Asymptotic expansions for additive measures of branching Brownian motions *

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Abstract

Let $N(t)$ be the collection of particles alive at time t in a branching Brownian motion in $\mathbb{R}^d$, and for $u \in N(t)$, let $\mathbf{X}_u(t)$ be the position of particle u at time t. For $\theta \in \mathbb{R}^d$, we define the additive measures of the branching Brownian motion by

$$\mu_t^\theta(\mathrm{d}\mathbf{x}) := e^{-(1+\frac{\|\theta\|^2}{2})t} \sum_{u \in N(t)} e^{-\theta \cdot \mathbf{X}_u(t)} \delta_{(\mathbf{X}_u(t)+\theta t)}(\mathrm{d}\mathbf{x}).$$

In this paper, under some conditions on the offspring distribution, we give asymptotic expansions of arbitrary order for $\mu_t^\theta((\mathbf{a},\mathbf{b}])$ and $\mu_t^\theta((-\infty,\mathbf{a}])$ for $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$. These expansions sharpen the asymptotic results of Asmussen and Kaplan (1976) and Kang (1999), and are analogs of the expansions in Gao and Liu (2021) and Révész, Rosen and Shi (2005) for branching Wiener processes (a particular class of branching random walks) corresponding to $\theta = \mathbf{0}$.

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1 Introduction and main results

1.1 Introduction

A branching random walk in $\mathbb{R}^d$ is a discrete-time Markov process which can be defined as follows: at time 0, there is a particle at $\mathbf{0} \in \mathbb{R}^d$; at time 1, this particle is replaced by a random number of particle distributed according to a point process $\mathcal{L}$; at time 2, each individual, of generation 1, if located at $\mathbf{x} \in \mathbb{R}^d$, is replaced by a point process $\mathbf{x} + \mathcal{L}_\mathbf{x}$, where $\mathcal{L}_\mathbf{x}$ is an independent copy of $\mathcal{L}$. This procedure goes on. We use Z_n to denote the point process formed by the positions of the particles of generation n.

Biggins [5] studied the L^p convergence of the additive martingale

$$W_n(\theta) := \frac{1}{m(\theta)^n} \int e^{-\theta \cdot \mathbf{x}} Z_n(\mathrm{d}\mathbf{x}),$$

where $m(\theta) := \mathbb{E}\left(\int e^{-\theta \cdot \mathbf{x}} Z_1(\mathrm{d}\mathbf{x})\right)$. He used the L^p convergence of the additive martingale to study the asymptotic behavior of $Z_n(n\mathbf{c} + I)$ for fixed $\mathbf{c}$ and bounded interval I. To describe Biggins'

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result, we introduce the following additive measure $\mu_n^{Z,\theta}$ of the branching random walk, which is a shifted version of the measure introduced before Theorem 4 in [5]:

$$\mu_n^{Z,\theta}(A) := m(\theta)^{-n} \int e^{-\theta\cdot\mathbf{y}} 1_A(\mathbf{y} - \mathbf{c}_\theta n) Z_n(\mathrm{d}\mathbf{y}), \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{1.1}$$

with $(\mathbf{c}_\theta)_i := m(\theta)^{-1}\mathbb{E}\left(\int x_i e^{-\theta\cdot\mathbf{x}} Z_1(\mathrm{d}\mathbf{x})\right)$. [5, Theroem 4] implies that, in the weak disorder regime (i.e., $-\log m(\theta) < -\theta\cdot\nabla m(\theta)/m(\theta)$), if there exists $\gamma > 1$ such that $\mathbb{E}(W_1(\theta)^\gamma) < \infty$, then for $\mathbf{x} \in \mathbb{R}^d$ and $h > 0$, as $n \to \infty$,

$$n^{d/2}\mu_n^{Z,\theta}(\mathbf{x} + I_h) \longrightarrow \frac{(2h)^d W_\infty(\theta)}{(2\pi \det(\Sigma_\theta))^{d/2}}, \quad \text{a.s.}$$

where $W_\infty(\theta) := \lim_{n\to\infty} W_n(\theta), I_h = [-h, h]^d$ and

$$(\Sigma_\theta)_{i,j} = m(\theta)^{-1}\mathbb{E}\Big(\int (x_i - (\mathbf{c}_\theta)_i)(x_j - (\mathbf{c}_\theta)_j) e^{-\theta\cdot\mathbf{x}} Z_1(\mathrm{d}\mathbf{x})\Big), \quad i, j \in \{1, ..., d\}.$$

In the case $d = 1$, Pain proved that, see [15, (1.14)], in the weak disorder regime, if there exists $\gamma > 1$ such that $\mathbb{E}(W_1(\theta)^\gamma) < \infty$, then for any $b \in \mathbb{R}$, as $n \to \infty$,

$$\mu_n^{Z,\theta}\left((-\infty, b\Sigma_\theta\sqrt{n}]\right) \to W_\infty(\theta)\Phi(b) \quad \text{in probability,}$$

where $\Phi(b) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^b e^{-z^2/2}\mathrm{d}z$.

For the case $\theta = \mathbf{0}$, there are many further asymptotic results. In the case when $d = 1$ and the point process $\mathcal{L}$ is given by $\mathcal{L} = \sum_{i=1}^B \delta_{X_i}$, where X_i are iid with common distribution G and B is an independent $\mathbb{N}$-valued random variable with $\mathbb{P}(B = k) = p_k$ and $\mu := \sum_k kp_k > 1$, Asmussen and Kaplan [2, 3] proved that if G has mean 0, variance 1 and $\sum_{k=2}^\infty k(\log k)^{1+\varepsilon}p_k < \infty$ for some $\varepsilon > 0$, then conditioned on survival, for any $b \in \mathbb{R}$,

$$\mu_n^{Z,0}((-\infty, b\sqrt{n}]) \overset{n\to\infty}{\longrightarrow} W_\infty(0)\Phi(b), \quad \text{a.s.} \tag{1.2}$$

They also proved that if G has finite 3rd moment and $\sum_{k=2}^\infty k(\log k)^{3/2+\varepsilon}p_k < \infty$ for some $\varepsilon > 0$, then, for any $a < b \in \mathbb{R}$, conditioned on survival,

$$\sqrt{2\pi n}\mu_n^{Z,0}([a, b]) \overset{n\to\infty}{\longrightarrow} (b - a)W_\infty(0), \quad \text{a.s.} \tag{1.3}$$

Gao and Liu [10] gave first and second order expansions of $\mu_n^{Z,0}((-\infty, b\sqrt{n}])$. A third order expansion was proved by Gao and Liu [9, 11], where branching random walks in (time) random environment were studied. They also conjectured the form of asymptotic expansion of arbitrary order for $\mu_n^{Z,0}((-\infty, b\sqrt{n}])$. For general branching random walks, results similar to (1.2) and (1.3) were proved in Biggins [4].

When the point process $\mathcal{L}$ is given by $\mathcal{L} = \sum_{i=1}^B \delta_{\mathbf{X}_i}$ where $\mathbf{X}_1, \mathbf{X}_2, ...$ are independent d-dimensional standard normal random variables and B is an independent $\mathbb{N}$-valued random variable with $\mathbb{P}(B = k) = p_k$ and $\mu := \sum_k kp_k > 1$, Z_n is called a supercritical branching Wiener process. Révész [17] first proved the analogs of (1.2) and (1.3) for branching Wiener processes, then Chen [6] studied the corresponding convergence rates. Gao and Liu [8] proved that, for each $m \in \mathbb{N}$, when $\sum_{k=1}^\infty k(\log k)^{1+\lambda}p_k < \infty$ for some $\lambda > 3\max\{(m + 1), dm\}$, there exist random variables $\{V_\mathbf{a}, |\mathbf{a}| \le m\}$ such that for each $\mathbf{t} \in \mathbb{R}^d$,

$$\frac{1}{\mu^n} Z_n((-\infty, \mathbf{t}\sqrt{n}]) = \Phi_d(\mathbf{t})V_\mathbf{0} + \sum_{\ell=1}^m \frac{(-1)^\ell}{n^{\ell/2}} \sum_{|\mathbf{a}|=\ell} \frac{D^\mathbf{a}\Phi_d(\mathbf{t})}{\mathbf{a}!} V_\mathbf{a} + o(n^{-m/2}), \quad \text{a.s.}$$

where for $\mathbf{a} = (a_1, ..., a_d)$, $|\mathbf{a}| = a_1 + ... + a_d$, $\mathbf{a}! = a_1! \cdots a_d!$, $\Phi_d(\mathbf{t})$ is the distribution function of a d-dimensional standard normal random vector and $D^{\mathbf{a}}\Phi_d(\mathbf{t}) := \partial_{t_1}^{a_1} ... \partial_{t_d}^{a_d} \Phi_d(\mathbf{t})$. For the local limit theorem (1.3), Révész, Rosen and Shi [18] proved that, when $\sum_{k=1}^{\infty} k^2 p_k < \infty$, for any bounded Borel set $A \subset \mathbb{R}^d$,

$$(2\pi n)^{d/2} \frac{1}{\mu^n} Z_n(A) = \sum_{\ell=0}^{m} \frac{(-1)^\ell}{(2n)^\ell} \sum_{|\mathbf{a}|=\ell} \frac{1}{\mathbf{a}!} \sum_{\mathbf{b} \le 2\mathbf{a}} C_{2\mathbf{a}}^{\mathbf{b}} (-1)^{|\mathbf{b}|} M_{\mathbf{b}}(A) V_{2\mathbf{a}-\mathbf{b}} + o(n^{-m}), \quad \text{a.s.}, \quad (1.4)$$

where $\mathbf{b} \le 2\mathbf{a}$ means that $b_i \le 2a_i$ for all $1 \le i \le d$, $C_{2\mathbf{a}}^{\mathbf{b}} := C_{2a_1}^{b_1} \cdots C_{2a_d}^{b_d}$ and $M_{\mathbf{b}}(A) := \int_A x_1^{b_1} \cdots x_d^{b_d} \mathrm{d}x_1 ... \mathrm{d}x_d$.

For the lattice case, analogs of (1.2) and (1.3) can be found in [7, 12], and an asymptotic expansion similar to (1.4) for $Z_n(\{k\})$ was given by Grübel and Kabluchko[12].

In this paper, we are concerned with branching Brownian motions in $\mathbb{R}^d$. A branching Brownian motion in $\mathbb{R}^d$ is a continuous-time Markov process defined as follows: initially there is a particle at $\mathbf{0} \in \mathbb{R}^d$, it moves according to a d-dimensional standard Brownian motion and its lifetime is an exponential random variable of parameter 1, independent of the spatial motion. At the end of its lifetime, it produces k offspring with probability p_k for $k \in \mathbb{N}$ and the offspring move independently according to a d-dimensional standard Brownian motion from the death location of their parent, and repeat their parent's behavior independently. This procedure goes on. We will use $\mathbb{P}$ to denote the law of branching Brownian motion and $\mathbb{E}$ to denote the corresponding expectation. Without loss of generality, we assume that

$$\sum_{k=0}^{\infty} k p_k = 2.$$

Let $N(t)$ be the set of particles alive at time t and for $u \in N(t)$, we use $\mathbf{X}_u(t)$ to denote the position of particle u at time t. Define

$$Z_t := \sum_{u \in N(t)} \delta_{\mathbf{X}_u(t)}.$$

For $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$,

$$W_t(\theta) := e^{-(1+\frac{\|\theta\|^2}{2})t} \sum_{u \in N(t)} e^{-\theta \cdot \mathbf{X}_u(t)}$$

is a non-negative martingale and is called the additive martingale of the branching Brownian motion. When θ is the zero vector, $W_t(\theta)$ reduces to $e^{-t} Z_t(\mathbb{R}^d)$. It is well-known that (for $d = 1$, see Kyprianou[14]), for each $\theta \in \mathbb{R}^d$, $W_t(\theta)$ converges to a non-trivial limit $W_\infty(\theta)$ if and only if $\|\theta\| < \sqrt{2}$ and

$$\sum_{k=1}^{\infty} k(\log k) p_k < \infty. \quad (1.5)$$

From now on, we will only consider $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$. For any set $A \subset \mathbb{R}$ and $a \in \mathbb{R}$, we use $|A|$ to denote the Lebesgue measure of A and $aA := \{ax : x \in A\}$. Asmussen and Kaplan [3, Part 5] proved that when $d = 1$, under the assumption $\sum_{k=1}^{\infty} k^2 p_k < \infty$, for any Borel set B with $|\partial B| = 0$, as $t \to \infty$,

$$e^{-t} Z_t\left(\sqrt{t} B\right) \longrightarrow \frac{W_\infty(0)}{\sqrt{2\pi}} \int_B e^{-z^2/2} \mathrm{d}z, \quad \mathbb{P}\text{-a.s.} \quad (1.6)$$

and that for any bounded Borel set B with $|\partial B| = 0$, as $t \to \infty$,

$$\sqrt{2\pi t} e^{-t} Z_t(B) \longrightarrow |B| W_\infty(0), \quad \mathbb{P}\text{-a.s.}$$

Kang [13, Theorem 1] weakened the moment condition and proved that (1.6) holds with $B = (-\infty, b]$ under condition (1.5).

Similar to (1.1), we define the additive measure μ_t^θ of branching Brownian motion as

$$\mu_t^\theta(\mathrm{d}\mathbf{x}) := e^{-(1+\frac{\|\theta\|^2}{2})t} \sum_{u \in N(t)} e^{-\theta \cdot \mathbf{X}_u(t)} \delta_{(\mathbf{X}_u(t)+\theta t)}(\mathrm{d}\mathbf{x}).$$

The aim of this paper is to prove asymptotic expansions of arbitrary order for μ_t^θ for $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$, see Theorems 1.1 and 1.2 below. These expansions sharpen the asymptotic results of [3, 13, Part 5] mentioned above. The asymptotic expansions of [8, 18] are for the additive measure $\mu_n^{Z,0}$ of branching Wiener processes, while the asymptotic expansions of Theorems 1.1 and 1.2 are for the additive measure μ_t^θ of branching Brownian motions with θ not necessarily $\mathbf{0}$.

One might expect that the asymptotic expansions for branching Wiener processes, when considered along $\{t_n = n\delta, n \in \mathbb{N}\}$, can be used to used to get the expansions of this paper by letting $\delta \to 0$. However, it seems that this idea does not work due to two different reasons. One of the reasons is that values along $\{n\delta, n \in \mathbb{N}\}$ are not good enough to control the behavior between the time intervals $[t_n, t_{n+1}]$. Another reason is that $\{Z_{n\delta} : n \in \mathbb{N}\}$ is not a branching Wiener process since in $Z_\delta = \sum_{u \in N(\delta)} \delta_{\mathbf{X}_u(\delta)}$, for $u, v \in N(\delta), u \neq v$, $\mathbf{X}_u(\delta)$ and $\mathbf{X}_v(\delta)$ are not independent.

1.2 Notation

We list here some notation that will be used repeatedly below. Throughout this paper, $\mathbb{N} = \{0, 1, \dots\}$. Recall that $N(t)$ is the set of the particles alive at time t and that for $u \in N(t)$, $\mathbf{X}_u(t)$ is the position of u. For $u \in N(t)$, we use d_u and O_u to denote the death time and the offspring number of u respectively. For v and u, we will use $v < u$ to denote that v is an ancestor of u. The notation $v \leq u$ means that $v = u$ or $v < u$.

For $\mathbf{a} = (a_1, ..., a_d) \in \mathbb{R}^d$, define $(\mathbf{a})_j := a_j$ and $(-\infty, \mathbf{a}] := (-\infty, a_1] \times \cdots \times (-\infty, a_d]$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, we use $\mathbf{a} < \mathbf{b}$ ($\mathbf{a} \leq \mathbf{b}$) to denote that $(\mathbf{a})_j < (\mathbf{b})_j$ ($(\mathbf{a})_j < (\mathbf{b})_j$) for all $1 \leq j \leq d$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ with $\mathbf{a} < \mathbf{b}$, define $(\mathbf{a}, \mathbf{b}] := (a_1, b_1] \times \cdots \times (a_d, b_d]$. The definition of $[\mathbf{a}, \mathbf{b}]$ is similar. For $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{N}^d$, set $|\mathbf{k}| := k_1 + ... + k_d$ and $\mathbf{k}! := k_1! \cdots k_d!$. For a function f on $\mathbb{R}^d$, $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{k} \in \mathbb{N}^d$, let $D^{\mathbf{k}} f(\mathbf{x}) := \partial_{x_1}^{k_1} ... \partial_{x_d}^{k_d} f(\mathbf{x})$. We also use the notation $\phi(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ and $\Phi_d(\mathbf{x}) := \prod_{j=1}^d \int_{-\infty}^{x_j} \phi(z) \mathrm{d}z$. Sometimes we write $\Phi(y)$ for $\Phi_1(y)$.

1.3 Main results

We will assume that

$$\sum_{k=1}^\infty k (\log k)^{1+\lambda} p_k < \infty \tag{1.7}$$

for appropriate $\lambda > 0$. Let H_k be the k-th order Hermite polynomial: $H_0(x) := 1$ and for $k \geq 1$,

$$H_k(x) := \sum_{j=0}^{[k/2]} \frac{k!(-1)^j}{2^j j!(k-2j)!} x^{k-2j}.$$

It is well known that if $\{(B_t)_{t\geq 0}, \Pi_0\}$ is a standard Brownian motion, then for any $k \geq 0$, $\{t^{k/2}H_k(B_t/\sqrt{t}), \sigma(B_s : s \leq t),\ \Pi_0\}$ is a martingale. Now for $\mathbf{k} \in \mathbb{N}^d$ and $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$, we define

$$M_t^{(\mathbf{k},\theta)} := e^{-(1+\frac{\|\theta\|^2}{2})t} \sum_{u\in N(t)} e^{-\theta\cdot\mathbf{X}_u(t)} t^{|\mathbf{k}|/2} \prod_{j=1}^{d} H_{k_j}\Big(\frac{(\mathbf{X}_u(t))_j + \theta_j t}{\sqrt{t}}\Big), \quad t \geq 0.$$

Note that $M_t^{(\mathbf{0},\theta)} = W_t(\theta)$. We will prove in Proposition 2.6 below that if (1.7) holds for λ large enough, $M_t^{(\mathbf{k},\theta)}$ will converges almost surely and in L^1 to a limit $M_\infty^{(\mathbf{k},\theta)}$. Here are the main results of this paper:

Theorem 1.1 *Suppose $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$. For any given $m \in \mathbb{N}$, if (1.7) holds for some $\lambda > \max\{3m+8, d(3m+5)\}$, then for any $\mathbf{b} \in \mathbb{R}^d$, $\mathbb{P}$-almost surely, as $s \to \infty$,*

$$\begin{aligned}\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) &= \sum_{\mathbf{k}:|\mathbf{k}|\leq m} \frac{(-1)^{|\mathbf{k}|}}{\mathbf{k}!} \frac{1}{s^{|\mathbf{k}|/2}} D^{\mathbf{k}}\Phi_d(\mathbf{b}) M_\infty^{(\mathbf{k},\theta)} + o(s^{-m/2}) \\ &= \sum_{\ell=0}^{m} \frac{(-1)^\ell}{s^{\ell/2}} \sum_{\mathbf{k}:|\mathbf{k}|=\ell} \frac{D^{\mathbf{k}}\Phi_d(\mathbf{b})}{\mathbf{k}!} M_\infty^{(\mathbf{k},\theta)} + o(s^{-m/2}).\end{aligned}$$

Theorem 1.2 *Suppose $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$. For any given $m \in \mathbb{N}$, if (1.7) holds for some $\lambda > \max\{d(3m+5), 3m+3d+8\}$, then for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ with $\mathbf{a} < \mathbf{b}$, $\mathbb{P}$-almost surely, as $s \to \infty$,*

$$\begin{aligned}&s^{d/2}\mu_s^\theta((\mathbf{a}, \mathbf{b}]) \\ &= \sum_{\ell=0}^{m} \frac{1}{s^{\ell/2}} \sum_{j=0}^{\ell} (-1)^j \sum_{\mathbf{k}:|\mathbf{k}|=j} \frac{M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}!} \sum_{\mathbf{i}:|\mathbf{i}|=\ell-j} \frac{D^{\mathbf{k}+\mathbf{i}+\mathbf{1}}\Phi_d(\mathbf{0})}{\mathbf{i}!} \int_{[\mathbf{a},\mathbf{b}]} \prod_{j=1}^{d} z_j^{i_j} \mathrm{d}z_1 ... \mathrm{d}z_d + o(s^{-m/2}),\end{aligned}$$

where $\mathbf{1} := (1, ..., 1)$.

Remark 1.3 *Note that we only dealt with the case that the branching rate is 1 and the mean number of offspring is 2 in the two theorems above. In the general case when the branching rate is $\beta > 0$ and the mean number of offspring is $\mu > 1$, one can use the same argument to prove the following counterpart of Theorem 1.1: Suppose $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2\beta(\mu-1)}$. For any given $m \in \mathbb{N}$, if (1.7) holds for some $\lambda > \max\{3m+8, d(3m+5)\}$, then for any $\mathbf{b} \in \mathbb{R}^d$, $\mathbb{P}$-almost surely, as $s \to \infty$,*

$$\begin{aligned}\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) &:= e^{-(\beta(\mu-1)+\frac{\|\theta\|^2}{2})t} \sum_{u\in N(t)} e^{-\theta\cdot\mathbf{X}_u(t)} 1_{(-\infty,\mathbf{b}\sqrt{s}]}\left(\mathbf{X}_u(t) + \theta t\right) \\ &= \sum_{\ell=0}^{m} \frac{(-1)^\ell}{s^{\ell/2}} \sum_{\mathbf{k}:|\mathbf{k}|=\ell} \frac{D^{\mathbf{k}}\Phi_d(\mathbf{b})}{\mathbf{k}!} M_\infty^{(\mathbf{k},\theta)} + o(s^{-m/2}),\end{aligned}$$

with $M_\infty^{(\mathbf{k},\theta)}$ given by

$$M_\infty^{(\mathbf{k},\theta)} := \lim_{t\to\infty} e^{-(\beta(\mu-1)+\frac{\|\theta\|^2}{2})t} \sum_{u\in N(t)} e^{-\theta\cdot\mathbf{X}_u(t)} t^{|\mathbf{k}|/2} \prod_{j=1}^{d} H_{k_j}\Big(\frac{(\mathbf{X}_u(t))_j + \theta_j t}{\sqrt{t}}\Big). \tag{1.8}$$

In the general case, the counterpart of Theorem 1.2 is as follows: Suppose $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2\beta(\mu-1)}$. For any given $m \in \mathbb{N}$, if (1.7) holds for some $\lambda > \max\{d(3m+5), 3m+3d+8\}$, then for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ with $\mathbf{a} < \mathbf{b}$, $\mathbb{P}$-almost surely, as $s \to \infty$,

$$s^{d/2}\mu_s^\theta((\mathbf{a},\mathbf{b}]) = e^{-(\beta(\mu-1)+\frac{\|\theta\|^2}{2})t} \sum_{u \in N(t)} e^{-\theta \cdot \mathbf{X}_u(t)} 1_{(\mathbf{a},\mathbf{b}]}(\mathbf{X}_u(t) + \theta t)$$
$$= \sum_{\ell=0}^{m} \frac{1}{s^{\ell/2}} \sum_{j=0}^{\ell} (-1)^j \sum_{\mathbf{k}:|\mathbf{k}|=j} \frac{M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}!} \sum_{\mathbf{i}:|\mathbf{i}|=\ell-j} \frac{D^{\mathbf{k}+\mathbf{i}+\mathbf{1}}\Phi_d(\mathbf{0})}{\mathbf{i}!} \int_{[\mathbf{a},\mathbf{b}]} \prod_{j=1}^{d} z_j^{i_j} \mathrm{d}z_1 \ldots \mathrm{d}z_d + o(s^{-m/2}),$$

with $M_\infty^{(\mathbf{k},\theta)}$ given in (1.8).

Remark 1.4 *One could also consider asymptotic expansions for the additive measure $\mu_n^{Z,\theta}$ for branching random walks. Using the tools established in [9], it is possible to get fixed order expansions. However, getting asymptotic expansions of arbitrary order may be difficult.*

We end this section with a few words about the strategy of the proofs and the organization of the paper. In Section 2, we introduce the spine decomposition and gather some useful facts. We also study the convergence rate of the martingales $M_t^{(\mathbf{k},\theta)}$ and moments of the additive martingale $W_t(\theta)$. In Section 3, we prove Theorems 1.1 and 1.2. To prove Theorem 1.1, we choose a sequence of discrete time $r_n = n^{1/\kappa}$ for some $\kappa > 1$. To control the behavior of particles alive in (r_n, r_{n+1}), we need $r_{n+1} - r_n \to 0$. This is the reason we do not choose $r_n = n\delta$. We prove in Lemma 3.1 that $\mu_{r_n}^\theta\left((-\infty, \mathbf{b}\sqrt{r_n}]\right) \approx \mathbb{E}\left[\mu_{r_n}^\theta\left((-\infty, \mathbf{b}\sqrt{r_n}]\right) \big| \mathcal{F}_{\sqrt{r_n}}\right]$, where $\mathcal{F}_t$ is the σ-field generated by the branching Brownian motion up to time t. To deal with $s \in (r_n, r_{n+1})$, we adapt some ideas from from [3, Lemma 8] and [13, paragraph below (13)]. We prove in Lemma 3.2 that, for $s \in (r_n, r_{n+1})$, $\mu_s^\theta((-\infty, \mathbf{b}\sqrt{s}]) \approx \mathbb{E}\left[\mu_s^\theta((-\infty, \mathbf{b}\sqrt{s}]) \big| \mathcal{F}_{\sqrt{r_n}}\right]$. We complete the proof of Theorem 1.1 by using a series of identities proved in [8]. The proof of Theorem 1.2 is similar.

2 Preliminaries

2.1 Spine decomposition

Define

$$\left.\frac{\mathrm{d}\mathbb{P}^{-\theta}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := W_t(\theta). \tag{2.1}$$

Then under $\mathbb{P}^{-\theta}$, the evolution of our branching Brownian motion can be described as follows (spine decomposition) (see [14] for the case $d = 1$ or see [16] for a more general case):

(i) there is an initial marked particle at $\mathbf{0} \in \mathbb{R}^d$ which moves according to the law of $\{\mathbf{B}_t - \theta t, \Pi_\mathbf{0}\}$, where $\{\mathbf{B}_t, \Pi_\mathbf{0}\}$ is a d-dimensional standard Brownian motion;

(ii) the branching rate of this marked particle is 2;

(iii) when the marked particle dies at site $\mathbf{y}$, it gives birth to $\widehat{L}$ children with $\mathbb{P}^{-\theta}(\widehat{L} = k) = kp_k/2$;

(iv) one of these children is uniformly selected and marked, and the marked child evolves as its parent independently and the other children evolve independently with law $\mathbb{P}_\mathbf{y}$, where $\mathbb{P}_\mathbf{y}$ denotes the law of a branching Brownian motion starting at $\mathbf{y}$.

Let d_i be the i-th splitting time of the spine and O_i be the number of children produced by the spine at time d_i. According to the spine decomposition, it is easy to see that $\{d_i : i \geq 1\}$ are the atoms for a Poisson point process with rate 2, $\{O_i : i \geq 1\}$ are iid with common law $\widehat{L}$ given by $\mathbb{P}^{-\theta}(\widehat{L} = k) = kp_k/2$, and that $\{d_i : i \geq 1\}$ and $\{O_i : i \geq 1\}$ and $\mathbf{X}_\xi$ are independent. This fact will be used repeatedly.

We use ξ_t and $\mathbf{X}_\xi(t)$ to denote the marked particle at time t and the position of this marked particle respectively. By [16, Theorem 2.11], we have that, for $u \in N(t)$,

$$\mathbb{P}^{-\theta}\left(\xi_t = u \middle| \mathcal{F}_t\right) = \frac{e^{-\theta\cdot\mathbf{X}_u(t)}}{\sum_{u\in N(t)} e^{-\theta\cdot\mathbf{X}_u(t)}} = \frac{e^{-(1+\frac{\|\theta\|^2}{2})t}e^{-\theta\cdot\mathbf{X}_u(t)}}{W_t(\theta)}. \tag{2.2}$$

Using (2.2), we can get the following many-to-one formula.

Lemma 2.1 *For any $t > 0$ and $u \in N(t)$, let $H(u,t)$ be a non-negative $\mathcal{F}_t$-measurable random variable. Then*

$$\mathbb{E}\Big(\sum_{u\in N(t)} H(u,t)\Big) = e^{(1+\frac{\|\theta\|^2}{2})t}\mathbb{E}^{-\theta}\left(e^{\theta\cdot\mathbf{X}_\xi(t)}H(\xi_t,t)\right).$$

Proof: Combining (2.1) and (2.2), we get

$$\mathbb{E}\left(\sum_{u\in N(t)} H(u,t)\right) = \mathbb{E}^{-\theta}\left(\sum_{u\in N(t)}\frac{H(u,t)}{W_t(\theta)}\right)$$
$$= \mathbb{E}^{-\theta}\left(\sum_{u\in N(t)} H(u,t)e^{(1+\frac{\|\theta\|^2}{2})t}e^{\theta\cdot\mathbf{X}_u(t)}\mathbb{P}^{-\theta}\left(\xi_t = u\middle|\mathcal{F}_t\right)\right)$$
$$= e^{(1+\frac{\|\theta\|^2}{2})t}\mathbb{E}^{-\theta}\left(\mathbb{E}^{-\theta}\left(\sum_{u\in N(t)} 1_{\{\xi_t=u\}}H(u,t)e^{\theta\cdot\mathbf{X}_u(t)}\middle|\mathcal{F}_t\right)\right)$$
$$= e^{(1+\frac{\|\theta\|^2}{2})t}\mathbb{E}^{-\theta}\left(H(\xi_t,t)e^{\theta\cdot\mathbf{X}_\xi(t)}\sum_{u\in N(t)} 1_{\{\xi_t=u\}}\right) = e^{(1+\frac{\|\theta\|^2}{2})t}\mathbb{E}^{-\theta}\left(e^{\theta\cdot\mathbf{X}_\xi(t)}H(\xi_t,t)\right).$$

□

2.2 Some useful facts

In this subsection, we gather some useful facts that will be used later.

Lemma 2.2 *(i) Let $\ell \in [1,2]$ be a fixed constant. Then for any finite family of independent centered random variables $\{X_i : i = 1,\dots,n\}$ with $\mathrm{E}|X_i|^\ell < \infty$ for all $i = 1,\dots,n$, it holds that*

$$\mathrm{E}\Big|\sum_{i=1}^n X_i\Big|^\ell \leq 2\sum_{i=1}^n \mathrm{E}|X_i|^\ell.$$

(ii) For any $\ell \in [1,2]$ and any random variable X with $\mathrm{E}|X|^2 < \infty$,

$$\mathrm{E}\left|X - \mathrm{E}X\right|^\ell \lesssim \mathrm{E}|X|^\ell \leq (\mathrm{E}X^2)^{\ell/2}.$$

Proof: For (i), see [19, Theorem 2]. (ii) follows easily from Jensen's inequality. □

Lemma 2.3 *For any $\rho \in (0,1), b, x \in \mathbb{R}$, it holds that*

$$\Phi\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right) = \Phi(b) - \phi(b)\sum_{k=1}^{\infty}\frac{\rho^k}{k!}H_{k-1}(b)H_k(x).$$

Proof: See [8, Lemma 4.2.]. □

To prove Theorem 1.1, we will define $r_n := n^{\frac{1}{\kappa}}$ for some $\kappa > 1$. For $s \in [r_n, r_{n+1})$, applying Lemma 2.3 with $\rho = \sqrt{\sqrt{r_n}/s}$ and $x = r_n^{-1/4}y$, we get that for any $b, y \in \mathbb{R}$,

$$\Phi\Big(\frac{b\sqrt{s}-y}{\sqrt{s-\sqrt{r_n}}}\Big) = \Phi(b) - \phi(b)\sum_{k=1}^{\infty}\frac{1}{k!}\frac{1}{s^{k/2}}H_{k-1}(b)\Big(r_n^{k/4}H_k\Big(\frac{y}{r_n^{1/4}}\Big)\Big). \tag{2.3}$$

Recall that (see [8, (4.1)]) for any $k \geq 1$ and $x \in \mathbb{R}$,

$$|H_k(x)| \leq 2\sqrt{k!}e^{x^2/4}. \tag{2.4}$$

Lemma 2.4 *For a given $m \in \mathbb{N}$, let $\kappa = m+3$ and $r_n = n^{1/\kappa}$. Let $K > 0$ be a fixed constant and J be an integer such that $J > 2m + K\kappa$. For any $b, y \in \mathbb{R}$ and $s \in [r_n, r_{n+1})$, it holds that*

$$\Phi\Big(\frac{b\sqrt{s}-y}{\sqrt{s-\sqrt{r_n}}}\Big) = \Phi(b) - \phi(b)\sum_{k=1}^{J}\frac{1}{k!}\frac{1}{s^{k/2}}H_{k-1}(b)\Big(r_n^{k/4}H_k\Big(\frac{y}{r_n^{1/4}}\Big)\Big) + \varepsilon_{m,y,b,s},$$

and that

$$\sup\Big\{s^{m/2}|\varepsilon_{m,y,b,s}| : \ s \in [r_n, r_{n+1}), \ |y| \leq \sqrt{K\sqrt{r_n}\log n}, b \in \mathbb{R}\Big\} \overset{n\to\infty}{\longrightarrow} 0.$$

Proof: It follows from (2.4) that there exists a constant C such that for all $b \in \mathbb{R}$, $n \geq 2$, $s \in [r_n, r_{n+1})$, $|y| \leq \sqrt{K\sqrt{r_n}\log n}$ and $k \geq m$,

$$s^{m/2}\frac{1}{k!}\frac{1}{s^{k/2}}\left(\phi(b)\,|H_{k-1}(b)|\right)\Big|r_n^{k/4}H_k\Big(\frac{y}{r_n^{1/4}}\Big)\Big| \leq C\frac{n^{k/(4\kappa)}}{n^{(k-m)/(2\kappa)}}n^{K/4}.$$

Combining this with (2.3), we get that for $J > 2m + K\kappa$, $n \geq 2$, $s \in [r_n, r_{n+1})$, $b \in \mathbb{R}$ and $|y| \leq \sqrt{K\sqrt{r_n}\log n}$,

$$s^{m/2}|\varepsilon_{m,y,b,s}| \leq C\sum_{k=J+1}^{\infty} n^{-(k-2m-K\kappa)/(4\kappa)} \lesssim n^{-(J+1-2m-K\kappa)/(4\kappa)}.$$

Thus the assertions of the lemma are valid. □

Now we give a result of similar flavor which will be used to prove Theorem 1.2. Taking derivative with respect to b in Lemma 2.3, and using the fact that

$$\frac{\mathrm{d}^k}{\mathrm{d}b^k}\Phi(b) = (-1)^{k-1}H_{k-1}(b)\phi(b),$$

we get that

$$\frac{1}{\sqrt{1-\rho^2}}\phi\Big(\frac{b-\rho x}{\sqrt{1-\rho^2}}\Big) = \phi(b) + \phi(b)\sum_{k=1}^{\infty}\frac{\rho^k}{k!}H_k(b)H_k(x). \tag{2.5}$$

Now letting $\rho = \sqrt{\sqrt{r_n}/s}$, $b = z/\sqrt{s}$ and $x = r_n^{-1/4} y$ in (2.5), we get that for any $z, y \in \mathbb{R}$,

$$\frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}}\phi\Big(\frac{z-y}{\sqrt{s-\sqrt{r_n}}}\Big) = \phi\Big(\frac{z}{\sqrt{s}}\Big)\Big(1+\sum_{k=1}^{\infty}\frac{1}{k!}\frac{1}{s^{k/2}}H_k\Big(\frac{z}{\sqrt{s}}\Big)r_n^{k/4}H_k\Big(\frac{y}{r_n^{1/4}}\Big)\Big).$$

The proof of the following result is similar to that of Lemma 2.4, we omit the details.

Lemma 2.5 *For a given $m \in \mathbb{N}$, let $\kappa = m+3$ and $r_n = n^{1/\kappa}$. Let $K > 0$ be a fixed constant and J be an integer such that $J > 2m + K\kappa$. For any $a < b \in \mathbb{R}$, $y, z \in \mathbb{R}$ and $s \in [r_n, r_{n+1})$, it holds that*

$$\begin{aligned}&\frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}}\phi\Big(\frac{z-y}{\sqrt{s-\sqrt{r_n}}}\Big)\\ &= \phi\Big(\frac{z}{\sqrt{s}}\Big)\Big(1+\sum_{k=1}^{J}\frac{1}{k!}\frac{1}{s^{k/2}}H_k\Big(\frac{z}{\sqrt{s}}\Big)r_n^{k/4}H_k\Big(\frac{y}{r_n^{1/4}}\Big)\Big) + \varepsilon_{m,y,z,s},\end{aligned}$$

and that

$$\sup\Big\{s^{m/2}\,|\varepsilon_{m,y,z,s}|:\ s\in[r_n, r_{n+1}),\ z\in[a,b],\ |y|\le\sqrt{K\sqrt{r_n}\log n}\Big\}\overset{n\to\infty}{\longrightarrow} 0.$$

2.3 Convergence rate for the martingales

Proposition 2.6 *For any $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$ and $\mathbf{k} \in \mathbb{N}^d$, $\{M_t^{(\mathbf{k},\theta)}, t \ge 0; \mathbb{P}\}$ is a martingale. If (1.7) holds for some $\lambda > |\mathbf{k}|/2$, then $M_t^{(\mathbf{k},\theta)}$ converges to a limit $M_\infty^{(\mathbf{k},\theta)}$ $\mathbb{P}$-a.s. and in L^1. Moreover, for any $\eta \in (0, \lambda - |\mathbf{k}|/2)$, as $t \to \infty$,*

$$M_t^{(\mathbf{k},\theta)} - M_\infty^{(\mathbf{k},\theta)} = o(t^{-(\lambda-|\mathbf{k}|/2)+\eta}), \quad \mathbb{P}\text{-a.s.}$$

Proof: By Lemma 2.1, it is easy to see that $\{M_t^{(\mathbf{k},\theta)}, t \ge 0; \mathbb{P}\}$ is a martingale.

Now we fix $\mathbf{k} \in \mathbb{N}^d$ and assume (1.7) holds for some $\lambda > |\mathbf{k}|/2$. We first look at the case when $t \to \infty$ along integers. Let $t = n \in \mathbb{N}$. Recall that $N(n+1)$ is the set of particles alive at time $n+1$. For $u \in N(n+1)$, define $B_{n,u}$ to be the event that, for all $v < u$ with $d_v \in (n, n+1)$, it holds that $O_v \le e^{c_0 n}$, where $c_0 > 0$ is a small constant to be determined later. Set

$$\begin{aligned}M_{n+1}^{(\mathbf{k},\theta),B} &:= e^{-(1+\frac{\|\theta\|^2}{2})(n+1)}\\ &\times \sum_{u\in N(n+1)} e^{-\theta\cdot\mathbf{X}_u(n+1)}(n+1)^{|\mathbf{k}|/2}\prod_{j=1}^{d}H_{k_j}\Big(\frac{(\mathbf{X}_u(n+1))_j+\theta_j(n+1)}{\sqrt{n+1}}\Big)1_{B_{n,u}}.\end{aligned}$$

Since $|H_k(x)| \lesssim |x|^k + 1$ for all $x \in \mathbb{R}$ and $(|x|+|y|)^k \lesssim |x|^k + |y|^k$ for all $x, y \in \mathbb{R}$, we have

$$(n+1)^{k/2}\Big|H_k\Big(\frac{x+z}{\sqrt{n+1}}\Big)\Big| \lesssim (|x|+|z|)^k + (n+1)^{k/2} \lesssim |x|^k + |z|^k + n^{k/2},$$

which implies that for all $j \in \{1, ..., d\}$,

$$\begin{aligned}&(n+1)^{k_j/2}\Big|H_{k_j}\Big(\frac{(\mathbf{X}_u(n+1))_j+\theta_j(n+1)}{\sqrt{n+1}}\Big)\Big|\\ &\lesssim |(\mathbf{X}_u(n))_j+\theta_j n|^{k_j} + n^{k_j/2} + |(\mathbf{X}_u(n+1))_j - (\mathbf{X}_u(n))_j + \theta_j|^{k_j}.\end{aligned} \tag{2.6}$$

Therefore,

$$
\begin{aligned}
&\Big|M^{(\mathbf{k},\theta)}_{n+1}-M^{(\mathbf{k},\theta),B}_{n+1}\Big| \le e^{-(1+\frac{\|\theta\|^2}{2})(n+1)}\\
&\quad\times\sum_{u\in N(n+1)} e^{-\theta\cdot\mathbf{X}_u(n+1)}\Big|(n+1)^{|\mathbf{k}|/2}\prod_{j=1}^d H_{k_j}\Big(\frac{(\mathbf{X}_u(n+1))_j+\theta_j(n+1)}{\sqrt{n+1}}\Big)\Big|1_{(B_{n,u})^c}\\
&\lesssim e^{-(1+\frac{\|\theta\|^2}{2})(n+1)}\sum_{u\in N(n)} e^{-\theta\cdot\mathbf{X}_u(n)}\sum_{v\in N(n+1):u\le v} e^{-\theta\cdot(\mathbf{X}_v(n+1)-\mathbf{X}_u(n))}\\
&\quad\times\prod_{j=1}^d\Big(|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}+|(\mathbf{X}_v(n+1))_j-(\mathbf{X}_u(n))_j+\theta_j|^{k_j}\Big)1_{(B_{n,v})^c}.
\end{aligned}
$$

By the branching property and the Markov property, we get that

$$
\begin{aligned}
&\mathbb{E}\Big(\Big|M^{(\mathbf{k},\theta)}_{n+1}-M^{(\mathbf{k},\theta),B}_{n+1}\Big|\,|\mathcal{F}_n\Big)\lesssim e^{-(1+\frac{\|\theta\|^2}{2})n}\sum_{u\in N(n)} e^{-\theta\cdot\mathbf{X}_u(n)}\\
&\times\mathbb{E}\Big(e^{-(1+\frac{\|\theta\|^2}{2})}\sum_{v\in N(1)} e^{-\theta\cdot\mathbf{X}_v(1)}\prod_{j=1}^d\Big(|(\mathbf{X}_v(1))_j+\theta_j|^{k_j}+y_j\Big)1_{(D_{n,v})^c}\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}}\\
&=: e^{-(1+\frac{\|\theta\|^2}{2})n}\sum_{u\in N(n)} e^{-\theta\cdot\mathbf{X}_u(n)}F(\mathbf{y})\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}},
\end{aligned}
\tag{2.7}
$$

where, for $v\in N(1)$, $D_{n,v}$ denotes the event that, for all $w<v$, it holds that $O_w\le e^{c_0 n}$. Recall that d_i is the i-th splitting time of the spine and O_i is the number of children produced by the spine at time d_i. Define D_{n,ξ_1} to be the event that, for all i with $d_i<1$, it holds that $O_i\le e^{c_0 n}$. By Lemma 2.1,

$$
F(\mathbf{y})=\mathbb{E}^{-\theta}\Big(\prod_{j=1}^d\Big(|(\mathbf{X}_\xi(1))_j+\theta_j|^{k_j}+y_j\Big)1_{(D_{n,\xi_1})^c}\Big).
$$

Using the independence of $\{d_i:i\ge 1\}$, $\{O_i:i\ge 1\}$ and $\mathbf{X}_\xi$, we have that D_{n,ξ_1} is independent of $\mathbf{X}_\xi$, which implies that for $y_j\ge 1$,

$$
\begin{aligned}
F(\mathbf{y})&=\prod_{j=1}^d\Big(\Pi_0(|B_1|^{k_j})+y_j\Big)\mathbb{P}^{-\theta}\left(D^c_{n,\xi_1}\right)\le\prod_{j=1}^d\Big(\Pi_0(|B_1|^{k_j})+y_j\Big)\mathbb{E}^{-\theta}\Big(\sum_{i:d_i\le 1}1_{\{O_i>e^{c_0 n}\}}\Big)\\
&\lesssim\Big(\prod_{j=1}^d y_j\Big)\mathbb{P}^{-\theta}(\widehat{L}>e^{c_0 n})\lesssim\Big(\prod_{j=1}^d y_j\Big)\mathbb{P}^{-\theta}(\widehat{L}>e^{c_0 n})\lesssim\frac{\prod_{j=1}^d y_j}{n^{1+\lambda}},
\end{aligned}
\tag{2.8}
$$

here in the first equality, we also used the fact that $\{\mathbf{X}_\xi(t)+\theta t,\mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, and in the last inequality, we used $\mathbb{E}^{-\theta}(\log_+^{1+\lambda}\widehat{L})<\infty$ (which follows from (1.7)). By (2.7) and (2.8), we have

$$
\begin{aligned}
&\mathbb{E}\Big(\Big|M^{(\mathbf{k},\theta)}_{n+1}-M^{(\mathbf{k},\theta),B}_{n+1}\Big|\,|\mathcal{F}_n\Big)\\
&\lesssim e^{-(1+\frac{\|\theta\|^2}{2})n}\sum_{u\in N(n)} e^{-\theta\cdot\mathbf{X}_u(n)}\frac{\prod_{j=1}^d\left(|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}\right)}{n^{1+\lambda}}.
\end{aligned}
$$

Taking expectation with respect to $\mathbb{P}$, by Lemma 2.1, we obtain that

$$\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_{n+1}^{(\mathbf{k},\theta),B}\right|\right) \lesssim \frac{1}{n^{1+\lambda}} \prod_{j=1}^{d} \Pi_0 \left(|B_n|^{k_j} + n^{k_j/2}\right) \lesssim \frac{n^{|\mathbf{k}|/2}}{n^{1+\lambda}}. \tag{2.9}$$

On the other hand, by the branching property,

$$M_{n+1}^{(\mathbf{k},\theta),B} = e^{-(1+\frac{\|\theta\|^2}{2})n} \sum_{u\in N(n)} e^{-\theta\cdot\mathbf{X}_u(n)} J_{n,u},$$

where

$$J_{n,u} := \sum_{v\in N(n+1):u\le v} e^{-\theta\cdot(\mathbf{X}_v(n+1)-\mathbf{X}_u(n))}(n+1)^{|\mathbf{k}|/2} \prod_{j=1}^{d} H_{k_j}\Big(\frac{(\mathbf{X}_v(n+1))_j + \theta_j(n+1)}{\sqrt{n+1}}\Big)1_{B_{n,v}}$$

are independent given $\mathcal{F}_n$. For any fixed $1 < \ell < \min\{2/\|\theta\|^2, 2\}$, Applying Lemma 2.2 (i) to the finite family $\{J_{n,u} - \mathbb{E}(J_{n,u}|\mathcal{F}_n) : u \in N(n)\}$ and Lemma 2.2 (ii) to $J_{n,u}$, together with (2.6), we get that

$$\mathbb{E}\Big(\Big|M_{n+1}^{(\mathbf{k},\theta),B} - \mathbb{E}\Big(M_{n+1}^{(\mathbf{k},\theta),B}\Big|\mathcal{F}_n\Big)\Big|^{\ell}\Big|\mathcal{F}_n\Big) \lesssim e^{-\ell(1+\frac{\|\theta\|^2}{2})n} \sum_{u\in N(n)} e^{-\ell\theta\cdot\mathbf{X}_u(n)} \left(M_{n,u}\right)^{\ell/2}, \tag{2.10}$$

where $M_{n,u}$ is given by

$$M_{n,u} := \mathbb{E}\Big(e^{-2(1+\frac{\|\theta\|^2}{2})}\Big(\sum_{v\in N(1)} e^{-\theta\cdot\mathbf{X}_v(1)} S_v(\mathbf{y},1)1_{D_{n,v}}\Big)^2\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}}$$

with $S_v(\mathbf{y},r) := \prod_{j=1}^{d}\Big(|(\mathbf{X}_v(r))_j + \theta_j r|^{k_j} + y_j\Big)$. Set

$$T_{n,u} := S_u(\mathbf{y},1)1_{D_{n,u}} e^{-(1+\frac{\|\theta\|^2}{2})} \sum_{v\in N(1)} e^{-\theta\cdot\mathbf{X}_v(1)} S_v(\mathbf{y},1)1_{D_{n,v}}.$$

By Lemma 2.1, we have

$$\begin{aligned}
M_{n,u} &= e^{-(1+\frac{\|\theta\|^2}{2})}\mathbb{E}\Big(\sum_{u\in N(1)} e^{-\theta\cdot\mathbf{X}_u(1)} T_{n,u}\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}} \\
&= \mathbb{E}^{-\theta}\left(T_{n,\xi_1}\right)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}} \\
&= \mathbb{E}^{-\theta}\Big(1_{D_{n,\xi_1}} S_{\xi_1}(\mathbf{y},1) e^{-(1+\frac{\|\theta\|^2}{2})} \sum_{v\in N(1)} e^{-\theta\cdot\mathbf{X}_v(1)} S_v(\mathbf{y},1)1_{D_{n,v}}\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}} \\
&\le \mathbb{E}^{-\theta}\Big(1_{D_{n,\xi_1}} S_{\xi_1}(\mathbf{y},1) e^{-(1+\frac{\|\theta\|^2}{2})} \sum_{v\in N(1)} e^{-\theta\cdot\mathbf{X}_v(1)} S_v(\mathbf{y},1)\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}}.
\end{aligned} \tag{2.11}$$

Conditioned on $\mathcal{G} := \sigma(\mathbf{X}_\xi, d_i, O_i : i \ge 1)$, by the branching property, on the set D_{n,ξ_1},

$$e^{-(1+\frac{\|\theta\|^2}{2})}\mathbb{E}^{-\theta}\Big(\sum_{v\in N(1)} e^{-\theta\cdot\mathbf{X}_v(1)} S_v(\mathbf{y},1)\big|\mathcal{G}\Big)$$

$$
\begin{aligned}
&= \sum_{i:d_i\le 1}(O_i-1)e^{-(1+\frac{\|\theta\|^2}{2})d_i}e^{-\theta\cdot\mathbf{X}_\xi(d_i)}\Pi_{\mathbf{0}}\Big(\prod_{j=1}^d\Big(|(\mathbf{B}_{1-d_i})_j+z_j|^{k_j}+y_j\Big)\Big)\Big|_{z_j=(\mathbf{X}_\xi(d_i))_j+\theta_j d_i}\\
&\lesssim e^{c_0 n}\sum_{i:d_i\le 1}e^{-\theta\cdot\mathbf{X}_\xi(d_i)}\prod_{j=1}^d\Big(y_j+|(\mathbf{X}_\xi(d_i))_j+\theta_j d_i|^{k_j}\Big)=e^{c_0 n}\sum_{i:d_i\le 1}e^{-\theta\cdot\mathbf{X}_\xi(d_i)}S_{\xi_{d_i}}(\mathbf{y},d_i), \quad (2.12)
\end{aligned}
$$

where in the first equality we also used Lemma 2.1. Plugging (2.12) into (2.11), noting that $\mathbf{X}_\xi$ and d_i are independent, we get that

$$
\begin{aligned}
&\mathbb{E}^{-\theta}\Big(1_{D_{n,\xi_1}}S_{\xi_1}(\mathbf{y},1)e^{-(1+\frac{\|\theta\|^2}{2})}\sum_{v\in N(1)}e^{-\theta\cdot\mathbf{X}_v(1)}S_v(\mathbf{y},1)\big|\mathbf{X}_\xi\Big)\\
&\lesssim S_{\xi_1}(\mathbf{y},1)e^{c_0 n}\mathbb{E}^{-\theta}\Big(\sum_{i:d_i\le 1}e^{-\theta\cdot\mathbf{X}_\xi(d_i)}S_{\xi_{d_i}}(\mathbf{y},d_i)\big|\mathbf{X}_\xi\Big)\\
&=2S_{\xi_1}(\mathbf{y},1)e^{c_0 n}\int_0^1 e^{-\theta\cdot\mathbf{X}_\xi(s)}S_{\xi_s}(\mathbf{y},s)\mathrm{d}s\\
&\lesssim e^{c_0 n}\prod_{j=1}^d\left\{\Big(y_j+\sup_{s<1}|(\mathbf{X}_\xi(s))_j+\theta_j s|^{k_j}\Big)^2 e^{\|\theta\|\sup_{s<1}|(\mathbf{X}_\xi(s))_j+\theta_j s|}\right\}. \quad (2.13)
\end{aligned}
$$

Since $\{\mathbf{X}_\xi(s)+\theta s,\mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, combining with (2.11) and (2.13), we conclude that

$$
\begin{aligned}
M_{n,u}&\lesssim e^{c_0 n}\prod_{j=1}^d\Pi_{\mathbf{0}}\Big(\big(y_j+\sup_{s<1}|(\mathbf{B}_s)_j|^{k_j}\big)^2 e^{\|\theta\|\sup_{s<1}|(\mathbf{B}_s)_j|}\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}}\\
&\lesssim\prod_{j=1}^d\Big(|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}\Big)^2 e^{c_0 n}. \quad (2.14)
\end{aligned}
$$

Plugging (2.14) into (2.10), we conclude that

$$
\begin{aligned}
&\mathbb{E}\Big(\Big|M_{n+1}^{(\mathbf{k},\theta),B}-\mathbb{E}\big(M_{n+1}^{(\mathbf{k},\theta),B}\big|\mathcal{F}_n\big)\Big|^\ell\,\Big|\mathcal{F}_n\Big)\\
&\lesssim e^{c_0\ell n/2}e^{-\ell(1+\frac{\|\theta\|^2}{2})n}\sum_{u\in N(n)}e^{-\ell\theta\cdot\mathbf{X}_u(n)}\prod_{j=1}^d\Big(|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}\Big)^\ell\\
&=e^{-\big((\ell-1)(1-\|\theta\|^2\ell/2)-c_0\ell/2\big)n}e^{-(1+\ell^2\|\theta\|^2/2)n}\\
&\quad\times\sum_{u\in N(n)}e^{-\ell\theta\cdot\mathbf{X}_u(n)}\prod_{j=1}^d\Big(|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}\Big)^\ell. \quad (2.15)
\end{aligned}
$$

Choose $c_0>0$ small so that $c_0\ell/2<(\ell-1)(1-\|\theta\|^2\ell/2)$ and set $c_1:=(\ell-1)(1-\|\theta\|^2\ell/2)-c_0\ell/2>0$. Taking expectation with respect to $\mathbb{P}$ in (2.15), by Lemma 2.1 with θ replaced to $\ell\theta$, we get that

$$
\begin{aligned}
&\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta),B}-\mathbb{E}\left(M_{n+1}^{(\mathbf{k},\theta),B}\big|\mathcal{F}_n\right)\right|^\ell\right)\\
&\lesssim e^{-c_1 n}\prod_{j=1}^d\Pi_{\mathbf{0}}\left(|(\mathbf{B}_n)_j-(\ell-1)\theta_j n|^{k_j}+n^{k_j/2}\right)^\ell\lesssim\left(\frac{n^{|\mathbf{k}|/2}}{n^{1+\lambda}}\right)^\ell. \quad (2.16)
\end{aligned}
$$

Now combining (2.9) and (2.16), using the inequality:

$$\begin{aligned}&\mathrm{E}\left(|X-\mathrm{E}(X|\mathcal{F})|\right) \le \mathrm{E}\left(|X-Y|\right) + \mathrm{E}\left(|Y-\mathrm{E}(Y|\mathcal{F})|\right) + \mathrm{E}\left(\left|\mathrm{E}\left(X-Y\middle|\mathcal{F}\right)\right|\right)\\ &\le 2\mathrm{E}\left(|X-Y|\right) + \mathrm{E}\left(|Y-\mathrm{E}(Y|\mathcal{F})|^{\ell}\right)^{1/\ell},\end{aligned}$$

and the fact that $M_n^{(\mathbf{k},\theta)} = \mathbb{E}\left(M_{n+1}^{(\mathbf{k},\theta)}\middle|\mathcal{F}_n\right)$, we get that

$$\begin{aligned}\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right|\right) &\le 2\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_{n+1}^{(\mathbf{k},\theta),B}\right|\right)\\ &+ \mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta),B} - \mathbb{E}\left(M_{n+1}^{(\mathbf{k},\theta),B}\middle|\mathcal{F}_n\right)\right|^{\ell}\right)^{1/\ell} \lesssim \frac{n^{|\mathbf{k}|/2}}{n^{1+\lambda}}.\end{aligned} \tag{2.17}$$

Since $\lambda > |\mathbf{k}|/2$, we have $\sum_{n=1}^{\infty}\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right|\right) < \infty$, which implies that $M_n^{(\mathbf{k},\theta)}$ converges to a limit $M_\infty^{(\mathbf{k},\theta)}$ $\mathbb{P}$-almost surely and in L^1. Therefore, $M_n^{(\mathbf{k},\theta)} = \mathbb{E}\left(M_\infty^{(\mathbf{k},\theta)}\middle|\mathcal{F}_n\right)$, $n \ge 1$.

For $s \in (n, n+1)$, $M_s^{(\mathbf{k},\theta)} = \mathbb{E}\left(M_{n+1}^{(\mathbf{k},\theta)}\middle|\mathcal{F}_s\right) = \mathbb{E}\left(M_\infty^{(\mathbf{k},\theta)}\middle|\mathcal{F}_s\right)$, thus the second assertion of the proposition is valid.

Now we prove the last assertion of the proposition. For any $\eta \in (0, \lambda - |\mathbf{k}|/2)$, by (2.17),

$$\sum_{n=1}^{\infty} n^{-|\mathbf{k}|/2+\lambda-\eta}\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right|\right) \lesssim \sum_{n=1}^{\infty}\frac{1}{n^{1+\eta}} < \infty,$$

which implies that

$$\sum_{n=1}^{\infty} n^{\lambda-|\mathbf{k}|/2-\eta}\left(M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right) \text{ converges a.s.}$$

Thus $n^{\lambda-|\mathbf{k}|/2-\eta}\left(M_n^{(\mathbf{k},\theta)} - M_\infty^{(\mathbf{k},\theta)}\right) \stackrel{n\to\infty}{\longrightarrow} 0$, $\mathbb{P}$-a.s. (see for example [1, Lemma 2]). For $s \in [n, n+1]$, by Doob's inequality, for any $\varepsilon > 0$,

$$\begin{aligned}&\sum_{n=1}^{\infty}\mathbb{P}\Big(n^{-|\mathbf{k}|/2+\lambda-\eta}\sup_{n\le s\le n+1}\left|M_s^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right| > \varepsilon\Big)\\ &\le \frac{1}{\varepsilon}\sum_{n=1}^{\infty} n^{-|\mathbf{k}|/2+\lambda-\eta}\mathbb{E}\Big(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right|\Big) < \infty.\end{aligned}$$

Therefore, $n^{-|\mathbf{k}|/2+\lambda-\eta}\sup_{n\le s\le n+1}\left|M_s^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right| \stackrel{n\to\infty}{\longrightarrow} 0, \mathbb{P}$-a.s. Hence $\mathbb{P}$-almost surely,

$$\begin{aligned}&\sup_{n\le s\le n+1} s^{-|\mathbf{k}|/2+\lambda-\eta}\left|M_s^{(\mathbf{k},\theta)} - M_\infty^{(\mathbf{k},\theta)}\right|\\ &\le (n+1)^{-|\mathbf{k}|/2+\lambda-\eta}\sup_{n\le s\le n+1}\left|M_s^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right| + (n+1)^{-|\mathbf{k}|/2+\lambda-\eta}\left|M_n^{(\mathbf{k},\theta)} - M_\infty^{(\mathbf{k},\theta)}\right|\\ &\stackrel{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

which completes the proof of the last assertion of the proposition. □

2.4 Moment estimate for the additive martingale

In this subsection, we give an upper bound for $W_t(\theta)$ which will be used later.

Lemma 2.7 *Suppose $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$. If (1.7) holds for some $\lambda > 0$, then there exists a constant $C_{\theta,\lambda}$ such that for all $t > 0$,*

$$\mathbb{E}\left((W_t(\theta)+1)\log^{1+\lambda}(W_t(\theta)+1)\right) \le C_{\theta,\lambda}(t+1).$$

Proof: Since $\mathbb{E}W_t(\theta) = 1$, it suffices to prove that there exists a constant $C_{\theta,\lambda}$ such that for all $t > 0$, $\mathbb{E}\left(W_t(\theta)\log_+^{1+\lambda}(W_t(\theta))\right) \le C_{\theta,\lambda}(t+1)$. By using an projection argument, we can easily reduce to the one dimensional case. So we will only deal with the case $d = 1$. By (2.1), we have $\mathbb{E}\left((W_t(\theta))\log_+^{1+\lambda}(W_t(\theta))\right) = \mathbb{E}^{-\theta}\left(\log_+^{1+\lambda}(W_t(\theta))\right)$. Using the spine decomposition, we have

$$W_t(\theta) = e^{-(1+\frac{\theta^2}{2})t}e^{-\theta X_\xi(t)} + \sum_{i:d_i\le t} e^{-(1+\frac{\theta^2}{2})d_i}e^{-\theta X_\xi(d_i)}\sum_{j=1}^{O_i-1} W^{i,j}_{t-d_i},$$

here d_i, O_i are the i-th fission time and the number of offspring of the spine at time d_i respectively. Given all the information $\mathcal{G}$ about the spine, $(W^{i,j}_{t-d_i})_{j\ge1}$ are independent with the same law as $W_{t-d_i}(\theta)$ under $\mathbb{P}$.

Using elementary analysis one can easily show that there exists $A = A_\lambda > 1$ such that for any $x, y > A$,

$$\log_+^{1+\lambda}(x+y) \le \log_+^{1+\lambda}(x) + \log_+^{1+\lambda}(y). \tag{2.18}$$

We set

$$K_1 := e^{-(1+\frac{\theta^2}{2})t}e^{-\theta X_\xi(t)},$$

$$K_2 := \sum_{i:d_i\le t} e^{-(1+\frac{\theta^2}{2})d_i}e^{-\theta X_\xi(d_i)}\sum_{j=1}^{O_i-1} W^{i,j}_{t-d_i}1_{\left\{e^{-(1+\frac{\theta^2}{2})d_i}e^{-\theta X_\xi(d_i)}\sum_{j=1}^{O_i-1}W^{i,j}_{t-d_i}\le A\right\}} \le A\sum_{i:d_i\le t}1,$$

$$K_3 := \sum_{i:d_i\le t} e^{-(1+\frac{\theta^2}{2})d_i}e^{-\theta X_\xi(d_i)}\sum_{j=1}^{O_i-1} W^{i,j}_{t-d_i}1_{\left\{e^{-(1+\frac{\theta^2}{2})d_i}e^{-\theta X_\xi(d_i)}\sum_{j=1}^{O_i-1}W^{i,j}_{t-d_i}> A\right\}}.$$

Note that $\log_+^{1+\lambda}(x+y+z) \le \log_+^{1+\lambda}(3x) + \log_+^{1+\lambda}(3y) + \log_+^{1+\lambda}(3z)$, $\log_+^{1+\lambda}(xy) \le (\log_+ x + \log_+ y)^{1+\lambda} \lesssim \log_+^{1+\lambda}(x) + \log_+^{1+\lambda}(y)$ and $\log_+^{1+\lambda}(x) \lesssim x$. By (2.18), we have

$$\begin{aligned}
&\log_+^{1+\lambda}(W_t(\theta)) = \log_+^{1+\lambda}(K_1+K_2+K_3) \le \log_+^{1+\lambda}(3K_1) + \log_+^{1+\lambda}(3K_2) + \log_+^{1+\lambda}(3K_3)\\
&\lesssim 1 + \log_+^{1+\lambda}(K_1) + \Big(\sum_{i:d_i\le t}1\Big) + \sum_{i:d_i\le t}\log_+^{1+\lambda}\Big(e^{-(1+\frac{\theta^2}{2})d_i}e^{-\theta X_\xi(d_i)}\sum_{j=1}^{O_i-1}W^{i,j}_{t-d_i}\Big)\\
&\lesssim 1 + \log_+^{1+\lambda}(K_1) + \Big(\sum_{i:d_i\le t}1 + \log_+^{1+\lambda}\big(e^{-(1+\frac{\theta^2}{2})d_i}e^{-\theta X_\xi(d_i)}\big)\Big)\\
&\qquad + \sum_{i:d_i\le t}\log_+^{1+\lambda}\Big(\sum_{j=1}^{O_i-1}W^{i,j}_{t-d_i}\Big).
\end{aligned} \tag{2.19}$$

Put $\gamma := 1-\theta^2/2 > 0$. Recalling that $\{X_\xi(t)+\theta t, \mathbb{P}^{-\theta}\}$ is a standard Brownian motion and $\{d_i : i\geq 1\}$ are the atoms of a Poisson point process with rate 2 independent of X_ξ, we have

$$\begin{aligned}
&\mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}(K_1) + \big(\sum_{i:d_i\leq t} 1 + \log_+^{1+\lambda}\big(e^{-(1+\frac{\theta^2}{2})d_i} e^{-\theta X_\xi(d_i)}\big)\big)\Big) \\
&= \mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}(K_1) + 2\int_0^t \big(1+\log_+^{1+\lambda}\big(e^{-(1+\frac{\theta^2}{2})s} e^{-\theta X_\xi(s)}\big)\mathrm{d}s\big)\Big) \\
&\lesssim \Pi_0\Big((-\theta B_t - \gamma t)_+^{1+\lambda} + \int_0^t \big(1+(-\theta B_s-\gamma s)_+^{1+\lambda}\big)\mathrm{d}s\Big) \lesssim t+1,
\end{aligned} \tag{2.20}$$

where the last inequality follows from the following estimate:

$$\begin{aligned}
&\Pi_0\big((-\theta B_s - \gamma s)_+^{1+\lambda}\big) = s^{(1+\lambda)/2}\Pi_0\big(\big(|\theta| B_1 - \gamma\sqrt{s}\big)^{1+\lambda} 1_{\{|\theta|B_1>\gamma\sqrt{s}\}}\big) \\
&\leq \gamma^{1+\lambda}(s+1)^{1+\lambda} e^{-\gamma\sqrt{s}}\Pi_0\big((|\theta||B_1|+1)\, e^{|\theta|B_1}\big) \lesssim 1.
\end{aligned}$$

For the last term on the right-hand side of (2.19), conditioned on $\{d_i, O_i : i\geq 1\}$, we get

$$\begin{aligned}
&\mathbb{E}^{-\theta}\Big(\sum_{i:d_i\leq t} \log_+^{1+\lambda}\big(\sum_{j=1}^{O_i-1} W_{t-d_i}^{i,j}\big) \big| d_i, O_i, i\geq 1\Big) \\
&\lesssim \sum_{i:d_i\leq t} \log_+^{1+\lambda}(O_i-1) + \sum_{i:d_i\leq t} \mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}\big(\max_{j\leq O_i-1} W_{t-d_i}^{i,j}\big) \big| d_i, O_i, i\geq 1\Big).
\end{aligned} \tag{2.21}$$

Note that

$$\begin{aligned}
&\mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}\big(\max_{j\leq O_i-1} W_{t-d_i}^{i,j}\big) \big| d_i, O_i, i\geq 1\Big) \\
&= (1+\lambda)\int_0^\infty y^\lambda \mathbb{P}^{-\theta}\Big(\max_{j\leq O_i-1} W_{t-d_i}^{i,j} > e^y \Big| d_i, O_i, i\geq 1\Big)\mathrm{d}y \\
&= (1+\lambda)\int_0^\infty y^\lambda \Big(1 - \prod_{j=1}^{O_i-1}\big(1-\mathbb{P}^{-\theta}\big(W_{t-d_i}^{i,j} > e^y \big| d_i, O_i, i\geq 1\big)\big)\Big)\mathrm{d}y \\
&\lesssim \int_0^\infty y^\lambda \big(1-\big(1-e^{-y}\big)^{O_i-1}\big)\mathrm{d}y,
\end{aligned} \tag{2.22}$$

where in the inequality we used Markov's inequality. When $O_i - 1 \geq e^{y/2}$ (which is equivalent to $y\leq 2\log(O_i-1)$), we have $y^\lambda\left(1-(1-e^{-y})^{O_i-1}\right) \lesssim \log^\lambda(O_i-1)$; when $O_i-1 < e^{y/2}$, by the inequality $(1-x)^n \geq 1-nx$, we get

$$y^\lambda\big(1-\big(1-e^{-y}\big)^{O_i-1}\big) \leq y^\lambda (O_i-1)e^{-y} \leq y^\lambda e^{-y/2}.$$

Thus, by (2.22),

$$\mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}\big(\max_{j\leq O_i-1} W_{t-d_i}^{i,j}\big) \big| d_i, O_i, i\geq 1\Big) \lesssim \log_+^{1+\lambda}(O_i-1)+1.$$

Plugging this back to (2.21) and taking expectation with respect to $\mathbb{P}^{-\theta}$, we conclude that

$$\begin{aligned}
&\mathbb{E}^{-\theta}\Big(\sum_{i:d_i\leq t} \log_+^{1+\lambda}\big(\sum_{j=1}^{O_i-1} W_{t-d_i}^{i,j}\big)\Big) \\
&\lesssim \mathbb{E}^{-\theta}\Big(\sum_{i:d_i\leq t} \big(\log_+^{1+\lambda}(O_i-1)+1\big)\Big) \lesssim t+1.
\end{aligned} \tag{2.23}$$

Combining (2.19), (2.20) and (2.23), we get the desired result. □

3 Proof of the main results

Let $\kappa > 1$ be fixed. Define

$$r_n := n^{\frac{1}{\kappa}}, \quad n \in \mathbb{N}$$

Lemma 3.1 *For any given $\alpha, \beta > 0$ and $\delta \in (0,1]$, assume that (1.7) holds for λ with $\lambda\delta - \alpha > \kappa(1+\beta)$.*

(i) For each n, let $a_n \le n^\beta$ and $\{Y_{n,u} : u \in N(r_n^\delta)\}$ be a family of random variables such that $\mathbb{E}\left(Y_{n,u} \middle| \mathcal{F}_{r_n^\delta}\right) = 0$, and conditioned on $\mathcal{F}_{r_n^\delta}$, $Y_{n,u}, u \in N(r_n^\delta)$, are independent. If $|Y_{n,u}| \le W_{a_n}(\theta; u) + 1$ for all n and $u \in N(r_n^\delta)$, with $\left(W_{a_n}(\theta; u), \mathbb{P}(\cdot|\mathcal{F}_{r_n^\delta})\right)$ being a copy of $W_{a_n}(\theta)$, then

$$r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta} \sum_{u \in N(r_n^\delta)} e^{-\theta \cdot \mathbf{X}_u(r_n^\delta)} Y_{n,u} \stackrel{n\to\infty}{\longrightarrow} 0, \quad a.s.$$

(ii) Consequently, if $\lambda\delta - \alpha > \kappa + 1$, then for any sequence $\{A_n\}$ of Borel sets in $\mathbb{R}^d$,

$$r_n^\alpha \left| \mu_{r_n}^\theta(A_n) - \mathbb{E}\left[\mu_{r_n}^\theta(A_n) \middle| \mathcal{F}_{r_n^\delta}\right] \right| \stackrel{n\to\infty}{\longrightarrow} 0, \quad a.s.$$

Proof: (i) Define

$$\overline{Y}_{n,u} := Y_{n,u} 1_{\{|Y_{n,u}| \le e^{c_* r_n^\delta}\}}, \quad Y'_{n,u} = \overline{Y}_{n,u} - \mathbb{E}\left(\overline{Y}_{n,u} \middle| \mathcal{F}_{r_n^\delta}\right),$$

where $c_* > 0$ is a constant to be chosen later. Then for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\Big(\Big| r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta} \sum_{u \in N(r_n^\delta)} e^{-\theta \cdot \mathbf{X}_u(r_n^\delta)} Y_{n,u}\Big| > \varepsilon \Big| \mathcal{F}_{r_n^\delta}\Big) \\
&\le \mathbb{P}\Big(\Big| r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta} \sum_{u \in N(r_n^\delta)} e^{-\theta \cdot \mathbf{X}_u(r_n^\delta)} \left(Y_{n,u} - \overline{Y}_{n,u}\right)\Big| > \frac{\varepsilon}{3} \Big| \mathcal{F}_{r_n^\delta}\Big) \\
&\quad + \mathbb{P}\Big(r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta} \Big| \sum_{u \in N(r_n^\delta)} e^{-\theta \cdot \mathbf{X}_u(r_n^\delta)} Y'_{n,u}\Big| > \frac{\varepsilon}{3} \Big| \mathcal{F}_{r_n^\delta}\Big) \\
&\quad + 1_{\left\{ r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta} \left|\sum_{u \in N(r_n^\delta)} e^{-\theta \cdot \mathbf{X}_u(r_n^\delta)} \mathbb{E}\left(\overline{Y}_{n,u} \middle| \mathcal{F}_{r_n^\delta}\right)\right| > \frac{\varepsilon}{3}\right\}} =: I + II + III.
\end{aligned} \tag{3.1}$$

Using the inequality

$$|Y_{n,u} - \overline{Y}_{n,u}| = |Y_{n,u}| 1_{\{|Y_{n,u}| > e^{c_* r_n^\delta}\}} \le \left(W_{a_n}(\theta; u) + 1\right) 1_{\left\{W_{a_n}(\theta;u)+1 > e^{c_* r_n^\delta}\right\}}$$

and Markov's inequality, we have

$$\begin{aligned}
I &\le \frac{3}{\varepsilon} r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta} \sum_{u \in N(r_n^\delta)} e^{-\theta \cdot \mathbf{X}_u(r_n^\delta)} \mathbb{E}\left(\left|Y_{n,u} - \overline{Y}_{n,u}\right| \middle| \mathcal{F}_{r_n^\delta}\right) \\
&\lesssim r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta} \sum_{u \in N(r_n^\delta)} e^{-\theta \cdot \mathbf{X}_u(r_n^\delta)} \mathbb{E}\left(\left(W_{a_n}(\theta) + 1\right) 1_{\{W_{a_n}(\theta)+1 > e^{c_* r_n^\delta}\}}\right) \\
&\le \frac{r_n^\alpha}{(c_* r_n^\delta)^{\lambda+1}} e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta} \sum_{u \in N(r_n^\delta)} e^{-\theta \cdot \mathbf{X}_u(r_n^\delta)} \mathbb{E}\left(\left(W_{a_n}(\theta) + 1\right) \log_+^{1+\lambda}\left(W_{a_n}(\theta) + 1\right)\right)
\end{aligned}$$

$$\lesssim \frac{r_n^{\alpha}}{(c_* r_n^{\delta})^{\lambda+1}} e^{-(1+\frac{\|\theta\|^2}{2})r_n^{\delta}} \sum_{u\in N(r_n^{\delta})} e^{-\theta\cdot\mathbf{X}_u(r_n^{\delta})} n^{\beta}, \tag{3.2}$$

where in the last inequality we used Lemma 2.7. For $1<\ell<\min\{2/\|\theta\|^2,2\}$, set $b:=e^{(\ell-1)(1-\|\theta\|^2\ell/2)/2}\in(1,e)$ and $c_*:=\ln b$. Using Markov's inequality, Lemma 2.2, the conditional independence of $Y'_{n,u}$, and the fact that $|\overline{Y}_{n,u}|^{\ell}\le e^{c_*(\ell-1)r_n^{\delta}}|Y_{n,u}|\le e^{c_* r_n^{\delta}}\left(W_{a_n}(\theta;u)+1\right)$, we have

$$\begin{aligned}
II &\le \frac{3^{\ell}}{\varepsilon^{\ell}} r_n^{\alpha\ell} e^{-\ell(1+\frac{\|\theta\|^2}{2})r_n^{\delta}} \mathbb{E}\Big(\Big|\sum_{u\in N(r_n^{\delta})} e^{-\theta\cdot\mathbf{X}_u(r_n^{\delta})} Y'_{n,u}\Big|^{\ell}\Big|\mathcal{F}_{r_n^{\delta}}\Big)\\
&\lesssim \frac{3^{\ell}}{\varepsilon^{\ell}} r_n^{\ell\alpha} e^{-\ell(1+\frac{\|\theta\|^2}{2})r_n^{\delta}} \sum_{u\in N(r_n^{\delta})} e^{-\ell\theta\cdot\mathbf{X}_u(r_n^{\delta})} \mathbb{E}\left(\left|\overline{Y}_{n,u}\right|^{\ell}\left|\mathcal{F}_{r_n^{\delta}}\right.\right)\\
&\le \frac{3^{\ell}}{\varepsilon^{\ell}} r_n^{\ell\alpha} e^{-\ell(1+\frac{\|\theta\|^2}{2})r_n^{\delta}} \sum_{u\in N(r_n^{\delta})} e^{-\ell\theta\cdot\mathbf{X}_u(r_n^{\delta})} e^{c_* r_n^{\delta}} \mathbb{E}\left(W_{a_n}(\theta;u)+1\middle|\mathcal{F}_{r_n^{\delta}}\right)\\
&\lesssim r_n^{\ell\alpha} b^{-2r_n^{\delta}} e^{-(1+\frac{\ell^2\|\theta\|^2}{2})r_n^{\delta}} b^{r_n^{\delta}} \sum_{u\in N(r_n^{\delta})} e^{-\ell\theta\cdot\mathbf{X}_u(r_n^{\delta})},
\end{aligned} \tag{3.3}$$

where in the last inequality, we used the identities $\mathbb{E}\left(W_{a_n}(\theta;u)+1\middle|\mathcal{F}_{r_n^{\delta}}\right)=\mathbb{E}\left(W_{a_n}(\theta)+1\right)=2$, $e^{-\ell(1+\frac{\|\theta\|^2}{2})r_n^{\delta}}=b^{-2r_n^{\delta}}e^{-(1+\frac{\ell^2\|\theta\|^2}{2})r_n^{\delta}}$, and $e^{c_*}=b$. Therefore, by (3.3), we get

$$II \lesssim r_n^{\ell\alpha} b^{-r_n^{\delta}} W_{r_n^{\delta}}(\ell\theta). \tag{3.4}$$

Now taking expectation with respect to $\mathbb{P}$ in (3.1), and using (3.2) and (3.4), we get that

$$\begin{aligned}
&\mathbb{P}\Big(\Big|r_n^{\alpha} e^{-(1+\frac{\|\theta\|^2}{2})r_n^{\delta}} \sum_{u\in N(r_n^{\delta})} e^{-\theta\cdot\mathbf{X}_u(r_n^{\delta})} Y_{n,u}\Big|>\varepsilon\Big)\\
&\lesssim \frac{r_n^{\alpha} n^{\beta}}{(r_n^{\delta})^{\lambda+1}} + r_n^{\ell\alpha} b^{-r_n^{\delta}} + \mathbb{P}\left(r_n^{\alpha} e^{-(1+\frac{\|\theta\|^2}{2})r_n^{\delta}}\Big|\sum_{u\in N(r_n^{\delta})} e^{-\theta\cdot\mathbf{X}_u(r_n^{\delta})}\mathbb{E}\left(\overline{Y}_{n,u}\middle|\mathcal{F}_{r_n^{\delta}}\right)\Big|>\frac{\varepsilon}{3}\right).
\end{aligned} \tag{3.5}$$

By Markov's inequality and the fact that $\mathbb{E}\left(\overline{Y}_{n,u}\middle|\mathcal{F}_{r_n^{\delta}}\right)=-\mathbb{E}\left(Y_{n,u}1_{\{|Y_{n,u}|>e^{c_* r_n^{\delta}}\}}\middle|\mathcal{F}_{r_n^{\delta}}\right)$, the third term in right-hand side of (3.5) is bounded from above by

$$\begin{aligned}
&\frac{3}{\varepsilon} r_n^{\alpha} e^{-(1+\frac{\|\theta\|^2}{2})r_n^{\delta}} \mathbb{E}\Big(\sum_{u\in N(r_n^{\delta})} e^{-\theta\cdot\mathbf{X}_u(r_n^{\delta})}\mathbb{E}\big(|Y_{n,u}|1_{\{|Y_{n,u}|>e^{c_* r_n^{\delta}}\}}\big|\mathcal{F}_{r_n^{\delta}}\big)\Big)\\
&\le \frac{3}{\varepsilon} r_n^{\alpha}(c_* r_n^{\delta})^{-\lambda-1}\mathbb{E}\left((W_{a_n}(\theta)+1)\log_+^{\lambda+1}(1+W_{a_n}(\theta))\right) \lesssim r_n^{\alpha}(r_n^{\delta})^{-\lambda-1} n^{\beta},
\end{aligned}$$

where in the last inequality we used Lemma 2.7. Plugging the upper bound above into (3.5) and recalling $r_n=n^{\frac{1}{\kappa}}$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{P}\Big(\Big|r_n^{\alpha} e^{-(1+\frac{\|\theta\|^2}{2})r_n^{\delta}} \sum_{u\in N(r_n^{\delta})} e^{-\theta\cdot\mathbf{X}_u(r_n^{\delta})} Y_{n,u}\Big|>\varepsilon\Big)\\
&\lesssim \sum_{n=1}^{\infty}\Big(\frac{r_n^{\alpha} n^{\beta}}{(r_n^{\delta})^{\lambda+1}} + r_n^{\ell\alpha} b^{-r_n^{\delta}} + r_n^{-((\lambda+1)\delta-\alpha-\kappa\beta)}\Big),
\end{aligned}$$

which is summable since $\lambda\delta-\alpha>\kappa(1+\beta)$. This completes the proof of (i).

(ii) By the Markov property and Lemma 2.1,

$$\begin{aligned}&\mathbb{E}\left[\mu^{\theta}_{r_n}(A_n)\,\middle|\mathcal{F}_{r_n^\delta}\right]\\&=e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)}e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}\mathbb{P}^{-\theta}\left(\mathbf{X}_\xi(r_n-r_n^\delta)+\theta r_n+\mathbf{y}\in A_n\right)\big|_{\mathbf{y}=\mathbf{X}_u(r_n^\delta)}.\end{aligned}$$

Since $\{\mathbf{X}_\xi(t)+\theta t,\mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, we have

$$\begin{aligned}&\mathbb{E}\left[\mu^{\theta}_{r_n}(A_n)\,\middle|\mathcal{F}_{r_n^\delta}\right]\\&=e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)}e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}\Pi_{\mathbf{0}}\left(\mathbf{B}_{r_n-r_n^\delta}+\mathbf{y}+\theta r_n^\delta\in A_n\right)\big|_{\mathbf{y}=\mathbf{X}_u(r_n^\delta)}.\end{aligned}$$

Therefore,

$$\mu^{\theta}_{r_n}(A_n)-\mathbb{E}\left[\mu^{\theta}_{r_n}(A_n)\,\middle|\mathcal{F}_{r_n^\delta}\right]=:e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)}e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}Y_{n,u},$$

where

$$\begin{aligned}Y_{n,u}:=\;&e^{-(1+\frac{\|\theta\|^2}{2})(r_n-r_n^\delta)}\sum_{v\in N(r_n):u\le v}e^{-\theta\cdot\left(\mathbf{X}_v(r_n)-\mathbf{X}_u(r_n^\delta)\right)}1_{\{\mathbf{X}_v(r_n)+\theta r_n\in A_n\}}\\&-\Pi_{\mathbf{0}}\left(\mathbf{B}_{r_n-r_n^\delta}+\mathbf{y}+\theta r_n^\delta\in A_n\right)\big|_{\mathbf{y}=\mathbf{X}_u(r_n^\delta)}.\end{aligned}$$

By the branching property, we see that, conditioned on $\mathcal{F}_{r_n^\delta}$, $\{Y_{n,u}:u\in N(r_n^\delta)\}$ is a family of centered independent random variables. Furthermore, it holds that

$$\begin{aligned}|Y_{n,u}|&\le e^{-(1+\frac{\|\theta\|^2}{2})(r_n-r_n^\delta)}\sum_{v\in N(r_n):u\le v}e^{-\theta\cdot\left(\mathbf{X}_v(r_n)-\mathbf{X}_u(r_n^\delta)\right)}+1\\&=W_{r_n-r_n^\delta}(\theta;u)+1.\end{aligned}$$

Therefore, the second result is valid by (i) by taking $\beta=1/\kappa$ and $a_n=r_n-r_n^\delta$. □

Now we treat the case $s\in[r_n,r_{n+1})$. We will take $\delta=1/2,\beta=1/\kappa$ and $\alpha=m/2$ for $m\in\mathbb{N}$. Then the condition $\lambda\delta-\alpha>\kappa(1+\beta)$ is equivalent to $\lambda>m+2(\kappa+1)$.

Lemma 3.2 *For $\mathbf{b}\in\mathbb{R}^d$, let $\mathbf{b}_s:=\mathbf{b}\sqrt{s}$ or $\mathbf{b}_s:=\mathbf{b}$. For any given $m\in\mathbb{N}$, assume that $\kappa>m+2$ and that* (1.7) *holds for some $\lambda>m+2(\kappa+1)$. Define $k_s:=\sqrt{r_n}$ for $s\in[r_n,r_{n+1})$. Then for any $\mathbf{b}\in\mathbb{R}^d$,*

$$s^{m/2}\left|\mu^{\theta}_s((-\infty,\mathbf{b}_s])-\mathbb{E}\left[\mu^{\theta}_s((-\infty,\mathbf{b}_s])\,\middle|\mathcal{F}_{k_s}\right]\right|\stackrel{s\to\infty}{\longrightarrow}0,\quad\mathbb{P}\text{-}a.s.$$

Proof: Step 1: In this step, we prove that almost surely,

$$\sup_{r_n\le s<r_{n+1}}s^{m/2}\left|\mathbb{E}\left[\mu^{\theta}_s((-\infty,\mathbf{b}_s])\,\middle|\mathcal{F}_{k_s}\right]-\mathbb{E}\left[\mu^{\theta}_{r_n}((-\infty,\mathbf{b}_{r_n}])\,\middle|\mathcal{F}_{k_s}\right]\right|\stackrel{n\to\infty}{\longrightarrow}0.\qquad(3.6)$$

By the Markov property and Lemma 2.1, we see that

$$\begin{aligned}
&\mathbb{E}\left[\mu_s^\theta\left((-\infty,\mathbf{b}_s]\right)\big|\mathcal{F}_{k_s}\right]\\
&= e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}\Pi_{\mathbf{0}}\left(\mathbf{B}_{s-\sqrt{r_n}}+\mathbf{y}+\theta\sqrt{r_n}\le\mathbf{b}_s\right)\Big|_{\mathbf{y}=\mathbf{X}_u(\sqrt{r_n})}\\
&= e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}\Phi_d\Big(\frac{\mathbf{b}_s-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big),
\end{aligned}$$

Thus, for $s\in[r_n,r_{n+1})$, it holds that

$$\begin{aligned}
&s^{m/2}\left|\mathbb{E}\left[\mu_s^\theta\left((-\infty,b_s]\right)\big|\mathcal{F}_{k_s}\right]-\mathbb{E}\left[\mu_{r_n}^\theta\left((-\infty,b_{r_n}]\right)\big|\mathcal{F}_{k_s}\right]\right|\\
&\le r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}\\
&\qquad\times\left|\Phi_d\Big(\frac{\mathbf{b}_s-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big)-\Phi_d\Big(\frac{\mathbf{b}_{r_n}-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{r_n-\sqrt{r_n}}}\Big)\right|\\
&=: r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}R(u,s).
\end{aligned}$$

Note that, on the event $\cup_{j=1}^d\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{r_n}\}$, we use the trivial upper-bound $\sup_{r_n\le s<r_{n+1}}R(u,s)\le 2$. Using Lemma 2.1 and the fact that $\{\mathbf{X}_\xi(t)+\theta t,\mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, we have

$$\begin{aligned}
&\sum_{n=1}^\infty r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\mathbb{E}\Big(\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{\{\cup_{j=1}^d\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{r_n}\}\}}\sup_{r_n\le s<r_{n+1}}R(u,s)\Big)\\
&\le 2\sum_{n=1}^\infty r_{n+1}^{m/2}\Pi_{\mathbf{0}}\left(\cup_{j=1}^d\left\{|(\mathbf{B}_1)_j|>r_n^{1/4}\right\}\right)\le 2d\Pi_{\mathbf{0}}(e^{|(\mathbf{B}_1)_1|})\sum_{n=1}^\infty r_{n+1}^{m/2}e^{-r_n^{1/4}}<\infty,
\end{aligned}$$

which implies that $\mathbb{P}$-almost surely,

$$r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{\{\cup_{j=1}^d\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{r_n}\}\}}R(u,s)\overset{s\to\infty}{\longrightarrow}0. \tag{3.7}$$

On the other hand, on the event $\cap_{j=1}^d\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{r_n}\}$, in the case $\mathbf{b}_s=\mathbf{b}\sqrt{s}$, using the trivial inequality

$$|\Phi_d(\mathbf{a})-\Phi_d(\mathbf{b})|\le\sum_{j=1}^d|\Phi(a_j)-\Phi(b_j)|\le\frac{1}{\sqrt{2\pi}}\sum_{j=1}^d|a_j-b_j|,$$

we get that, uniformly for $s\in[r_nr_{n+1})$,

$$\begin{aligned}
R(u,s)\le\sum_{j=1}^d\Bigg(&\frac{|b_j|}{\sqrt{2\pi}}\left|\frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}}-\frac{\sqrt{r_n}}{\sqrt{r_n-\sqrt{r_n}}}\right|\\
&+\frac{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|}{\sqrt{2\pi}}\left|\frac{1}{\sqrt{s-\sqrt{r_n}}}-\frac{1}{\sqrt{r_n-\sqrt{r_n}}}\right|\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{\left|\sqrt{s(r_n-\sqrt{r_n})}-\sqrt{r_n(s-\sqrt{r_n})}\right|}{\sqrt{s-\sqrt{r_n}}\sqrt{r_n-\sqrt{r_n}}}+\sqrt{r_n}\frac{\left|\sqrt{s-\sqrt{r_n}}-\sqrt{r_n-\sqrt{r_n}}\right|}{\sqrt{s-\sqrt{r_n}}\sqrt{r_n-\sqrt{r_n}}}\\
&\lesssim \frac{1}{r_n}\frac{\left|s(r_n-\sqrt{r_n})-r_n(s-\sqrt{r_n})\right|}{r_n}+\frac{\sqrt{r_n}}{r_n^{3/2}}(r_{n+1}-r_n)\lesssim \frac{1}{r_n}(r_{n+1}-r_n).
\end{aligned}
$$

In the case $\mathbf{b}_s=\mathbf{b}$, we have

$$
\begin{aligned}
R(u,s) \le& \sum_{j=1}^{d}\left(\frac{|b_j|}{\sqrt{2\pi}}\left|\frac{1}{\sqrt{s-\sqrt{r_n}}}-\frac{1}{\sqrt{r_n-\sqrt{r_n}}}\right|\right.\\
&\left.+\frac{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|}{\sqrt{2\pi}}\left|\frac{1}{\sqrt{s-\sqrt{r_n}}}-\frac{1}{\sqrt{r_n-\sqrt{r_n}}}\right|\right)\\
\lesssim& \frac{|s-r_n|}{\sqrt{s-\sqrt{r_n}}\sqrt{r_n-\sqrt{r_n}}\left(\sqrt{s-\sqrt{r_n}}+\sqrt{r_n-\sqrt{r_n}}\right)}+\sqrt{r_n}\frac{\left|\sqrt{s-\sqrt{r_n}}-\sqrt{r_n-\sqrt{r_n}}\right|}{\sqrt{s-\sqrt{r_n}}\sqrt{r_n-\sqrt{r_n}}}\\
\lesssim& \frac{1}{r_n^{3/2}}(r_{n+1}-r_n)+\frac{\sqrt{r_n}}{r_n^{3/2}}(r_{n+1}-r_n)\lesssim\frac{1}{r_n}(r_{n+1}-r_n).
\end{aligned}
$$

Thus in both cases, we have that

$$
\begin{aligned}
&r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})}e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{\{\cap_{j=1}^d\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{r_n}\}\}}\sup_{r_n\le s<r_{n+1}}R(u,s)\\
&\lesssim r_{n+1}^{m/2}\frac{1}{r_n}(r_{n+1}-r_n)W_{\sqrt{r_n}}(\theta). \qquad (3.8)
\end{aligned}
$$

We claim that right-hand side of (3.8) goes to 0 almost surely as $s\to\infty$. In fact,

$$
r_{n+1}^{m/2}\frac{1}{r_n}(r_{n+1}-r_n)\lesssim r_n^{(-2+m)/2}(r_{n+1}-r_n)=n^{(-2+m)/(2\kappa)}\left((n+1)^{1/\kappa}-n^{1/\kappa}\right).
$$

By the mean value theorem, the right-hand side above is equal to

$$
n^{(-2+m)/(2\kappa)}\xi^{-1+\frac{1}{\kappa}}\lesssim n^{(m-2\kappa)/(2\kappa)}.
$$

Since $\kappa>m+2$, the claim is valid. Combining this with (3.7) and (3.8), we get (3.6).

Step 2: In this step, we prove that

$$
\sup_{r_n\le s<r_{n+1}} s^{m/2}\left|\mu_s^\theta\left((-\infty,\mathbf{b}_s]\right)-\mu_{r_n}^\theta\left((-\infty,\mathbf{b}_{r_n}]\right)\right|\overset{n\to\infty}{\longrightarrow}0,\quad \mathbb{P}\text{-a.s.} \qquad (3.9)
$$

Once we get (3.9), we can combine (3.6) and Lemma 3.1 (ii) (with $A_n=(-\infty,\mathbf{b}_{r_n}]$ and $\delta=1/2$) to get the assertion of the lemma.

To prove (3.9), we first prove that

$$
\liminf_{n\to\infty}\inf_{r_n\le s<r_{n+1}} s^{m/2}\left(\mu_s^\theta\left((-\infty,\mathbf{b}_s]\right)-\mu_{r_n}^\theta\left((-\infty,\mathbf{b}_{r_n}]\right)\right)\ge 0,\quad \mathbb{P}\text{-a.s.} \qquad (3.10)
$$

Define $\varepsilon_n:=\sqrt{r_{n+1}-r_n}$. For $u\in N(r_n)$, let G_u be the event that u does not split before r_{n+1} and that $\max_{s\in(r_n,r_{n+1})}\|\mathbf{X}_u(s)-\mathbf{X}_u(r_n)\|\le\sqrt{r_n}\varepsilon_n$. Then

$$
\mathbb{P}\left(G_u\middle|\mathcal{F}_{r_n}\right)=e^{-(r_{n+1}-r_n)}\Pi_{\mathbf{0}}\Big(\max_{r\le r_{n+1}-r_n}\|\mathbf{B}_r\|\le\sqrt{r_n}\varepsilon_n\Big)=e^{-(r_{n+1}-r_n)}\Pi_{\mathbf{0}}\Big(\max_{r\le1}\|\mathbf{B}_r\|\le\sqrt{r_n}\Big).
$$

Recalling that $\mathbf{1} := (1, ..., 1)$, it holds that

$$\begin{aligned}
&\mu_s^\theta((-\infty, \mathbf{b}_s]) = e^{-(1+\frac{\|\theta\|^2}{2})s} \sum_{u\in N(r_n)} e^{-\theta\cdot\mathbf{X}_u(r_n)} \sum_{v\in N(s): u\le v} e^{-\theta\cdot(\mathbf{X}_v(s)-\mathbf{X}_u(r_n))} 1_{\{\mathbf{X}_v(s)+\theta s\le \mathbf{b}_s\}} \\
&\ge e^{-(1+\frac{\|\theta\|^2}{2})r_{n+1}} e^{-\|\theta\|\sqrt{r_n}\varepsilon_n} \sum_{u\in N(r_n)} e^{-\theta\cdot\mathbf{X}_u(r_n)} 1_{\{\mathbf{X}_u(r_n)+\theta r_n \le \mathbf{b}_{r_n}-\varepsilon_n\sqrt{r_n}\mathbf{1}-\|\theta\|(r_{n+1}-r_n)\mathbf{1}\}} 1_{G_u} \\
&= e^{-(1+\frac{\|\theta\|^2}{2})r_{n+1}-\|\theta\|\sqrt{r_n}\varepsilon_n} \sum_{u\in N(r_n)} e^{-\theta\cdot\mathbf{X}_u(r_n)} \\
&\quad\times 1_{\{\mathbf{X}_u(r_n)+\theta r_n \le \mathbf{b}_{r_n}-\varepsilon_n\sqrt{r_n}\mathbf{1}-\|\theta\|(r_{n+1}-r_n)\mathbf{1}\}} \left(1_{G_u} - \mathbb{P}\left(G_u \middle| \mathcal{F}_{r_n}\right)\right) \\
&\quad + e^{-(1+\frac{\|\theta\|^2}{2})r_{n+1}-\|\theta\|\sqrt{r_n}\varepsilon_n} \sum_{u\in N(r_n)} e^{-\theta\cdot\mathbf{X}_u(r_n)} \\
&\quad\times 1_{\{\mathbf{X}_u(r_n)+\theta r_n \le \mathbf{b}_{r_n}-\varepsilon_n\sqrt{r_n}\mathbf{1}-\|\theta\|(r_{n+1}-r_n)\mathbf{1}\}} \mathbb{P}\left(G_u \middle| \mathcal{F}_{r_n}\right) =: I + II. \qquad (3.11)
\end{aligned}$$

For I, we will apply Lemma 3.1 (i) with $\alpha = m/2$, $\delta = 1, a_n = 0, \beta = 0$ and

$$Y_{n,u} := 1_{\{\mathbf{X}_u(r_n)+\theta r_n \le \mathbf{b}_{r_n}-\varepsilon_n\sqrt{r_n}\mathbf{1}-\|\theta\|(r_{n+1}-r_n)\mathbf{1}\}} \left(1_{G_u} - \mathbb{P}\left(G_u \middle| \mathcal{F}_{r_n}\right)\right).$$

It is easy to see that $|Y_{n,u}| \le 2$, $r_{n+1} - r_n \to 0$ and $\sqrt{r_n}\varepsilon_n \lesssim \sqrt{n^{(2-\kappa)/\kappa}} \to 0$. Since $\lambda > m+2(\kappa+1)$, we have

$$\sup_{r_n\le s<r_{n+1}} s^{m/2}|I| \stackrel{n\to\infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.} \qquad (3.12)$$

If we can prove that

$$\sup_{r_n\le s<r_{n+1}} s^{m/2} \left| II - \mu_{r_n}^\theta((-\infty, \mathbf{b}_{r_n}]) \right| \stackrel{n\to\infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.}, \qquad (3.13)$$

then (3.10) will follow from (3.11), (3.12) and (3.13). Now we prove (3.13). Since $\kappa > m + 2$, we have $r_n^{m/2}(r_{n+1} - r_n) \lesssim n^{-1+(m+2)/(2\kappa)} \to 0$. Thus,

$$\begin{aligned}
&s^{m/2}\left|1 - \mathbb{P}\left(G_u \middle| \mathcal{F}_{r_n}\right)\right| \le r_{n+1}^{m/2}(1 - e^{-(r_{n+1}-r_n)}) + r_{n+1}^{m/2}\Pi_{\mathbf{0}}\Big(\max_{r\le 1}\|\mathbf{B}_r\| > \sqrt{r_n}\Big) \\
&\lesssim r_{n+1}^{m/2}(r_{n+1} - r_n) + r_{n+1}^{m/2} e^{-\sqrt{r_n}} \to 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&s^{m/2}\left| e^{(1+\frac{\|\theta\|^2}{2})(r_{n+1}-r_n)} e^{\|\theta\|\sqrt{r_n}\varepsilon_n} II - \mu_{r_n}^\theta((-\infty, \mathbf{b}_{r_n} - \varepsilon_n\sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1} - r_n)\mathbf{1}]) \right| \\
&\lesssim W_{r_n}(\theta)\left(r_{n+1}^{m/2}(r_{n+1} - r_n) + r_{n+1}^{m/2} e^{-\sqrt{r_n}}\right) \stackrel{s\to\infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.} \qquad (3.14)
\end{aligned}$$

Note that $0 \le e^{(1+\frac{\|\theta\|^2}{2})(r_{n+1}-r_n)} e^{\|\theta\|\sqrt{r_n}\varepsilon_n} II \le W_{r_n}(\theta)$, $e^{(1+\frac{\|\theta\|^2}{2})(r_{n+1}-r_n)} = 1 + O(r_{n+1} - r_n) = 1 + o(r_n^{-m/2})$, and that $e^{\|\theta\|\sqrt{r_n}\varepsilon_n} = 1 + O(\sqrt{r_n}\sqrt{r_{n+1} - r_n}) = 1 + O(n^{1/\kappa-1/2}) = 1 + o(r_n^{-m/2})$ by the assumption that $\kappa > m + 2$. Therefore, (3.14) implies that

$$s^{m/2}\left| II - \mu_{r_n}^\theta((-\infty, \mathbf{b}_{r_n} - \varepsilon_n\sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1} - r_n)\mathbf{1}]) \right| \stackrel{s\to\infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.} \qquad (3.15)$$

Now we put $A_n = (-\infty, \mathbf{b}_{r_n}] \setminus (-\infty, \mathbf{b}_{r_n} - \varepsilon_n \sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1} - r_n)\mathbf{1}] \subset \cup_{j=1}^d C_{n,j}$ where $C_{n,j} := \left\{\mathbf{x} = (x_1, ..., x_d) : x_j \in ((\mathbf{b}_{r_n})_j - \varepsilon_n \sqrt{r_n} - \|\theta\|(r_{n+1} - r_n), (\mathbf{b}_{r_n})_j]\right\}$. Then by Lemma 2.1 and the inequality $\Pi_{\mathbf{0}}\left(\mathbf{B}_t + \mathbf{y} \in C_{n,j}\right) \le \frac{\varepsilon_n \sqrt{r_n} + \|\theta\|(r_{n+1} - r_n)}{\sqrt{2\pi t}}$, we obtain that

$$
\begin{aligned}
& r_n^{m/2} \mathbb{E}\left[\mu_{r_n}^\theta\left(A_n\right) \middle| \mathcal{F}_{\sqrt{r_n}}\right] \\
& = r_n^{m/2} e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta \cdot \mathbf{X}_u(\sqrt{r_n})} \mathbb{P}^{-\theta}\left(\mathbf{X}_\xi(r_n - \sqrt{r_n}) + \mathbf{y} + \theta r_n \in A_n\right)|_{\mathbf{y} = \mathbf{X}_u(\sqrt{r_n})} \\
& \le \sum_{j=1}^d r_n^{m/2} e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta \cdot \mathbf{X}_u(\sqrt{r_n})} \Pi_{\mathbf{0}}\left(\mathbf{B}_{r_n - \sqrt{r_n}} + \mathbf{y} + \theta\sqrt{r_n} \in C_{n,j}\right)|_{\mathbf{y} = \mathbf{X}_u(\sqrt{r_n})} \\
& \le d W_{\sqrt{r_n}}(\theta) \frac{r_n^{m/2}\left(\varepsilon_n \sqrt{r_n} + \|\theta\|(r_{n+1} - r_n)\right)}{\sqrt{2\pi(r_n - \sqrt{r_n})}} \stackrel{s\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Here the last assertion about the limit being 0 follows from the following argument:

$$
\frac{r_n^{m/2}\left(\varepsilon_n \sqrt{r_n} + \|\theta\|(r_{n+1} - r_n)\right)}{\sqrt{2\pi(r_n - \sqrt{r_n})}} \lesssim r_n^{m/2}\varepsilon_n = n^{m/(2\kappa)} \sqrt{(n+1)^{1/\kappa} - n^{1/\kappa}} \lesssim n^{(m+1-\kappa)/(2\kappa)} \to 0.
$$

Using Lemma 3.1 (ii), we immediately get that $r_n^{m/2}\mu_{r_n}^\theta(A_n) \to 0$, $\mathbb{P}$-almost surely. Then by (3.15), we conclude that (3.13) holds.

Applying similar arguments for the interval $(\mathbf{b}_s, +\infty)$, we can also get

$$
\liminf_{n\to\infty} \inf_{r_n \le s < r_{n+1}} s^{m/2}\left(\mu_s^\theta\left((\mathbf{b}_s, +\infty)\right) - \mu_{r_n}^\theta\left((\mathbf{b}_{r_n}, +\infty)\right)\right) \ge 0, \quad \mathbb{P}\text{-a.s.} \tag{3.16}
$$

Using Proposition 2.6 with $\mathbf{k} = \mathbf{0}$ and $\eta = 2(\kappa + 1)$, and the assumption $\lambda > m + 2(\kappa + 1)$, we get

$$
\lim_{n\to\infty} \sup_{r_n \le s < r_{n+1}} s^{m/2}\left|\mu_s^\theta\left(\mathbb{R}^d\right) - \mu_{r_n}^\theta\left(\mathbb{R}^d\right)\right| = \lim_{n\to\infty} \sup_{r_n \le s < r_{n+1}} s^{m/2}\left|W_s(\theta) - W_{r_n}(\theta)\right| = 0. \tag{3.17}
$$

Now we prove (3.9) follows from (3.10), (3.16) and (3.17). Indeed, for any $\varepsilon > 0$, (3.10), (3.16) and (3.17) imply that one can find a random time N such that for all $n > N$ and $r_n \le s < r_{n+1}$,

$$
\begin{aligned}
& s^{m/2}\left(\mu_s^{m/2}\left((-\infty, \mathbf{b}_s]\right) - \mu_{r_n}^\theta\left((-\infty, \mathbf{b}_{r_n}]\right)\right) > -\varepsilon, \\
& s^{m/2}\left(\mu_s^\theta\left((\mathbf{b}_s, +\infty)\right) - \mu_{r_n}^\theta\left((\mathbf{b}_{r_n}, +\infty)\right)\right) > -\varepsilon \quad \text{and} \quad s^{m/2}\left|\mu_s^\theta\left(\mathbb{R}^d\right) - \mu_{r_n}^\theta\left(\mathbb{R}^d\right)\right| < \varepsilon.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& s^{m/2}\left(\mu_s^\theta\left((-\infty, \mathbf{b}_s]\right) - \mu_{r_n}^\theta\left((-\infty, \mathbf{b}_{r_n}]\right)\right) \\
& = s^{m/2}\left(\mu_s^\theta\left(\mathbb{R}^d\right) - \mu_{r_n}^\theta\left(\mathbb{R}^d\right)\right) - s^{m/2}\left(\mu_s^\theta\left((\mathbf{b}_s, +\infty)\right) - \mu_{r_n}^\theta\left((\mathbf{b}_{r_n}, +\infty)\right)\right) < 2\varepsilon.
\end{aligned}
$$

Hence we have that when $n > N$ and $r_n \le s < r_{n+1}$,

$$
s^{m/2}\left|\mu_s^\theta\left((-\infty, \mathbf{b}_s]\right) - \mu_{r_n}^\theta\left((-\infty, \mathbf{b}_{r_n}]\right)\right| < 2\varepsilon,
$$

which implies (3.9). □

For any given $m \in \mathbb{N}$, we will take $\kappa := m+3$ in the remainder of this section. It follows from Lemma 3.2 that, if (1.7) holds for some $\lambda > m + 2(\kappa+1) = 3m+8$, then $\mathbb{P}$-almost surely for all $s \in [r_n, r_{n+1})$ and $\mathbf{b}_s = \mathbf{b}\sqrt{s}$,

$$\begin{aligned}\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) &= \mathbb{E}\left[\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) \big| \mathcal{F}_{k_s}\right] + o(s^{-m/2}) \\ &= e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} \Phi_d\Big(\frac{\mathbf{b}\sqrt{s} - \theta\sqrt{r_n} - \mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big) \\ &\quad + o(s^{-m/2}). \end{aligned} \tag{3.18}$$

Note that, for any $\mathbf{a} < \mathbf{b}$, $(\mathbf{a}, \mathbf{b}] = \prod_{j=1}^d (a_j, b_j]$ can be expressed in terms $\prod_{j=1}^d E_j$ where $E_j \in \{(-\infty, a_j], (-\infty, b_j]\}$ using a finite number of set theoretic operations. Thus, applying Lemma 3.2 to $\prod_{j=1}^d E_j$, we get that, if (1.7) holds for some $\lambda > 3(m+d)+8 = 3m+3d+8$, then

$$\begin{aligned}\mu_s^\theta\left((\mathbf{a}, \mathbf{b}]\right) &= o(s^{-(m+d)/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} \\ &\quad \times \prod_{j=1}^d \frac{1}{\sqrt{s-\sqrt{r_n}}} \int_{a_j}^{b_j} \phi\Big(\frac{z_j - \theta_j\sqrt{r_n} - (\mathbf{X}_u(\sqrt{r_n}))_j}{\sqrt{s-r_n}}\Big) \mathrm{d}z_j. \end{aligned} \tag{3.19}$$

Proof of Theorem 1.1: Let $m \in \mathbb{N}$ and assume (1.7) holds for some $\lambda > \max\{3m+8, d(3m+5)\}$. Recall that $r_n = n^{1/\kappa}$ and $\kappa = m+3$. Put $K := m/\kappa + 3$. Combining Lemma 2.1, $\sup_{\mathbf{z}\in\mathbb{R}^d} \Phi_d(\mathbf{z}) = 1$ and the fact that $\{(\mathbf{X}_\xi(t) + \theta t)_{t\geq 0}, \mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, we get that

$$\begin{aligned}&\sum_{n=2}^\infty r_n^{m/2} \mathbb{E}\Big(e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} \\ &\qquad \times \sup_{r_n \leq s < r_{n+1}} \Phi_d\Big(\frac{\mathbf{b}\sqrt{s} - \theta\sqrt{r_n} - \mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big) 1_{\left\{\cup_{j=1}^d \left\{|(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}| > \sqrt{K\sqrt{r_n}\log n}\right\}\right\}}\Big) \\ &\leq \sum_{n=2}^\infty n^{m/(2\kappa)} \sum_{j=1}^d \Pi_{\mathbf{0}}\left(|(\mathbf{B}_{\sqrt{r_n}})_j| > \sqrt{K\sqrt{r_n}\log n}\right) \\ &= d\sum_{n=2}^\infty n^{m/(2\kappa)} \Pi_{\mathbf{0}}\left(|(\mathbf{B}_1)_1| > \sqrt{K\log n}\right) \lesssim \sum_{n=1}^\infty n^{m/(2\kappa)} n^{-K/2} < \infty, \end{aligned} \tag{3.20}$$

where in the last inequality we used the fact that $\Pi_{\mathbf{0}}(|(\mathbf{B}_1)_1| > x) \lesssim e^{-x^2/2}$. Therefore, $\mathbb{P}$-almost surely,

$$\begin{aligned}&r_n^{m/2} e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{\left\{\cup_{j=1}^d \left\{|(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}| > \sqrt{K\sqrt{r_n}\log n}\right\}\right\}} \\ &\qquad \times \sup_{r_n \leq s < r_{n+1}} \Phi_d\Big(\frac{\mathbf{b}\sqrt{s} - \theta\sqrt{r_n} - \mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big) \stackrel{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.21}$$

Since $\lambda > 3m+8$, by (3.18) and (3.21), for any $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}, \mathbf{b} \in \mathbb{R}^d$ and $s \in [r_n, r_{n+1})$,

$$\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) = o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})}$$

$$\times \Phi_d\Big(\frac{\mathbf{b}\sqrt{s}-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big)1_{\big\{\cap_{j=1}^d\big\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{K\sqrt{r_n}\log n}\big\}\big\}}.$$

Put $J := 6m+10$. Then $J > 2m+K\kappa = 3m+3\kappa = 6m+9$. By Lemma 2.4, we get that for any $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}, \mathbf{b} \in \mathbb{R}^d$ and $s \in [r_n, r_{n+1})$,

$$\begin{aligned}
&\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right)\\
&= o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{\big\{\cap_{j=1}^d\big\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{K\sqrt{r_n}\log n}\big\}\big\}}\\
&\quad\times\prod_{j=1}^d \Big(\Phi(b_j)-\phi(b_j)\sum_{k=1}^J\frac{1}{k!}\frac{1}{s^{k/2}}H_{k-1}(b_j)\left(\left(\sqrt{r_n}\right)^{k/2}H_k\left(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\right)\right)+\varepsilon_{m,u,s,j}\Big)\\
&= o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{\big\{\cap_{j=1}^d\big\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{K\sqrt{r_n}\log n}\big\}\big\}}\\
&\quad\times\prod_{j=1}^d \Big(\Phi(b_j)-\phi(b_j)\sum_{k=1}^J\frac{1}{k!}\frac{1}{s^{k/2}}H_{k-1}(b_j)\Big(\left(\sqrt{r_n}\right)^{k/2}H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big)\Big),
\end{aligned}\tag{3.22}$$

where $\varepsilon_{m,u,s,j} = \varepsilon_{m,y,b,s}|_{y=\theta_j\sqrt{r_n}+(\mathbf{X}_u(\sqrt{r_n}))_j,\ b=b_j}$. To justify the last equality, we first apply Lemma 2.4 to get that, for each $u \in N(\sqrt{r_n})$, as $s \to \infty$, $\mathbb{P}$-almost surely,

$$\begin{aligned}
&s^{m/2}e^{-(1+\frac{\theta^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta X_u(\sqrt{r_n})}1_{\big\{\cap_{j=1}^d\big\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{K\sqrt{r_n}\log n}\big\}\big\}}|\varepsilon_{m,u,s,j}|\\
&\le W_{\sqrt{r_n}}(\theta)\times s^{m/2}\sup_{j\le d}\sup_{u\in N(\sqrt{r_n})}|\varepsilon_{m,u,s,j}|1_{\big\{\cap_{j=1}^d\big\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{K\sqrt{r_n}\log n}\big\}\big\}} \to 0.
\end{aligned}$$

Then note that by (2.4) and $|H_k(x)| \lesssim |x|^k+1$, on the set $\cap_{j=1}^d\left\{\left|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}\right| \le \sqrt{K\sqrt{r_n}\log n}\right\}$,

$$\begin{aligned}
|Q_{u,j}| &:= \Big|\Phi(b_j)-\phi(b_j)\sum_{k=1}^J\frac{1}{k!}\frac{1}{s^{k/2}}H_{k-1}(b_j)\Big(\left(\sqrt{r_n}\right)^{k/2}H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big)\Big|\\
&\lesssim 1+\sum_{k=1}^J\frac{1}{r_n^{k/2}}r_n^{k/4}\Big(1+\Big|\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big|^k\Big) \lesssim 1+\frac{(\log n)^{J/2}}{r_n^{1/4}} \lesssim 1.
\end{aligned}$$

Combining the two displays above, we get that, on the set $\cap_{j=1}^d\left\{\left|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}\right| \le \sqrt{K\sqrt{r_n}\log n}\right\}$,

$$\begin{aligned}
&\Big|\Phi_d\Big(\frac{\mathbf{b}\sqrt{s}-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big)-\prod_{j=1}^d Q_{u,j}\Big| \le \sum_{j=1}^d\prod_{\ell\ne j}|Q_{v,\ell}|\,|\varepsilon_{m,u,s,j}|\\
&\lesssim d\sup_{j\le d}\sup_{u\in N(\sqrt{r_n})}|\varepsilon_{m,u,s,j}|1_{\big\{\cap_{j=1}^d\big\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{K\sqrt{r_n}\log n}\big\}\big\}},
\end{aligned}$$

which implies (3.22).

Let $\varepsilon \in (0,1)$ be small enough so that $K(1-\varepsilon) \ge m/\kappa+2$. For any $\mathbf{k} \in \mathbb{N}^d$ with $1 \le |\mathbf{k}| \le J$, using the inequality $|H_k(x)| \lesssim 1+|x|^k$ first and then Lemma 2.1, we get

$$\sum_{n=2}^\infty r_n^{m/2}\mathbb{E}\sup_{r_n\le s<r_{n+1}}\Big|e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})}e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}$$

$$
\times 1_{\left\{\cup_{j=1}^d\left\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{K\sqrt{r_n}\log n}\right\}\right\}}\prod_{j=1}^d \frac{\left(\sqrt{r_n}\right)^{k_j/2}}{s^{k_j/2}}H_{k_j}\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big|
$$

$$
\lesssim \sum_{n=2}^\infty r_n^{m/2}\Pi_{\mathbf{0}}\Big(\sum_{j=1}^d 1_{\left\{|(\mathbf{B}_{\sqrt{r_n}})_j|>\sqrt{K\sqrt{r_n}\log n}\right\}}\prod_{\ell=1}^d \frac{1}{r_n^{k_\ell/4}}\Big(1+\left|\frac{(\mathbf{B}_{\sqrt{r_n}})_\ell}{\sqrt{\sqrt{r_n}}}\right|^{k_\ell}\Big)\Big)
$$

$$
=\sum_{j=1}^d\sum_{n=2}^\infty n^{(2m-|\mathbf{k}|)/(4\kappa)}\Pi_{\mathbf{0}}\Big(1_{\left\{|(\mathbf{B}_1)_j|>\sqrt{K\log n}\right\}}\prod_{\ell=1}^d\Big(1+|(\mathbf{B}_1)_\ell|^{k_\ell}\Big)\Big)
$$

$$
\lesssim \Pi_{\mathbf{0}}\left(\left(1+|B_1|^J\right)e^{(1-\varepsilon)|B_1|^2/2}\right)\sum_{n=2}^\infty n^{(2m-|\mathbf{k}|)/(4\kappa)}n^{-(1-\varepsilon)K/2}\lesssim\sum_{n=2}^\infty n^{-(4\kappa+1)/(4\kappa)}<\infty. \quad (3.23)
$$

Thus we have that $\mathbb{P}$-almost surely,

$$
\lim_{n\to\infty} r_n^{m/2}\sup_{r_n\le s<r_{n+1}}\Big|e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})}e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{\left\{\cup_{j=1}^d\left\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{K\sqrt{r_n}\log n}\right\}\right\}}
$$

$$
\times\prod_{j=1}^d\frac{\left(\sqrt{r_n}\right)^{k_j/2}}{s^{k_j/2}}H_{k_j}\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big|=0. \quad (3.24)
$$

Combining (3.22) and (3.24), we get

$$
\mu_s^\theta\left((-\infty,\mathbf{b}\sqrt{s}]\right)=o(s^{-m/2})+e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})}e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}
$$

$$
\times\prod_{j=1}^d\Big(\sum_{k=0}^J\frac{(-1)^k}{k!}\frac{1}{s^{k/2}}\frac{\mathrm{d}^k}{\mathrm{d}b_j^k}\Phi(b_j)\Big(\left(\sqrt{r_n}\right)^{k/2}H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big)\Big)
$$

$$
=o(s^{-m/2})+\sum_{\mathbf{k}:\mathbf{k}\le J\mathbf{1}}\frac{(-1)^{|\mathbf{k}|}}{\mathbf{k}!}\frac{1}{s^{|\mathbf{k}|/2}}D^{\mathbf{k}}\Phi_d(\mathbf{b})M_{\sqrt{r_n}}^{(\mathbf{k},\theta)}.
$$

Since $\lambda>|J\mathbf{1}|/2=d(3m+5)$, it follows from Proposition 2.6 that for any $\mathbf{k}\in\mathbb{N}^d$ with $m+1\le|\mathbf{k}|\le|J\mathbf{1}|$, $s^{-|\mathbf{k}|/2}M_{\sqrt{r_n}}^{(\mathbf{k},\theta)}=o(s^{-m/2})$. Thus

$$
\mu_s^\theta\left((-\infty,\mathbf{b}\sqrt{s}]\right)=\sum_{\mathbf{k}:|\mathbf{k}|\le m}\frac{(-1)^{|\mathbf{k}|}}{\mathbf{k}!}\frac{1}{s^{|\mathbf{k}|/2}}D^{\mathbf{k}}\Phi_d(\mathbf{b})M_{\sqrt{r_n}}^{(\mathbf{k},\theta)}+o(s^{-m/2}).
$$

Take $\eta>0$ sufficient small so that $\lambda>\frac{3m}{2}+\eta$. Then by Proposition 2.6, for any $\mathbf{k}\in\mathbb{N}^d$ with $0\le|\mathbf{k}|\le m$,

$$
M_{\sqrt{r_n}}^{(\mathbf{k},\theta)}-M_\infty^{(\mathbf{k},\theta)}=o(r_n^{-(\lambda-|\mathbf{k}|/2)/2+\eta/2})=o\left(r_n^{-m/2}r_n^{m/2-(\lambda-m/2)/2+\eta/2}\right)=o(r_n^{-m/2}),
$$

which implies that as $s\to\infty$,

$$
\mu_s^\theta\left((-\infty,\mathbf{b}\sqrt{s}]\right)=\sum_{\mathbf{k}:|\mathbf{k}|\le m}\frac{(-1)^{|\mathbf{k}|}}{\mathbf{k}!}\frac{1}{s^{|\mathbf{k}|/2}}D^{\mathbf{k}}\Phi_d(\mathbf{b})M_\infty^{(\mathbf{k},\theta)}+o(s^{-m/2}),\quad \mathbb{P}\text{-a.s.}
$$

Therefore, the assertion of the theorem is valid under the assumption $\lambda>\max\{3m+8,d(3m+5)\}$.
□

Proof of Theorem 1.2: Let $m\in\mathbb{N}$ and assume (1.7) holds for some $\lambda>\max\{d(3m+5),3m+3d+8\}$. Recall that $r_n=n^{1/\kappa}$ and $\kappa=m+3$. Put $K:=m/\kappa+3$ and define

$$Y_{s,n,u}:=e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}\prod_{j=1}^d\frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}}\int_{a_j}^{b_j}\phi\Big(\frac{z_j-\theta_j\sqrt{r_n}-(\mathbf{X}_u(\sqrt{r_n}))_j}{\sqrt{s-r_n}}\Big)\mathrm{d}z_j.$$

Since $\lambda>3m+3d+8$, by (3.19), for any $\theta\in\mathbb{R}^d$ with $\|\theta\|<\sqrt{2}$, any $\mathbf{a},\mathbf{b}\in\mathbb{R}^d$ with $\mathbf{a}<\mathbf{b}$ and $s\in[r_n,r_{n+1})$, $\mathbb{P}$-almost surely, as $s\to\infty$,

$$s^{d/2}\mu_s^\theta\left((\mathbf{a},\mathbf{b}]\right)=e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})}Y_{s,n,u}+o(s^{-m/2}).\tag{3.25}$$

Noticing that $0\le\sup_{r_n\le s<r_{n+1}}Y_{s,n,u}\le e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}\prod_{j=1}^d\frac{(b_j-a_j)\sqrt{r_{n+1}}}{\sqrt{r_n-\sqrt{r_n}}}\lesssim e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}$, and using (3.20), we get

$$\begin{aligned}&\sum_{n=2}^\infty r_n^{m/2}\mathbb{E}\Big(\sup_{r_n\le s<r_{n+1}}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})}Y_{s,n,u}1_{\left\{\cup_{j=1}^d\left\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{K\sqrt{r_n}\log n}\right\}\right\}}\Big)\\&\lesssim d\sum_{n=2}^\infty n^{m/(2\kappa)}\Pi_{\mathbf{0}}\left(|(\mathbf{B}_{\sqrt{r_n}})_1|>\sqrt{K\sqrt{r_n}\log n}\right)<\infty.\end{aligned}$$

Therefore, $\mathbb{P}$-almost surely,

$$s^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})}Y_{s,n,u}1_{\left\{\cup_{j=1}^d\left\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{K\sqrt{r_n}\log n}\right\}\right\}}\stackrel{s\to\infty}{\longrightarrow}0.$$

By (3.25), for $s\in[r_n,r_{n+1})$,

$$\begin{aligned}&s^{d/2}\mu_s^\theta\left((\mathbf{a},\mathbf{b}]\right)\\&=e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})}Y_{s,n,u}1_{\left\{\cap_{j=1}^d\left\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{K\sqrt{r_n}\log n}\right\}\right\}}+o(s^{-m/2}).\end{aligned}\tag{3.26}$$

Using Lemma 2.5, on the set $\cap_{j=1}^d\{\left|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}\right|\le\sqrt{K\sqrt{r_n}\log n}\}$, for $J=6m+10$,

$$\begin{aligned}&\frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}}\int_{a_j}^{b_j}\phi\Big(\frac{z_j-\theta_j\sqrt{r_n}-(\mathbf{X}_u(\sqrt{r_n}))_j}{\sqrt{s-r_n}}\Big)\mathrm{d}z_j\\&=\int_{a_j}^{b_j}\phi\left(\frac{z_j}{\sqrt{s}}\right)\Big(\sum_{k=0}^J\frac{1}{k!}\frac{1}{s^{k/2}}H_k\Big(\frac{z_j}{\sqrt{s}}\Big)r_n^{k/4}H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big)\mathrm{d}z_j+\varepsilon_{m,u,s,j}\\&=\sum_{k=0}^J\frac{1}{k!}\frac{1}{s^{k/2}}\Big(\int_{a_j}^{b_j}\phi\Big(\frac{z_j}{\sqrt{s}}\Big)H_k\Big(\frac{z_j}{\sqrt{s}}\Big)\mathrm{d}z\Big)r_n^{k/4}H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)+\varepsilon_{m,u,s,j},\end{aligned}$$

where

$$r_{n+1}^{m/2}\sup_{s\in[r_n,r_{n+1})}\sup_{j\le d}\sup_{u\in N(\sqrt{r_n})}|\varepsilon_{m,u,s,j}|1_{\left\{\cap_{j=1}^d\left\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{K\sqrt{r_n}\log n}\right\}\right\}}\to0.$$

Therefore, using (3.26) and an argument similar to that leading to (3.22), we get

$$\begin{aligned}
s^{d/2}\mu_s^\theta((\mathbf{a},\mathbf{b}]) &= o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \\
&\quad\times \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})} 1_{\left\{\cap_{j=1}^d\left\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{K\sqrt{r_n}\log n}\right\}\right\}} \\
&\quad\times \prod_{j=1}^d \Big\{\sum_{k=0}^J \frac{1}{k!}\frac{1}{s^{k/2}}\Big(\int_{a_j}^{b_j}\phi\Big(\frac{z_j}{\sqrt{s}}\Big)H_k\Big(\frac{z_j}{\sqrt{s}}\Big)\mathrm{d}z_j\Big) r_n^{k/4} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big\}. \quad (3.27)
\end{aligned}$$

By Lemma 2.1 and (3.23) and the fact that $\left|\int_a^b \phi\left(\frac{z}{\sqrt{s}}\right) H_k\left(\frac{z}{\sqrt{s}}\right)\mathrm{d}z\right| \lesssim |b-a|$, we have that for any $\mathbf{k}\in\mathbb{N}^d$ with $0\le|\mathbf{k}|\le J$,

$$\begin{aligned}
&\sum_{n=2}^\infty r_n^{m/2}\mathbb{E}\sup_{r_n\le s<r_{n+1}}\Big|e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{\left\{\cup_{j=1}^d\left\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{K\sqrt{r_n}\log n}\right\}\right\}} \\
&\quad\times\prod_{\ell=1}^d\Big(\int_{a_\ell}^{b_\ell}\phi\Big(\frac{z_\ell}{\sqrt{s}}\Big)H_{k_\ell}\Big(\frac{z_\ell}{\sqrt{s}}\Big)\mathrm{d}z_\ell\Big)\frac{\left(\sqrt{r_n}\right)^{k_\ell/2}}{s^{k_\ell/2}}H_{k_\ell}\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_\ell+\theta_\ell\sqrt{r_n}}{r_n^{1/4}}\Big)\Big| \\
&\lesssim \sum_{n=2}^\infty r_n^{m/2} r_n^{-|\mathbf{k}|/4}\Pi_{\mathbf{0}}\Big(1_{\{\cup_{j=1}^d\{|(\mathbf{B}_{\sqrt{r_n}})_j|>\sqrt{K\sqrt{r_n}\log n}\}\}}\prod_{\ell=1}^d\Big|H_{k_\ell}\Big(\frac{(\mathbf{B}_{\sqrt{r_n}})_\ell}{r_n^{1/4}}\Big)\Big|\Big) \\
&\lesssim \sum_{j=1}^d\sum_{n=2}^\infty n^{(2m-|\mathbf{k}|)/(4\kappa)}\Pi_{\mathbf{0}}\Big(1_{\{|(\mathbf{B}_1)_j|>\sqrt{K\log n}\}}\prod_{\ell=1}^d\left(1+|(\mathbf{B}_1)_\ell|^J\right)\Big)<\infty.
\end{aligned}$$

Thus $\mathbb{P}$-almost surely, as $s\to\infty$,

$$\begin{aligned}
&\lim_{n\to\infty} r_n^{m/2}\sup_{r_n\le s<r_{n+1}}\Big|e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{\left\{\cup_{j=1}^d\left\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{K\sqrt{r_n}\log n}\right\}\right\}} \\
&\quad\times\prod_{\ell=1}^d\Big(\int_{a_\ell}^{b_\ell}\phi\Big(\frac{z_\ell}{\sqrt{s}}\Big)H_{k_\ell}\Big(\frac{z_\ell}{\sqrt{s}}\Big)\mathrm{d}z_\ell\Big)\frac{\left(\sqrt{r_n}\right)^{k_\ell/2}}{s^{k_\ell/2}}H_{k_\ell}\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_\ell+\theta_\ell\sqrt{r_n}}{r_n^{1/4}}\Big)\Big| = 0.
\end{aligned}$$

Therefore, by (3.27), since $\lambda>3m+3d+8$,

$$\begin{aligned}
s^{d/2}\mu_s^\theta((\mathbf{a},\mathbf{b}]) &= o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})} \\
&\quad\times \prod_{j=1}^d \Big\{\sum_{k=0}^J \frac{1}{k!}\frac{1}{s^{k/2}}\Big(\int_{a_j}^{b_j}\phi\Big(\frac{z_j}{\sqrt{s}}\Big)H_k\Big(\frac{z_j}{\sqrt{s}}\Big)\mathrm{d}z_j\Big) r_n^{k/4} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big\} \\
&= o(s^{-m/2}) + \sum_{\mathbf{k}:\mathbf{k}\le J\mathbf{1}}\prod_{j=1}^d\frac{1}{k_j!}\frac{1}{s^{k_j/2}}\Big(\int_{a_j}^{b_j}\phi\Big(\frac{z_j}{\sqrt{s}}\Big)H_{k_j}\Big(\frac{z_j}{\sqrt{s}}\Big)\mathrm{d}z_j\Big)M_{\sqrt{r_n}}^{(\mathbf{k},\theta)}.
\end{aligned}$$

Since $\lambda>\max\{d(3m+5),3m+3d+8\}$, using Proposition 2.6 and argument similar to that used in the proof of Theorem 1.1, we get that

$$s^{d/2}\mu_s^\theta((\mathbf{a},\mathbf{b}])$$

$$= o(s^{-m/2}) + \sum_{\mathbf{k}:|\mathbf{k}|\le m} \prod_{j=1}^{d} \frac{1}{k_j!} \frac{1}{s^{k_j/2}} \Big(\int_{a_j}^{b_j} \phi\Big(\frac{z_j}{\sqrt{s}}\Big) H_{k_j}\Big(\frac{z_j}{\sqrt{s}}\Big) \mathrm{d}z_j \Big) M_\infty^{(\mathbf{k},\theta)}. \tag{3.28}$$

By Taylor's expansion, as $x \to 0$,

$$\phi(x) = \sum_{j=0}^{m} \frac{\phi^{(j)}(0)}{j!} x^j + o(x^m). \tag{3.29}$$

Note that $\phi^{(k)}(x) = (-1)^k H_k(x)\phi(x)$ and that, for each $1 \le k \le m$,

$$\phi(x) H_k(x) = (-1)^k \sum_{j=0}^{m} \frac{\phi^{(k+j)}(0)}{j!} x^j + o(x^m). \tag{3.30}$$

Combining (3.29) and (3.30), we get

$$\begin{aligned}
&\sum_{\mathbf{k}:|\mathbf{k}|\le m} \prod_{j=1}^{d} \frac{1}{k_j!} \frac{1}{s^{k_j/2}} \Big(\int_{a_j}^{b_j} \phi\Big(\frac{z_j}{\sqrt{s}}\Big) H_{k_j}\Big(\frac{z_j}{\sqrt{s}}\Big) \mathrm{d}z_j \Big) M_\infty^{(\mathbf{k},\theta)} \\
&= o(s^{-m/2}) + \sum_{\mathbf{k}:|\mathbf{k}|\le m} \frac{(-1)^{|\mathbf{k}|} M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}! s^{|\mathbf{k}|/2}} \int_{[\mathbf{a},\mathbf{b}]} \sum_{\mathbf{i}:|\mathbf{i}|\le m} \frac{\prod_{j=1}^{d} \phi^{(k_j+i_j)}(0) z_j^{i_j}}{\mathbf{i}! s^{|\mathbf{i}|/2}} \mathrm{d}z_1 ... \mathrm{d}z_d \\
&= o(s^{-m/2}) + \sum_{\mathbf{k}:|\mathbf{k}|\le m} \frac{(-1)^{|\mathbf{k}|} M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}! s^{|\mathbf{k}|/2}} \int_{[\mathbf{a},\mathbf{b}]} \sum_{\mathbf{i}:|\mathbf{i}+\mathbf{k}|\le m} \frac{\prod_{j=1}^{d} \phi^{(k_j+i_j)}(0) z_j^{i_j}}{\mathbf{i}! s^{|\mathbf{i}|/2}} \mathrm{d}z_1 ... \mathrm{d}z_d.
\end{aligned}$$

Therefore, by (3.28), we conclude that

$$\begin{aligned}
s^{d/2} \mu_s^\theta((\mathbf{a},\mathbf{b}]) &= o(s^{-m/2}) + \sum_{\mathbf{k}:|\mathbf{k}|\le m} \frac{(-1)^{|\mathbf{k}|} M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}! s^{|\mathbf{k}|/2}} \int_{[\mathbf{a},\mathbf{b}]} \sum_{\mathbf{i}:|\mathbf{i}+\mathbf{k}|\le m} \frac{\prod_{j=1}^{d} \phi^{(k_j+i_j)}(0) z_j^{i_j}}{\mathbf{i}! s^{|\mathbf{i}|/2}} \mathrm{d}z_1 ... \mathrm{d}z_d \\
&= \sum_{\ell=0}^{m} \frac{1}{s^{\ell/2}} \sum_{j=0}^{\ell} (-1)^j \sum_{\mathbf{k}:|\mathbf{k}|=j} \frac{M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}!} \sum_{\mathbf{i}:|\mathbf{i}|=\ell-j} \frac{\prod_{j=1}^{d} \phi^{(k_j+i_j)}(0)}{\mathbf{i}!} \int_{[\mathbf{a},\mathbf{b}]} \prod_{j=1}^{d} z_j^{i_j} \mathrm{d}z_1 ... \mathrm{d}z_d.
\end{aligned}$$

□

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