}\mathrm{d}s\leq Ct^{5/2}e^{-(q+2\alpha\delta\epsilon)t}\to 0, \tag{3.41}$$

if we choose ϵ small enough such that $q+2\alpha\delta\epsilon>0$. The proof is now complete. □

A Appendix

Lemma A.1 *For $k \geq 1$,*

$$\mathbf{P}(\|Z_t\| \leq k) \leq ke^{-qt}.$$

Proof: Let Z'_t be a continuous time branching process with branching rate q, and when a particle dies, it splits into two particles. Then Z'_t is a pure birth process, and the distribution of Z'_t is given by

$$\mathbf{P}(Z'_t \leq k) = 1 - (1 - e^{-qt})^k.$$

According to the definition of Z_t, each particle splits into at least two children ($p_0 = p_1 = 0$), then we get that

$$\mathbf{P}(\|Z_t\| \leq k) \leq \mathbf{P}(Z'_t \leq k) = 1 - (1 - e^{-qt})^k \leq ke^{-qt}.$$

□

Proof of Lemma 2.10: Since $v(t, m_t - z) \leq 1$, it is clear that the desired result is valid for $z < 1$. In the following, we only need to consider the case $z \geq 1$. Put $a^* = \sqrt{2\alpha}(\rho - 1)/q$. Assume that $\eta \in (0, a^*/2)$ and $t \geq 1$.

(i) First we deal with the case $z > \frac{a^*}{\eta}\sqrt{t}$. Since for any θ,

$$q + \alpha\theta^2 - 2\alpha(\rho - 1)(1 - \theta) = \alpha(\rho - 1 + \theta)^2 \geq 0,$$

which implies that

$$q + \alpha\theta^2 \geq 2\alpha(\rho - 1)(1 - \theta). \tag{A.1}$$

Then by Lemma 2.9(2), one has that there exist $t_0 > 1$ and $c > 0$ such that, for any $t > t_0$ and $\theta < 1$,

$$v(t, \sqrt{2\alpha}\theta t - \sqrt{t}) \leq \mathbf{P}(M_t^Z \leq \sqrt{2\alpha}\theta t - \sqrt{t}) \leq cte^{-2\alpha(\rho-1)(1-\theta)t}, \tag{A.2}$$

where in the last inequality we used (A.1) to get an uniform upper bound for the two cases in Lemma 2.9(2). Thus, using the above inequality with $\theta = 1 - \frac{z-\sqrt{t}}{\sqrt{2\alpha}t} < 1$, we get that for any $t > t_0$,

$$\begin{aligned}\mathbf{P}(M_t^Z \leq m_t - z) &\leq \mathbf{P}(M_t^Z \leq \sqrt{2\alpha}t - z) = \mathbf{P}(M_t^Z \leq \sqrt{2\alpha}\theta t - \sqrt{t}) \\ &\leq cte^{-\sqrt{2\alpha}(\rho-1)(z-\sqrt{t})} \leq cz^2 e^{-\sqrt{2\alpha}(\rho-1)z} e^{q\eta z},\end{aligned} \tag{A.3}$$

where in the final inequality, we use the fact that $t \leq (\frac{\eta}{a^*}z)^2 \leq z^2$ and $\sqrt{2\alpha}(\rho - 1)\sqrt{t} = qa^*\sqrt{t} \leq q\eta z$.

(ii) Now we consider the case $z \in [1, \frac{a^*}{\eta}\sqrt{t}]$. Put $K := [a^*/\eta]$. Note that $K \geq 1$. Define $s_n = \eta z n$. In the following, we always assume that t is large enough such that $s_K < t$. Note that

$$\mathbf{P}(M_t^Z \leq m_t - z) \leq \mathbf{P}(\|Z_{s_K}\| \leq z^2) + \sum_{l=1}^{K} \mathbf{P}\left(\|Z_{s_{l-1}}\| \leq z^2 < \|Z_{s_l}\|, M_t^Z \leq m_t - z\right). \tag{A.4}$$

By Lemma A.1, we have that

$$\mathbf{P}(\|Z_{s_K}\| \le z^2) \le z^2 e^{-qs_K} = z^2 e^{-q\eta K z} \le z^2 e^{q\eta z} e^{-\sqrt{2\alpha}(\rho-1)z}. \tag{A.5}$$

Now we deal with the second part of the right-hand side of (A.4). For any $s \ge 0$, let $\mathcal{L}_s$ be the set of all particles of Z alive at time s. Suppose $1 \le l \le K$. Note that for any $u \in \mathcal{L}_{s_l}$, $z_u(s_l) \overset{d}{=} Y \sim N(0, s_l)$. Let $M_t^{Z,u} := \max_{v \in \mathcal{L}_t, u \preccurlyeq v} z_v(t) - z_u(s_l)$, for any $u \in \mathcal{L}_{s_l}$. By the branching property of Z, given $\sigma(\|Z_s\|, s \in [0, s_l])$, $\{M_t^{Z,u}, u \in \mathcal{L}_{s_l}\}$ are i.i.d. with the same distribution as $(M_{t-s_l}^Z, \mathbf{P})$, and independent of $\{z_u(s_l), u \in \mathcal{L}_{s_l}\}$. It is clear that

$$M_t^Z = \max_{u \in \mathcal{L}_{s_l}} [z_u(s_l) + M_t^{Z,u}].$$

It follows from [14, Lemma 5.1] that

$$\begin{aligned}
&\mathbf{P}\left(M_t^Z \le m_t - z \,\middle|\, \sigma(\|Z_s\|, s \in [0, s_l])\right) \\
\le& \mathbf{P}\left(Y + \max_{u \in \mathcal{L}_{s_l}} M_t^{Z,u} \le m_t - z \,\middle|\, \sigma(\|Z_s\|, s \in [0, s_l])\right)
\end{aligned}$$

Since Y is independent of $\sigma(\|Z_s\|, s \in [0, s_l]$, we continue the above estimation:

$$\begin{aligned}
&\mathbf{P}\left(M_t^Z \le m_t - z \,\middle|\, \sigma(\|Z_s\|, s \in [0, s_l])\right) \\
\le& \mathbf{P}(Y \le m_t - m_{t-s_l} - z) + \mathbf{P}\left(\max_{u \in \mathcal{L}_{s_l}} M_t^{Z,u} \le m_{t-s_l} | \sigma(\|Z_s\|, s \in [0, s_l])\right) \\
=& \mathbf{P}\left(Y \le m_t - m_{t-s_l} - z\right) + \left[\mathbf{P}(M_{t-s_l}^Z \le m_{t-s_l})\right]^{\|Z_{s_l}\|}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbf{P}\left(\|Z_{s_{l-1}}\| \le z^2 < \|Z_{s_l}\|, M_t^Z \le m_t - z\right) \\
\le& \mathbf{P}\left(\|Z_{s_{l-1}}\| \le z^2 < \|Z_{s_l}\|\right)\mathrm{P}(Y \le m_t - m_{t-s_l} - z\big) + \left[\mathbf{P}(M_{t-s_l}^Z \le m_{t-s_l})\right]^{z^2}.
\end{aligned}$$

Since $t - s_l \ge t - s_K \ge t - a^* z \ge t - \frac{(a^*)^2}{\eta}\sqrt{t} \to \infty$ as $t \to \infty$ and $\lim_{t\to\infty} \mathbf{P}(M_t^Z \le m_t) \in (0, 1)$, there exist $t(\eta) > 1$ and $c_0 > 0$ such that for all $t > t(\eta)$,

$$[\mathbf{P}(M_{t-s_l}^Z \le m_{t-s_l})]^{z^2} \le e^{-c_0 z^2}. \tag{A.6}$$

As $m_t - m_{t-s_l} - z \le -z(1 - \sqrt{2\alpha}\eta l)$ and $\sqrt{2\alpha}\eta l \le \sqrt{2\alpha}\eta K \le \frac{2\alpha(\rho-1)}{q} = \frac{2}{\rho+1} < 1$, we have by Lemma A.1,

$$\begin{aligned}
&\mathbf{P}\left(\|Z_{s_{l-1}}\| \le z^2 < \|Z_{s_l}\|\right)\mathbf{P}\left(Y \le m_t - m_{t-s_l} - z\right) \\
\le& z^2 e^{-q(l-1)\eta z}\mathrm{P}\left(B_{s_l} \le -z(1 - \sqrt{2\alpha}\eta l)\right) \\
=& z^2 e^{-q(l-1)\eta z}\mathrm{P}\left(B_1 \ge \sqrt{z}\Big(\frac{1}{\sqrt{\eta l}} - \sqrt{2\alpha}\sqrt{\eta l}\Big)\right) \\
\le& \frac{1}{\sqrt{2\pi}} z^{3/2} e^{-q(l-1)\eta z}\Big(\frac{1}{\sqrt{\eta l}} - \sqrt{2\alpha}\sqrt{\eta l}\Big)^{-1} e^{-\frac{1}{2}(\frac{1}{\sqrt{\eta l}} - \sqrt{2\alpha}\sqrt{\eta l})^2 z}
\end{aligned}$$

$$\leq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{a^*}}-\sqrt{2\alpha}\sqrt{a^*}\right)^{-1} z^{3/2}e^{q\eta z}e^{-\sqrt{2\alpha}(\rho-1)z}. \tag{A.7}$$

Here in the second inequality we used (2.38), and in the final inequality we used the facts that $\eta l \leq \eta K \leq a^*$ and $\frac{1}{2}(\frac{1}{\sqrt{\eta l}}-\sqrt{2\alpha}\sqrt{\eta l})^2+q\eta l=(\alpha+q)\eta l+\frac{1}{2l\eta}-\sqrt{2\alpha}\geq\sqrt{2\alpha}(\rho-1)$.

Combining (A.3)-(A.7), we get that for any $\eta\in(0,a^*/2)$, there exist t_η and $c_0, C>0$ such that for $t>t_\eta+t_0$ and $z\geq 1$,

$$\mathbf{P}(M_t^Z\leq m_t-z)\leq C(z^2e^{q\eta z}e^{-\sqrt{2\alpha}(\rho-1)z}+e^{-c_0z^2}).$$

Since $z^2\leq 2(q\eta)^{-2}e^{q\eta z}$, and $c_0z^2\geq qa^*z-\frac{(qa^*)^2}{4c_0}$, thus

$$\mathbf{P}(M_t^Z\leq m_t-z)\leq C\left(2(q\eta)^{-2}+e^{\frac{(qa^*)^2}{4c_0}}\right)e^{2q\eta z}e^{-\sqrt{2\alpha}(\rho-1)z}.$$

The proof is now complete. □

Lemma A.2 *For any $x\in(0,1)$ and $c\in\mathbb{R}$,*

$$q(1-x)+\frac{\alpha(x-c)^2}{1-x}\geq 2\alpha(\rho-1)(1-c)+\alpha\rho^2\Big(1-\frac{1-c}{\rho}-x\Big)^2.$$

Proof: Note that the function $(0,\infty)\ni x\to g(x)=a_1^2x+\frac{a_2^2}{x}$ achieves its minimum $2a_1a_2$ at the point $x=a_2/a_1$ and for any $x>0$,

$$g(x)=2a_1a_2+\frac{a_1^2}{x}(x-a_2/a_1)^2. \tag{A.8}$$

Then we have that for any $x\in(0,1)$

$$\begin{aligned}
q(1-x)+\frac{\alpha(x-c)^2}{1-x} &= (\alpha+q)(1-x)+\frac{\alpha(1-c)^2}{1-x}-2\alpha(1-c)\\
&=\alpha[\rho^2(1-x)+\frac{(1-c)^2}{1-x}-2(1-c)]\\
&=\alpha\Big[2(\rho-1)(1-c)+\frac{\rho^2}{1-x}\left(1-\frac{1-c}{\rho}-x\right)^2\Big]\\
&\geq 2\alpha(\rho-1)(1-c)+\alpha\rho^2\Big(1-\frac{1-c}{\rho}-x\Big)^2,
\end{aligned}$$

where in the third equality we used (A.8). □

Acknowledgments

The research of this project is supported by the National Key R&D Program of China (No. 2020YFA0712900). Yan-Xia Ren is supported by NSFC (Grant Nos. 11731009, 12071011 and 12231002) and the Fundamental Research Funds for Central Universities, Peking University LMEQF. Renming Song's research was supported in part by a grant from the Simons Foundation (#960480, Renming Song). Rui Zhang is supported by NSFC (Grant Nos. 11601354 and 12271374), Beijing Municipal Natural Science Foundation(Grant No. 1202004), and Academy for Multidisciplinary Studies, Capital Normal University.

We thank the referee for the very helpful comments. Part of the research for this paper was done while the second-named author was visiting Jiangsu Normal University, where he was partially supported by a grant from the National Natural Science Foundation of China (11931004, Yingchao Xie)

Statements and Declarations

We declare that the authors have no conflict of interest that might influence the results reported in this paper. All materials in the paper are available.

References

[1] E. Aïdékon, J. Berestycki, É. Brunet and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Related Fields* **157** (2013), 405–451.

[2] L.-P. Arguin, A. Bovier and N. Kistler, The extremal process of branching Brownian motion. *Probab. Theory Related Fields* **157** (2013), 535–574.

[3] J. Berestycki, A. E. Kyprianou and A. Murillo-Salas, The prolific backbone for supercritical superprocesses. *Stochastic Process. Appl.* **121** (2011), 1315–1331.

[4] M. Bramson, Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, **31** (1978), 531–581.

[5] B. Chauvin and A. Rouault, KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Theory Related Fields* **80** (1988), 299–314.

[6] B. Chauvin and A. Rouault, Supercritical branching Brownian motion and K-P-P equation in the critical speed-area. *Math. Nachr.* **149** (1990), 41–59.

[7] X. Chen, H. He and B. Mallein, Branching Brownian motion conditioned on small maximum, preprint, 2020. Available at arXiv:2007.00405.

[8] D. A. Dawson, *Measure-Valued Markov Processes*. In *École D'Été de Probabilités de Saint-Flour XXI*-1991. *Lecture Notes in Math.* **1541** 1–260. Springer, Berlin, 1993.

[9] B. Derrida and Z. Shi, Slower deviations of the branching Brownian motion and of branching random walks. *J. Phys. A* **50** (2017), no. 34, 344001, 13 pp.

[10] B. Derrida and Z. Shi, Large deviations for the rightmost position in a branching Brownian motion. In *Modern Problems of Stochastic Analysis and Statistics. MPSAS 2016, V. Panov (Ed.) Springer Proceedings in Mathematics* & *Statistics* 208. Springer, Cham, 2017.

[11] R. Durrett, *Probability: Theory and Examples.* Fourth edition. *Cambridge Series in Statistical and Probabilistic Mathematics, 31.* Cambridge University Press, Cambridge, 2010.

[12] E. B. Dynkin, Superprocesses and partial differential equations. *Ann. Probab.* **21** (1993), 1185–1262.

[13] E. B. Dynkin and S. E. Kuznetsov, $\mathbb{N}$-Measure for branching exit Markov system and their applications to differential equations. *Probab. Theory Related Fields* **130** (2004), 135–150.

[14] N. Gantert and T. Höfelsauer, Large deviations for the maximum of a branching random walk. *Electron. Commun. Probab.* **23** (2018), no 34, 12 pp.

[15] O. Kallenberg. *Random Measures, Theory and Applications.* Springer, Cham, 2017.

[16] A. E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications.* Springer, Berlin, 2006.

[17] A. E. Kyprianou, R.-L. Liu, A. Murillo-Salas and Y.-X. Ren, Supercritical super-Brownian motion with a general branching mechanism and travelling waves. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** (2012), 661–687.

[18] Z. Li, *Measure-Valued Branching Markov Processes.* Springer, Heidelberg, 2011.

[19] Y.-X. Ren, R. Song and R. Zhang, The extremal process of super-Brownian motion, *Stochastic Process. Appl.* **137** (2021) 1–34.