Asymptotic expansion for branching killed Brownian motion with drift*

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Abstract

Let $Z_t^{(0,\infty)}$ be the point process formed by the positions of all particles alive at time t in a branching Brownian motion with drift and killed upon reaching 0. We study the asymptotic expansions of $Z_t^{(0,\infty)}(A)$ for $A = (a,b)$ and $A = (a,\infty)$ under the assumption that $\sum_{k=1}^\infty k(\log k)^{1+\lambda}p_k < \infty$ for large λ in the regime of $\theta \in [0,\sqrt{2})$. These results extend and sharpen the results of Louidor and Saglietti [J. Stat. Phys, 2020] and Kesten [Stochastic Process. Appl., 1978].

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1 Introduction and main results

1.1 Introduction

A branching Brownian motion with drift is a continuous-time Markov process defined as follows: At time 0, there is a particle at site x and it moves according to a Brownian motion with drift $-\theta$, where $\theta \in \mathbb{R}$. After an exponential time of parameter 1 independent of the movement, this particle dies and it splits into k offspring with probability p_k. Each of the offspring independently repeats its parent's behavior from their birth-place. This procedure goes on. We use $\mathbb{P}_x$ and $\mathbb{E}_x$ to denote the law of this process and the corresponding expectation operator. Let $N(t)$ be the set of particles alive at time t and for each $u \in N(t)$, we will use $X_u(t)$ to denote the position of the particle. For $s < t$ and $u \in N(t)$, we will also use $X_u(s)$ to denote the position of the ancestor of u at time s.

Suppose now that $x > 0$ and that, once a particle hits $(-\infty, 0]$, we remove it (along with all its possible descendants) from the system. The resulting branching system is called a branching killed Brownian motion with drift. Let $Z_t^{(0,\infty)}$ denote the point process formed by the positions of all the particles alive at time t in the branching killed Brownian motion with drift, i.e.,

$$Z_t^{(0,\infty)} := \sum_{u\in N(t)} 1_{\{\min_{s\le t} X_u(s)>0\}}\delta_{X_u(t)}.$$

Assume that

$$\sum_{k=0}^\infty kp_k = 2.$$

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Kesten [17] proved that for any $\theta \in \mathbb{R}$, there exists a constant $C = C(x, \theta) > 0$ such that

$$\mathbb{E}_x\left(Z_t^{(0,\infty)}((0,\infty))\right) \sim \begin{cases} Ct^{-3/2}e^{(1-\frac{\theta^2}{2})t}, & \theta > 0, \\ Ct^{-1/2}e^t, & \theta = 0, \\ Ce^t, & \theta < 0. \end{cases}$$

Consequently, when $\theta \geq \sqrt{2}$, the branching killed Brownian motion with drift will die out. It was proved in [17] that, when $\theta < \sqrt{2}$ and $\sum_{k=1}^\infty k^2 p_k < \infty$, the branching killed Brownian motion with drift will survive with positive probability. In [17], Kesten also stated, without proof, that under the assumption $\sum_{k=0}^\infty k^2 p_k < \infty$, there exists a random variable $W(\theta)$ such that

$$\mathbb{P}_x\left(W(\theta) > 0 \middle| Z_t^{(0,\infty)}((0,\infty)) > 0,\ \forall t > 0\right) = 1 \tag{1.1}$$

and that

(i) If $\theta \in [0, \sqrt{2})$, then $\mathbb{P}_x$-a.s., simultaneously for all intervals $\Delta \subset (0,\infty)$ (finite or infinite), it holds that

$$\frac{Z_t^{(0,\infty)}(\Delta)}{\mathbb{E}_x\left(Z_t^{(0,\infty)}(\Delta)\right)} \overset{t\to\infty}{\longrightarrow} W(\theta). \tag{1.2}$$

(ii) If $\theta < 0$, then $\mathbb{P}_x$-a.s.,

$$e^{-t} Z_t^{(0,\infty)}((0,\infty)) \overset{t\to\infty}{\longrightarrow} W(\theta).$$

In [21], Louidor and Saglietti proved that (1.1) and (1.2) hold for the case $\theta \in (0, \sqrt{2})$.

The purpose of this paper is to extend and sharp the main result of [21]: We weaken the moment condition from $\sum_{k=1}^\infty k^2 p_k < \infty$ to $\sum_{k=1}^\infty k \log^{1+\lambda} k p_k < \infty$ for some $\lambda > 0$ and, for any $a \geq 0$, give asymptotic expansions of arbitrary order for

$$\frac{Z_t^{(0,\infty)}((a,\infty))}{t^{-3/2}e^{(1-\frac{\theta^2}{2})t}}$$

under this weaker assumption. We emphasize here that we do not use the results of [21] in this paper and that, as a consequence of Theorem 1.1 below, we give another proof of (1.2) under the weaker condition. We also include the case $\theta = 0$. It is natural to study similar problems for the case $\theta < 0$. We believe things might be different in this case and we plan to tackle this in a future work.

For other recent results on branching killed Brownian motion with drift, see [3, 4, 5, 6, 7, 14, 15, 19, 20, 22] and references therein. For asymptotic expansions for branching random walks and branching Wiener processes, see [8, 9, 10, 11, 12, 13, 24] and references therein.

1.2 Main results

We will assume that

$$\sum_{k=1}^\infty k(\log k)^{1+\lambda} p_k < \infty, \tag{1.3}$$

for some $\lambda > 0$. Let H_k be the k-th order Hermite polynomial: $H_0(x) := 1$ and for $k \geq 1$,

$$H_k(x) := \sum_{j=0}^{[k/2]} \frac{k!(-1)^j}{2^j j!(k-2j)!} x^{k-2j}.$$

It is well known that, if $\{(B_t)_{t\geq 0}, \Pi_0\}$ is a standard Brownian motion, then, for any $k \geq 1$, $\{t^{k/2} H_k(B_t/\sqrt{t}), \sigma(B_s : s \leq t), \Pi_0\}$ is a martingale. Throughout this paper $\mathbb{N} = \{0, 1, 2 \cdots\}$. Now for $\theta \in [0, \sqrt{2})$ and $k \in \mathbb{N}$, we define

$$M_t^{(2k+1,\theta)} := e^{-(1-\frac{\theta^2}{2})t} \sum_{u \in N(t)} 1_{\{\min_{s\leq t} X_u(s) > 0\}} e^{\theta X_u(t)} t^{(2k+1)/2} H_{2k+1}\left(\frac{X_u(t)}{\sqrt{t}}\right), \quad t \geq 0. \quad (1.4)$$

We will prove later (see Proposition 3.1) that, for any $x > 0$, $k \in \mathbb{N}$ and $\theta \in [0, \sqrt{2})$, $M_t^{(2k+1,\theta)}$ is a martingale, and if (1.3) holds for some large λ, then $M_t^{(2k+1,\theta)}$ converges to a limit $M_\infty^{(2k+1,\theta)}$ $\mathbb{P}_x$-almost surely and in $L^1(\mathbb{P}_x)$.

Theorem 1.1 *Assume the drift θ is in $(0, \sqrt{2})$. For any given $m \in \mathbb{N}$, if* (1.3) *holds for some $\lambda > 6m + 6$, then for any $x > 0, a \geq 0$, $\mathbb{P}_x$-almost surely, as $t \to \infty$,*

$$\begin{aligned}&\frac{Z_t^{(0,\infty)}((a,\infty))}{t^{-3/2} e^{(1-\frac{\theta^2}{2})t}} \\ &= -\sqrt{\frac{2}{\pi}} \sum_{\ell=0}^{m} \frac{H_{2\ell+2}(0)}{t^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,\theta)}}{(2k+1)!(2\ell-2k+1)!} \int_a^\infty z^{2\ell-2k+1} e^{-\theta z} \mathrm{d}z + o(t^{-m}).\end{aligned}$$

Note that, from Theorem 1.1, one can immediately get the asymptotic expansion of

$$\frac{Z_t^{(0,\infty)}((a,b))}{t^{-3/2} e^t}$$

for any finite interval $(a, b) \subset (0, \infty)$. For the case $\theta = 0$, the result is a little bit different. For finite intervals, the normalization function is the same as in Theorem 1.1. For infinite intervals, the normalization function is different.

Theorem 1.2 *Assume that the drift θ is 0. For any given $m \in \mathbb{N}$, if* (1.3) *holds for some $\lambda > 6m + 6$, then*

(i) for any $x > 0, b > a \geq 0$, $\mathbb{P}_x$-almost surely, as $t \to \infty$,

$$\begin{aligned}&\frac{Z_t^{(0,\infty)}((a,b))}{t^{-3/2} e^t} \\ &= -\sqrt{\frac{2}{\pi}} \sum_{\ell=0}^{m} \frac{H_{2\ell+2}(0)}{t^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k+1)!} \int_a^b z^{2\ell-2k+1} \mathrm{d}z + o(t^{-m});\end{aligned}$$

(ii) for any $x > 0, a \geq 0$, $\mathbb{P}_x$-almost surely, as $t \to \infty$,

$$\frac{Z_t^{(0,\infty)}((a,\infty))}{t^{-1/2} e^t} = \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^{m} \frac{H_{2\ell}(0)}{t^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k)!} a^{2\ell-2k} + o(t^{-m}).$$

Remark 1.3 *Note that we only dealt with the case that the branching rate is 1 and the mean number of offspring is 2 in the two theorems above. In the general case when the branching rate is* $\beta > 0$ *and the mean number of offspring is* $\mu > 1$*, one can use the same argument to prove the following counterpart of Theorem 1.1: Let* $\theta \in (0, \sqrt{2\beta(\mu-1)})$*. For any given* $m \in \mathbb{N}$*, if* (1.3) *holds for some* $\lambda > 6m+6$*, then for any* $x > 0, a \geq 0$*,* $\mathbb{P}_x$*-almost surely, as* $t \to \infty$*,*

$$\begin{aligned}&\frac{Z_t^{(0,\infty)}((a,\infty))}{t^{-3/2}e^{(\beta(\mu-1)-\frac{\theta^2}{2})t}}\\&= -\sqrt{\frac{2}{\pi}}\sum_{\ell=0}^{m}\frac{H_{2\ell+2}(0)}{t^\ell}\sum_{k=0}^{\ell}\frac{M_\infty^{(2k+1,\theta)}}{(2k+1)!(2\ell-2k+1)!}\int_a^\infty z^{2\ell-2k+1}e^{-\theta z}\mathrm{d}z + o(t^{-m}),\end{aligned}$$

with $M_\infty^{(2k+1,\theta)}$ *given by*

$$M_\infty^{(2k+1,\theta)} := \lim_{t\to\infty} e^{-(\beta(\mu-1)-\frac{\theta^2}{2})t}\sum_{u\in N(t)} 1_{\{\min_{s\leq t} X_u(s)>0\}} e^{\theta X_u(t)} t^{(2k+1)/2} H_{2k+1}\left(\frac{X_u(t)}{\sqrt{t}}\right). \quad (1.5)$$

The counterpart of Theorem 1.2 in the general case is as follows: For any given $m \in \mathbb{N}$*, if* (1.3) *holds for some* $\lambda > 6m+6$*, then*

(i) for any $x > 0, b > a \geq 0$*,* $\mathbb{P}_x$*-almost surely, as* $t \to \infty$*,*

$$\begin{aligned}&\frac{Z_t^{(0,\infty)}((a,b))}{t^{-3/2}e^{\beta(\mu-1)t}}\\&= -\sqrt{\frac{2}{\pi}}\sum_{\ell=0}^{m}\frac{H_{2\ell+2}(0)}{t^\ell}\sum_{k=0}^{\ell}\frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k+1)!}\int_a^b z^{2\ell-2k+1}\mathrm{d}z + o(t^{-m});\end{aligned}$$

(ii) for any $x > 0, a \geq 0$*,* $\mathbb{P}_x$*-almost surely, as* $t \to \infty$*,*

$$\frac{Z_t^{(0,\infty)}((a,\infty))}{t^{-1/2}e^{\beta(\mu-1)t}} = \sqrt{\frac{2}{\pi}}\sum_{\ell=0}^{m}\frac{H_{2\ell}(0)}{t^\ell}\sum_{k=0}^{\ell}\frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k)!}a^{2\ell-2k} + o(t^{-m}),$$

with $M_\infty^{(2k+1,0)}$ *given in* (1.5)*.*

Note that if $\sum_{k=1}^\infty k^{1+\epsilon}p_k < \infty$ for some $\epsilon > 0$, then the conclusions of Theorems 1.1 and 1.2 hold for all $m \in \mathbb{N}$.

Our strategy for proving these two theorems is as follows. We will choose appropriate $\kappa > 1$ and define

$$r_n := n^{\frac{1}{\kappa}}, \quad n \in \mathbb{N}.$$

We first study the asymptotic expansion along $\{r_n : n \in \mathbb{N}\}$, which is given by Proposition 3.3, and then control the behavior for $t \in (r_n, r_{n+1})$, see Lemma 3.5 below. Once we have Proposition 3.3 and Lemma 3.5, the proofs of Theorems 1.1 and 1.2 are straight-forward. To prove Proposition 3.3, we first show that $Z_{r_n}^{(0,\infty)}(A) \approx \mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A)\big|\mathcal{F}_{\sqrt{r_n}}\right)$, where $\mathcal{F}_t$ is the σ-field generated by the branching Brownian motion with drift up to t, see Lemma 3.2 below for more details. We then prove Proposition 3.3 with the help of two series expansions for the normal distribution (see Lemma 2.7 and Lemma 2.8) and the convergence rate for martingales $M_t^{(2k+1,\theta)}$ (see Proposition 3.1). To prove Lemma 3.5, we first give a lower bound of $Z_t^{(0,\infty)}(A)$ (see Lemma 3.4), then accomplish the proof of Lemma 3.5 by proving an upper bound of $Z_t^{(0,\infty)}((0,\infty))$.

2 Preliminaries

2.1 Spine decomposition

Define

$$\frac{\mathrm{d}\mathbf{P}_x}{\mathrm{d}\mathbb{P}_x}\Big|_{\mathcal{F}_t} := \frac{\sum_{u\in N(t)} 1}{e^t}, \tag{2.1}$$

then, under $\mathbf{P}_x$, the branching Brownian motion has the following spine decomposition(see [18], or [23] for a more general case):

(i) there is an initial marked particle at $x \in \mathbb{R}$ which moves according to standard Brownian motion with drift $-\theta$;

(ii) the branching rate of this marked particle is 2;

(iii) when the marked particle dies at site y, it gives birth to $\widehat{L}$ children with $\mathbf{P}_x(\widehat{L} = k) = kp_k/2$;

(iv) one of these children is uniformly selected and marked, and the marked child evolves as its parent and the other children evolve with law $\mathbb{P}_y$, where $\mathbb{P}_y$ denotes the law of a branching Brownian motion starting at y, and all the children evolve independently.

We use ξ_t and $X_\xi(t)$ to denote the marked particle at time t and the position of this marked particle respectively. By [23, Theorem 2.11], we can get that for $u \in N(t)$,

$$\mathbf{P}_x\left(\xi_t = u \middle| \mathcal{F}_t\right) = \frac{1}{\sum_{u\in N(t)} 1}. \tag{2.2}$$

Using (2.2), we get the following many-to-one formula:

Lemma 2.1 *For any $x \in \mathbb{R}$, $t > 0$ and $u \in N(t)$, let $\Gamma(u,t)$ be a non-negative $\mathcal{F}_t$-measurable random variable. Then*

$$\mathbb{E}_x\Big(\sum_{u\in N(t)} \Gamma(u,t)\Big) = e^t \mathbf{E}_x\left(\Gamma(\xi_t, t)\right).$$

Proof: Combining (2.1) and (2.2), we get

$$\begin{aligned}\mathbb{E}_x\Big(\sum_{u\in N(t)} \Gamma(u,t)\Big) &= e^t\mathbf{E}_x\Big(\sum_{u\in N(t)} \frac{\Gamma(u,t)}{\sum_{v\in N(t)} 1}\Big) = e^t\mathbf{E}_x\Big(\sum_{u\in N(t)} \Gamma(u,t)\mathbf{P}_x\Big(\xi_t = u\Big|\mathcal{F}_t\Big)\Big)\\ &= e^t\mathbf{E}_x\Big(\mathbf{E}_x\Big(\sum_{u\in N(t)} 1_{\{\xi_t=u\}}\Gamma(u,t)\Big|\mathcal{F}_t\Big)\Big) = e^t\mathbf{E}_x\Big(\Gamma(\xi_t,t)\sum_{u\in N(t)} 1_{\{\xi_t=u\}}\Big) = e^t\mathbf{E}_x\left(\Gamma(\xi_t,t)\right).\end{aligned}$$

□

2.2 Some useful facts

Lemma 2.2 *(i) Let $\ell \in [1,2]$. Then for any finite family of independent centered random variables $\{X_i : i = 1,\dots,n\}$ with $\mathrm{E}|X_i|^\ell < \infty$ for all $i = 1,\dots,n$, it holds that*

$$\mathrm{E}\Big|\sum_{i=1}^n X_i\Big|^\ell \le 2\sum_{i=1}^n \mathrm{E}|X_i|^\ell.$$

(ii) For any $\ell \in [1,2]$ and any random variable X with $\mathrm{E}|X|^\ell < \infty$,

$$\mathrm{E}\left|X - \mathrm{E}X\right|^\ell \lesssim \mathrm{E}|X|^\ell \le (\mathrm{E}X^2)^{\ell/2}.$$

Proof: For (i), see [25, Theorem 2]. (ii) follows easily from Jensen's inequality. □

We will use $\Pi_x^{-\theta}$ and Π_x to denote the laws of a Brownian motion with drift $-\theta$ starting from x and a standard Brownian motion starting from x respectively. Let $\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\Phi(x) := \int_{-\infty}^x \phi(y)\mathrm{d}y$.

Lemma 2.3 *(i) Let $(B_t, \Pi_x^{\uparrow})$ be a 3-dimensional Bessel process with transition density*

$$p_t^{\uparrow}(x,y) = 1_{\{y>0\}} \frac{y e^{-(y-x)^2/(2t)}}{x\sqrt{2\pi t}} \left(1 - e^{-2xy/t}\right) = 1_{\{y>0\}} \frac{y}{x\sqrt{t}} \left(\phi\left(\frac{y-x}{\sqrt{t}}\right) - \phi\left(\frac{y+x}{\sqrt{t}}\right)\right).$$

Then for any $\theta \in [0, \sqrt{2})$, we have

$$\left.\frac{\mathrm{d}\Pi_x^{\uparrow}}{\mathrm{d}\Pi_x^{-\theta}}\right|_{\sigma(B_s: s\leq t)} = \frac{B_t e^{\theta(B_t - x) + \frac{\theta^2}{2}t}}{x} 1_{\{\min_{s\leq t} B_s > 0\}}.$$

(ii) For any $t, x, y > 0$,

$$p_t^{\uparrow}(x,y) \lesssim \frac{y^2}{t^{3/2}} e^{-(x-y)^2/(2t)} \leq \frac{y^2}{t^{3/2}}.$$

Proof: (ii) follows from the inequality $1 - e^{-x} \leq x, x > 0$. For (i), note that under Π_x, both

$$\frac{B_t}{x} 1_{\{\min_{s\leq t} B_s > 0\}} \quad \text{and} \quad e^{-\theta(B_t - x) - \frac{\theta^2}{2}t}$$

are mean 1 non-negative martingales and that we have the following change-of-measure:

$$\left.\frac{\mathrm{d}\Pi_x^{\uparrow}}{\mathrm{d}\Pi_x}\right|_{\sigma(B_s: s\leq t)} = \frac{B_t}{x} 1_{\{\min_{s\leq t} B_s > 0\}} \quad \text{and} \quad \left.\frac{\mathrm{d}\Pi_x^{-\theta}}{\mathrm{d}\Pi_x}\right|_{\sigma(B_s: s\leq t)} = e^{-\theta(B_t - x) - \frac{\theta^2}{2}t}.$$

Therefore, (i) follows from

$$\left.\frac{\mathrm{d}\Pi_x^{\uparrow}}{\mathrm{d}\Pi_x^{-\theta}}\right|_{\sigma(B_s: s\leq t)} = \left.\frac{\mathrm{d}\Pi_x^{\uparrow}}{\mathrm{d}\Pi_x}\right|_{\sigma(B_s: s\leq t)} \times \left(\left.\frac{\mathrm{d}\Pi_x^{-\theta}}{\mathrm{d}\Pi_x}\right|_{\sigma(B_s: s\leq t)}\right)^{-1} = \frac{B_t e^{\theta(B_t - x) + \frac{\theta^2}{2}t}}{x} 1_{\{\min_{s\leq t} B_s > 0\}}.$$

□

Lemma 2.4 *(i) Let $\theta \in (0, \sqrt{2})$. For any $x, t > 0$ and Borel set $A \subset (0, \infty)$, it holds that*

$$\Pi_x^{-\theta}\left(\min_{s\leq t} B_s > 0, B_t \in A\right) = \sqrt{\frac{2}{\pi}} x e^{\theta x} t^{-3/2} e^{-\frac{\theta^2}{2}t} \left(\int_A y e^{-\theta y} \mathrm{d}y + \varepsilon_A(x,t)\right),$$

with $\varepsilon_A(x,t)$ satisfying

$$|\varepsilon_A(x,t)| \leq C_\theta \left(1 \wedge \frac{(x+1)^2}{t}\right)$$

for some constant C_θ depending on θ only. In particular, for any fixed $\theta \in (0, \sqrt{2})$,

$$\Pi_x^{-\theta}\Big(\min_{s\leq t} B_s > 0, B_t \in A\Big) \lesssim x e^{\theta x} t^{-3/2} e^{-\frac{\theta^2}{2}t}.$$

(ii) For any Borel set $A \subset (0, \infty)$ and $x, t > 0$, it holds that

$$\Pi_x\Big(\min_{s\leq t} B_s > 0, B_t \in A\Big) \lesssim x t^{-1/2} 1_{\{\sup\{y: y\in A\} = \infty\}} + x t^{-3/2} 1_{\{\sup\{y: y\in A\} < \infty\}}.$$

(iii) For any $x, t > 0$, $\theta \in [0, \sqrt{2})$ and Borel set $A \subset (0, \infty)$, it holds that

$$\Pi_x^{-\theta}\left(B_t \in A\right) \leq e^{\theta x} e^{-\frac{\theta^2}{2}t}.$$

Proof: For (i), see [21, Lemma 3.1]; for (ii), when $\sup\{y : y \in A\} = \infty$, by the reflection principle for Brownian motion, we have

$$\Pi_x\Big(\min_{s\le t} B_s > 0\Big) = \Pi_0(|B_t| \le x) = 2\int_0^x \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)}\mathrm{d}y \lesssim \frac{x}{\sqrt{t}}.$$

When $\sup\{y : y \in A\} < \infty$, by Lemma 2.3,

$$\Pi_x\Big(\min_{s\le t} B_s > 0, B_t \in A\Big) = x\Pi_x^{\uparrow}\left(\frac{1_A(B_t)}{B_t}\right) = x\int_A \frac{1}{y} p_t^{\uparrow}(x,y)\mathrm{d}y \lesssim \frac{x}{t^{3/2}}\int_A y\mathrm{d}y \lesssim \frac{x}{t^{3/2}}.$$

For (iii), by Girsanov's theorem,

$$\Pi_x^{-\theta}(B_t \in A) \le \Pi_x^{-\theta}(B_t > 0) = \Pi_x\Big(e^{-\theta(B_t - x) - \frac{\theta^2}{2}t} 1_{\{B_t>0\}}\Big) \le e^{\theta x} e^{-\frac{\theta^2}{2}t}.$$

□

Lemma 2.5 *(i) For any $k \ge 1$ and $x \in \mathbb{R}$,*

$$|H_k(x)| \le 2\sqrt{k!}e^{x^2/4}.$$

Consequently, it holds that

$$\sup_{y\in\mathbb{R}} |\phi(y)H_k(y)| \le \sqrt{\frac{2}{\pi}}\sqrt{k!}\sup_{y\in\mathbb{R}} e^{-y^2/4} = \sqrt{\frac{2}{\pi}}\sqrt{k!}.$$

(ii) For any $k \in \mathbb{N}$, there exists a constant $C(k)$ such that for all $x \in \mathbb{R}$,

$$|H_{2k+1}(x)| \le C(k)|x|\left(|x|^{2k} + 1\right).$$

Proof: For (i), see [12, (4.1)]; (ii) follows from the definition of $H_{2k+1}(x)$. □

Lemma 2.6 *For any $\rho \in (0,1), b, x \in \mathbb{R}$, it holds that*

$$\Phi\left(\frac{b - \rho x}{\sqrt{1-\rho^2}}\right) = \Phi(b) - \phi(b)\sum_{k=1}^{\infty} \frac{\rho^k}{k!} H_{k-1}(b)H_k(x).$$

Proof: See [12, Lemma 4.2.]. □

Recall that $r_n = n^{1/\kappa}$. Applying Lemma 2.6 with $\rho = r_n^{-1/4}$, $b = r_n^{-1/2}z$ and $x = r_n^{-1/4}y$, we get that for any $z, y \in \mathbb{R}$,

$$\Phi\left(\frac{z-y}{\sqrt{r_n - \sqrt{r_n}}}\right) = \Phi\left(\frac{z}{\sqrt{r_n}}\right) - \phi\left(\frac{z}{\sqrt{r_n}}\right)\sum_{k=1}^{\infty}\frac{1}{k!}\frac{1}{r_n^{k/2}} H_{k-1}\left(\frac{z}{\sqrt{r_n}}\right) r_n^{k/4} H_k\left(\frac{y}{r_n^{1/4}}\right).$$

Noting that, for any $k \in \mathbb{N}$, H_{2k} is an even function and H_{2k+1} is an odd function, we get that

$$\Phi\left(\frac{z+y}{\sqrt{r_n - \sqrt{r_n}}}\right) - \Phi\left(\frac{z-y}{\sqrt{r_n - \sqrt{r_n}}}\right)$$
$$= 2\phi\left(\frac{z}{\sqrt{r_n}}\right)\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\frac{1}{r_n^{(2k+1)/2}} H_{2k}\left(\frac{z}{\sqrt{r_n}}\right) r_n^{(2k+1)/4} H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right). \quad (2.3)$$

Lemma 2.7 *For any given $m \in \mathbb{N}$ and $\kappa > 1$, let $K > 0$ be a fixed constant and J be an integer such that $J > 2m + \frac{K\kappa - 1}{2}$. Then for any $y, z \in \mathbb{R}$, it holds that*

$$\Phi\left(\frac{z+y}{\sqrt{r_n - \sqrt{r_n}}}\right) - \Phi\left(\frac{z-y}{\sqrt{r_n - \sqrt{r_n}}}\right)$$
$$= 2\phi\left(\frac{z}{\sqrt{r_n}}\right) \sum_{k=0}^{J} \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k+1)/2}} H_{2k}\left(\frac{z}{\sqrt{r_n}}\right) r_n^{(2k+1)/4} H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right) + \varepsilon_{m,y,z,n,\kappa} \quad (2.4)$$

with

$$r_n^{(2m+1)/2} \sup\left\{|\varepsilon_{m,y,z,n,\kappa}| : \ z \in \mathbb{R}, \ |y| \le \sqrt{K\sqrt{r_n}\log n}\right\} \overset{n\to\infty}{\longrightarrow} 0.$$

Proof: By Lemma 2.5 (i), for all $k \ge 1$, $z \in \mathbb{R}$ and $|y| \le \sqrt{K\sqrt{r_n}\log n}$,

$$\frac{r_n^{(2m+1)/2}}{(2k+1)!} \frac{1}{r_n^{(2k+1)/2}} \left|\phi\left(\frac{z}{\sqrt{r_n}}\right) H_{2k}\left(\frac{z}{\sqrt{r_n}}\right)\right| \times \left|r_n^{(2k+1)/4} H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right)\right|$$
$$\le \frac{4 r_n^{(2m+1)/2}}{r_n^{(2k+1)/4}} \frac{1}{\sqrt{2\pi}} e^{y^2/\sqrt{r_n}} \le \frac{4}{\sqrt{2\pi}} \frac{1}{n^{(2k-1-4m)/(4\kappa)}} n^{K/4}.$$

Combining this with (2.3), we get that (2.4) holds with

$$r_n^m \sup_{z\in\mathbb{R}, |y| \le \sqrt{K\sqrt{r_n}\log n}} |\varepsilon_{m,y,z,n}|$$
$$\le \frac{4}{\sqrt{2\pi}} \sum_{k=J+1}^{\infty} \frac{1}{n^{(2k-1-4m)/(4\kappa)}} n^{K/4} \lesssim \frac{1}{n^{(2(J+1)-1-4m)/(4\kappa)}} n^{K/4},$$

which tends to 0 is since $J > 2m + \frac{K\kappa-1}{2}$. □

Taking derivative with respect to b in Lemma 2.6, and using the fact that

$$\frac{\mathrm{d}^k}{\mathrm{d}b^k} \Phi(b) = (-1)^{k-1} H_{k-1}(b)\phi(b), \quad (2.5)$$

we get that

$$\frac{1}{\sqrt{1-\rho^2}} \phi\left(\frac{b - \rho x}{\sqrt{1-\rho^2}}\right) = \phi(b) + \phi(b) \sum_{k=1}^{\infty} \frac{\rho^k}{k!} H_k(b) H_k(x). \quad (2.6)$$

Taking $\rho = r_n^{-1/4}$, $b = r_n^{-1/2} z$ and $x = r_n^{-1/4} y$ in (2.6), we get that for any $z, y \in \mathbb{R}$,

$$\frac{\sqrt{r_n}}{\sqrt{r_n - \sqrt{r_n}}} \phi\left(\frac{z-y}{\sqrt{r_n - \sqrt{r_n}}}\right) = \phi\left(\frac{z}{\sqrt{r_n}}\right)\left(1 + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{r_n^{k/2}} H_k\left(\frac{z}{\sqrt{r_n}}\right) r_n^{k/4} H_k\left(\frac{y}{r_n^{1/4}}\right)\right). \quad (2.7)$$

Noting that, for any $k \in \mathbb{N}$, H_{2k} is an even function and H_{2k+1} is an odd function, we deduce from (2.7) that

$$\frac{\sqrt{r_n}}{\sqrt{r_n - \sqrt{r_n}}} \left(\phi\left(\frac{z-y}{\sqrt{r_n - \sqrt{r_n}}}\right) - \phi\left(\frac{z+y}{\sqrt{r_n - \sqrt{r_n}}}\right)\right)$$

$$= 2\phi\left(\frac{z}{\sqrt{r_n}}\right)\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\frac{1}{r_n^{(2k+1)/2}}H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right)r_n^{(2k+1)/4}H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right).$$

Using an argument similar to that leading to Lemma 2.7, we also have the following lemma. We omit the proof.

Lemma 2.8 *For any given $m \in \mathbb{N}$ and $\kappa > 1$, let $K > 0$ be a fixed constant and J be an integer such that $J > 2m + \frac{K\kappa+1}{2}$. Then for any $y, z \in \mathbb{R}$, it holds that*

$$\frac{\sqrt{r_n}}{\sqrt{r_n - \sqrt{r_n}}}\left(\phi\left(\frac{z-y}{\sqrt{r_n - \sqrt{r_n}}}\right) - \phi\left(\frac{z+y}{\sqrt{r_n - \sqrt{r_n}}}\right)\right)$$
$$= 2\phi\left(\frac{z}{\sqrt{r_n}}\right)\sum_{k=0}^{J}\frac{1}{(2k+1)!}\frac{1}{r_n^{(2k+1)/2}}H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right)r_n^{(2k+1)/4}H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right) + \varepsilon_{m,y,z,n,\kappa}$$

with

$$r_n^{m+1}\sup\left\{|\varepsilon_{m,y,z,n,\kappa}| : \; z \in \mathbb{R}, \; |y| \le \sqrt{K\sqrt{r_n}\log n}\right\} \stackrel{n\to\infty}{\longrightarrow} 0.$$

3 Proofs of the main results

3.1 Convergence rate for the martingales

Proposition 3.1 *Suppose $x > 0$ and $\theta \in [0, \sqrt{2})$ (i) For any $k \in \mathbb{N}$, $\{M_t^{(2k+1,\theta)}, t \ge 0; \mathbb{P}_x\}$ is a martingale. (ii) If (1.3) holds for some $\lambda > k$, then $M_t^{(2k+1,\theta)}$ converges to a limit $M_\infty^{(2k+1,\theta)}$ $\mathbb{P}_x$-a.s. and in $L^1(\mathbb{P}_x)$. Moreover, for any $\eta \in (0, \lambda - k)$, as $t \to \infty$,*

$$M_t^{(2k+1,\theta)} - M_\infty^{(2k+1,\theta)} = o(t^{-(\lambda-k)+\eta}), \quad \mathbb{P}_x\text{-a.s.}$$

Proof: (i) We will use $v < u$ to denote that v is an ancestor of u and $v \le u$ to denote $v = u$ or $v < u$. By the Markov property and Lemma 2.1, for any $t, s > 0$,

$$\begin{aligned}
\mathbb{E}_x\left(M_{s+t}^{(2k+1,\theta)}\big|\mathcal{F}_t\right) &= e^{-(1-\frac{\theta^2}{2})(t+s)}\sum_{v\in N(t)} 1_{\{\min_{r\le t} X_v(r)>0\}}(t+s)^{(2k+1)/2} \\
&\quad \times \mathbb{E}_{X_v(t)}\left(\sum_{u\in N(t+s): v\le u} 1_{\{\min_{r\le s} X_v(t+r)>0\}} e^{\theta X_v(t+s)} H_{2k+1}\left(\frac{X_u(t+s)}{\sqrt{t+s}}\right)\Big|\mathcal{F}_t\right) \\
&= e^{-(1-\frac{\theta^2}{2})(t+s)}\sum_{v\in N(t)} 1_{\{\min_{r\le t} X_v(r)>0\}} \\
&\quad \times e^s \mathbf{E}_{X_v(t)}\left(1_{\{\min_{r\le s} X_\xi(r)>0\}} e^{\theta X_\xi(s)}(t+s)^{(2k+1)/2} H_{2k+1}\left(\frac{X_\xi(s)}{\sqrt{t+s}}\right)\right) \\
&=: e^{-(1-\frac{\theta^2}{2})(t+s)}\sum_{v\in N(t)} 1_{\{\min_{r\le t} X_v(r)>0\}} F(s,t,X_v(t)).
\end{aligned} \tag{3.1}$$

Note that $X_\xi(s)$ under $\mathbf{P}_x$ is a standard Brownian motion with drift $-\theta$. It follows from Lemma 2.3 (i) that

$$F(s,t,X_v(t)) = e^{(1-\frac{\theta^2}{2})s} X_v(t) e^{\theta X_v(t)} \Pi^{\uparrow}_{X_v(t)}\left(\frac{(t+s)^{(2k+1)/2}}{B_s} H_{2k+1}\left(\frac{B_s}{\sqrt{t+s}}\right)\right)$$

$$= e^{(1-\frac{\theta^2}{2})s} X_v(t) e^{\theta X_v(t)} \int_0^\infty \frac{(t+s)^{(2k+1)/2}}{y} H_{2k+1}\left(\frac{y}{\sqrt{t+s}}\right)$$
$$\times \frac{y}{X_v(t)\sqrt{s}} \left(\phi\left(\frac{y-X_v(t)}{\sqrt{s}}\right) - \phi\left(\frac{y+X_v(t)}{\sqrt{s}}\right)\right) \mathrm{d}y$$
$$= e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} \int_0^\infty (t+s)^{(2k+1)/2} H_{2k+1}\left(\frac{y}{\sqrt{t+s}}\right) \frac{\phi\left(\frac{y-X_v(t)}{\sqrt{s}}\right) - \phi\left(\frac{y+X_v(t)}{\sqrt{s}}\right)}{\sqrt{s}} \mathrm{d}y.$$

Using the fact that $H_{2k+1}(\cdot)$ is an odd function and that $\phi(\cdot)$ is an even function, we have

$$F(s,t,X_v(t)) = e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} \int_{-\infty}^\infty (t+s)^{(2k+1)/2} H_{2k+1}\left(\frac{y}{\sqrt{t+s}}\right) \frac{\phi\left(\frac{y-X_v(t)}{\sqrt{s}}\right)}{\sqrt{s}} \mathrm{d}y$$
$$= e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} \int_{-\infty}^\infty (t+s)^{(2k+1)/2} H_{2k+1}(z) \frac{\sqrt{t+s}}{\sqrt{s}} \phi\left(\frac{\sqrt{t+s}z - X_v(t)}{\sqrt{s}}\right) \mathrm{d}z.$$

Taking $\rho := \sqrt{t/(s+t)}, b = z, x = X_v(t)/\sqrt{t}$ in (2.6), we see that

$$\frac{\sqrt{t+s}}{\sqrt{s}} \phi\left(\frac{\sqrt{t+s}z - X_v(t)}{\sqrt{s}}\right) = \phi(z) + \phi(z) \sum_{\ell=1}^\infty \frac{t^{\ell/2}}{\ell!} \frac{1}{(t+s)^{\ell/2}} H_\ell(z) H_\ell\left(\frac{X_v(t)}{\sqrt{t}}\right).$$

Combining these with Lemma 2.5(i) and the fact $\phi(z)e^{z^2/4} \lesssim e^{-z^2/4}$, we can easily get that for any $s,t>0, y\in\mathbb{R}$, the series

$$H_{2k+1}(z)\,\phi(z) \sum_{\ell=1}^\infty \frac{t^{\ell/2}}{\ell!} \frac{1}{(t+s)^{\ell/2}} H_\ell(z) H_\ell\left(\frac{y}{\sqrt{t}}\right)$$

is uniformly convergent in $z\in\mathbb{R}$. Now applying the property $\int_{-\infty}^\infty H_m(z)H_n(z)\phi(z)\mathrm{d}z = \delta_{m,n} n!$, we get

$$F(s,t,X_v(t))$$
$$= e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} \int_{-\infty}^\infty (t+s)^{(2k+1)/2} \frac{t^{(2k+1)/2}}{(2k+1)!} \frac{H_{2k+1}(z)\,\phi(z)}{(t+s)^{(2k+1)/2}} H_{2k+1}(z) H_{2k+1}\left(\frac{X_v(t)}{\sqrt{t}}\right) \mathrm{d}z$$
$$= e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} t^{(2k+1)/2} H_{2k+1}\left(\frac{X_v(t)}{\sqrt{t}}\right).$$

Plugging this into (3.1), we get (i).

(ii) Suppose (1.3) holds for some $\lambda > k$. If the first assertion of (ii) holds along $t = n\in\mathbb{N}$, then it is valid along all t since for $t\in(n,n+1)$, $M_t^{(2k+1,\theta)} = \mathbb{E}_x\left(M_{n+1}^{(2k+1,\theta)} \middle| \mathcal{F}_t\right) = \mathbb{E}_x\left(M_\infty^{(2k+1,\theta)} \middle| \mathcal{F}_t\right)$. In the following we use two steps to prove the assertion of (ii) holds for the case $t = n\in\mathbb{N}$.

Step 1: In this step, we will define a truncated process $M_{n+1}^{(2k+1,\theta,B)}$ and give first moment estimate for $M_{n+1}^{(2k+1,\theta)} - M_{n+1}^{(2k+1,\theta,B)}$. Let d_v, O_v denote the death time and the number of offspring of v respectively. For $u\in N(n+1)$, let $B_{n,u}$ be the event that, for all $v<u$ with $d_v\in(n,n+1)$, it holds that $O_v \le e^{c_0 n}$, where $c_0>0$ is a small constant to be determined later. Define

$$M_{n+1}^{(2k+1,\theta,B)}$$
$$:= e^{-(1-\frac{\theta^2}{2})(n+1)} \sum_{u\in N(n+1)} 1_{\{\min_{s\le n+1} X_u(s)>0\}} e^{\theta X_u(n+1)} (n+1)^{(2k+1)/2} H_{2k+1}\left(\frac{X_u(n+1)}{\sqrt{n+1}}\right) 1_{B_{n,u}}.$$

By the branching property, it holds that

$$\begin{aligned}
&\left|M_{n+1}^{(2k+1,\theta)}-M_{n+1}^{(2k+1,\theta,B)}\right| \\
&\le e^{-(1-\frac{\theta^2}{2})(n+1)}\sum_{u\in N(n+1)}1_{\{\min_{s\le n+1}X_u(s)>0\}}e^{\theta X_u(n+1)}(n+1)^{(2k+1)/2}\left|H_{2k+1}\left(\frac{X_u(n+1)}{\sqrt{n+1}}\right)\right|1_{B^c_{n,u}} \\
&= e^{-(1-\frac{\theta^2}{2})(n+1)}\sum_{v\in N(n)}1_{\{\min_{s\le n}X_v(s)>0\}} \\
&\quad\times\sum_{u\in N(n+1):v\le u}1_{\{\min_{s\le 1}X_u(n+s)>0\}}e^{\theta X_u(n+1)}(n+1)^{(2k+1)/2}\left|H_{2k+1}\left(\frac{X_u(n+1)}{\sqrt{n+1}}\right)\right|1_{B^c_{n,u}}.
\end{aligned}$$

Using the Markov property first and then Lemma 2.5(ii), we get that

$$\begin{aligned}
&\mathbb{E}_x\left(\left|M_{n+1}^{(2k+1,\theta)}-M_{n+1}^{(2k+1,\theta,B)}\right|\Big|\mathcal{F}_n\right) \\
&\le e^{-(1-\frac{\theta^2}{2})(n+1)}\sum_{v\in N(n)}1_{\{\min_{s\le n}X_v(s)>0\}} \\
&\quad\times\mathbb{E}_{X_v(n)}\left(\sum_{u\in N(1)}1_{\{\min_{s\le 1}X_u(s)>0\}}e^{\theta X_u(1)}(n+1)^{(2k+1)/2}\left|H_{2k+1}\left(\frac{X_u(1)}{\sqrt{n+1}}\right)\right|1_{D^c_{n,u}}\right) \\
&\lesssim e^{-(1-\frac{\theta^2}{2})(n+1)}\sum_{v\in N(n)}1_{\{\min_{s\le n}X_v(s)>0\}} \\
&\quad\times\mathbb{E}_{X_v(n)}\left(\sum_{u\in N(1)}1_{\{\min_{s\le 1}X_u(s)>0\}}X_u(1)e^{\theta X_u(1)}\left((X_u(1))^{2k}+(n+1)^k\right)1_{D^c_{n,u}}\right),
\end{aligned}$$

where for $u\in N(1)$, $D_{n,u}$ denotes the event that, for all $w<u$ with $d_w<1$, it holds that $O_w\le e^{c_0 n}$. Let d_i be the i-th splitting time of the spine and O_i be the number of children produced by the spine at time d_i. Define D_{n,ξ_1} to be the event that, for all i with $d_i<1$, it holds that $O_i\le e^{c_0 n}$. By Lemma 2.1,

$$\begin{aligned}
&\mathbb{E}_x\left(\left|M_{n+1}^{(2k+1,\theta)}-M_{n+1}^{(2k+1,\theta,B)}\right|\Big|\mathcal{F}_n\right) \\
&\lesssim e^{-(1-\frac{\theta^2}{2})(n+1)}\sum_{v\in N(n)}1_{\{\min_{s\le n}X_v(s)>0\}} \\
&\quad\times e\mathbf{E}_{X_v(n)}\left(1_{\{\min_{s\le 1}X_\xi(s)>0\}}X_\xi(1)e^{\theta X_\xi(1)}\left(X_\xi(1)^{2k}+(n+1)^k\right)1_{D^c_{n,\xi_1}}\right) \\
&=: e^{1-(1-\frac{\theta^2}{2})(n+1)}\sum_{v\in N(n)}1_{\{\min_{s\le n}X_v(s)>0\}}R(X_v(n)).
\end{aligned}\tag{3.2}$$

Conditioned on the motion X_ξ, $\{d_i:i\ge 1\}$ are the atoms of a Poisson poin process with rate 2 and $\{O_i:i\in\mathbb{N}\}$ are iid copies of $\widehat{L}$ with law $\mathbf{P}_x(\widehat{L}=\ell)=\ell p_\ell/2$ which are independent of $\{d_i:i\in\mathbb{N}\}$. Therefore, D_{n,ξ_1} is independent of $X_\xi(t)$. Together with Lemma 2.3 (i), we get that

$$\begin{aligned}
R(X_v(n)) &= \mathbf{E}_{X_v(n)}\left(1_{\{\min_{s\le 1}X_\xi(s)>0\}}X_\xi(1)e^{\theta X_\xi(1)}\left(X_\xi(1)^{2k}+(n+1)^k\right)\right)\mathbf{E}_{X_v(n)}\left(1_{D^c_{n,\xi_1}}\right) \\
&\le e^{-\frac{\theta^2}{2}}X_v(n)e^{\theta X_v(n)}\Pi^{\uparrow}_{X_v(n)}\left(B_1^{2k}+(n+1)^k\right)\mathbf{E}_{X_v(n)}\left(\sum_{i:d_i\le 1}1_{\{O_i>e^{c_0 n}\}}\right)
\end{aligned}$$

$$= 2e^{-\frac{\theta^2}{2}} X_v(n) e^{\theta X_v(n)} \Pi^{\uparrow}_{X_v(n)} \left(B_1^{2k} + (n+1)^k \right) \mathbf{P}_x(\widehat{L} > e^{c_0 n}). \tag{3.3}$$

Noticing that $(B_t, \Pi^{\uparrow}_x)$ is a 3-dimensional Bessel process, we easily see that

$$\Pi^{\uparrow}_{X_v(n)} \left(B_1^{2k} + (n+1)^k \right) \le \Pi^{\uparrow}_0 \left((B_1 + y)^{2k} + (n+1)^k \right) \big|_{y = X_v(n)} \lesssim (X_v(n))^{2k} + n^k. \tag{3.4}$$

Noting that (1.3) implies $\mathbf{E}_x(\log_+^{1+\lambda} \widehat{L}) < \infty$, using (3.3) and (3.4), we obtain that

$$R(X_v(n)) \lesssim \frac{1}{n^{1+\lambda}} e^{-\frac{\theta^2}{2}} X_v(n) e^{\theta X_v(n)} \left((X_v(n))^{2k} + n^k \right).$$

Plugging this inequality into (3.2), we conclude that

$$\begin{aligned}
&\mathbb{E}_x \left(\left| M_{n+1}^{(2k+1,\theta)} - M_{n+1}^{(2k+1,\theta,B)} \right| \Big| \mathcal{F}_n \right) \\
&\lesssim \frac{e^{-(1-\frac{\theta^2}{2})n}}{n^{1+\lambda}} \sum_{v \in N(n)} 1_{\{\min_{s \le n} X_v(s) > 0\}} X_v(n) e^{\theta X_v(n)} \left((X_v(n))^{2k} + n^k \right).
\end{aligned}$$

Taking expectation with respect to $\mathbb{P}_x$, applying Lemma 2.1 first and then Lemma 2.3 (i), we get that

$$\begin{aligned}
&\mathbb{E}_x \left(\left| M_{n+1}^{(2k+1,\theta)} - M_{n+1}^{(2k+1,\theta,B)} \right| \right) \\
&\lesssim \frac{e^{-(1-\frac{\theta^2}{2})n}}{n^{1+\lambda}} \mathbb{E}_x \Big(\sum_{v \in N(n)} 1_{\{\min_{s \le n} X_v(s) > 0\}} X_v(n) e^{\theta X_v(n)} \left((X_v(n))^{2k} + n^k \right) \Big) \\
&= \frac{e^{\frac{\theta^2}{2} n}}{n^{1+\lambda}} \mathbf{E}_x \left(1_{\{\min_{s \le n} X_\xi(s) > 0\}} e^{\theta X_\xi(n)} X_\xi(n) \left((X_\xi(n))^{2k} + n^k \right) \right) \\
&= \frac{x e^{\theta x}}{n^{1+\lambda}} \Pi^{\uparrow}_x \left((B_n)^{2k} + n^k \right) \le \frac{x e^{\theta x}}{n^{1+\lambda}} \Pi^{\uparrow}_0 \left((B_n + x)^{2k} + n^k \right) \\
&\lesssim \frac{1}{n^{1+\lambda}} \Pi^{\uparrow}_0 \left((B_n)^{2k} + n^k \right) = \frac{n^k}{n^{1+\lambda}} \Pi^{\uparrow}_0 \left((B_1)^{2k} + 1 \right),
\end{aligned} \tag{3.5}$$

where in the last equality, we used the fact that $(B_t, \Pi^{\uparrow}_0) \overset{\mathrm{d}}{=} (\sqrt{t} B_1, \Pi^{\uparrow}_0)$.

Step 2: In this step, we will give an upper bound for the ℓ-th moment of $M_{n+1}^{(2k+1,\theta,B)} - \mathbb{E}_x \left(M_{n+1}^{(2k+1,\theta,B)} \big| \mathcal{F}_n \right)$ for appropriate $\ell \in (1,2)$. Combining this with Step 1 will yield the first result of (ii). For $v \in N(n)$, set

$$J_{n,v} := \sum_{u \in N(n+1): v \le u} 1_{\{\min_{s \le 1} X_u(n+s) > 0\}} e^{\theta X_u(n+1)} (n+1)^{(2k+1)/2} H_{2k+1} \left(\frac{X_u(n+1)}{\sqrt{n+1}} \right) 1_{B_{n,u}}.$$

By the branching property, $\{J_{n,v} : v \in N(n)\}$ are independent conditioned on $\mathcal{F}_n$. Thus, for any fixed $1 < \ell < \min\{2/\theta^2, 2\}$ with $(\ell-1)^2\theta^2/2 < (\ell-1)(1-\frac{\theta^2}{2})$, by Lemma 2.2,

$$\begin{aligned}
&\mathbb{E}_x \left(\left| M_{n+1}^{(2k+1,\theta,B)} - \mathbb{E}_x \left(M_{n+1}^{(2k+1,\theta,B)} \big| \mathcal{F}_n \right) \right|^{\ell} \big| \mathcal{F}_n \right) \\
&= e^{-\ell(1-\frac{\theta^2}{2})(n+1)} \mathbb{E}_x \left(\left| \sum_{v \in N(n)} 1_{\{\min_{s \le n} X_v(s) > 0\}} \left(J_{n,v} - \mathbb{E}_x \left(J_{n,v} \big| \mathcal{F}_n \right) \right) \right|^{\ell} \big| \mathcal{F}_n \right)
\end{aligned}$$

$$
\begin{aligned}
&\le 2e^{-\ell(1-\frac{\theta^2}{2})(n+1)} \sum_{v\in N(n)} 1_{\{X_v(n)>0\}} \mathbb{E}_x \left(\left| J_{n,v} - \mathbb{E}_x \left(J_{n,v} \middle| \mathcal{F}_n \right) \right|^\ell \middle| \mathcal{F}_n \right) \\
&\lesssim e^{-\ell(1-\frac{\theta^2}{2})(n+1)} \sum_{v\in N(n)} 1_{\{X_v(n)>0\}} \left(\mathbb{E}_x \left((J_{n,v})^2 \middle| \mathcal{F}_n \right) \right)^{\ell/2}. \qquad (3.6)
\end{aligned}
$$

Define

$$
J_n^* := \sum_{u\in N(1)} 1_{\{\min_{s\le 1} X_u(s)>0\}} e^{\theta X_u(1)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left(\frac{X_u(1)}{\sqrt{n+1}} \right) \right| 1_{D_{n,u}}.
$$

By the Markov property and Lemma 2.1,

$$
\begin{aligned}
&\mathbb{E}_x \left((J_{n,v})^2 \middle| \mathcal{F}_n \right) \le \mathbb{E}_{X_v(n)} \left((J_n^*)^2 \right) \\
&= \mathbb{E}_{X_v(n)} \left(\sum_{u\in N(1)} 1_{\{\min_{s\le 1} X_u(s)>0\}} e^{\theta X_u(1)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left(\frac{X_u(1)}{\sqrt{n+1}} \right) \right| 1_{D_{n,u}} J_n^* \right) \\
&= e\mathbf{E}_{X_v(n)} \left(1_{\{\min_{s\le 1} X_\xi(s)>0\}} e^{\theta X_\xi(1)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left(\frac{X_\xi(1)}{\sqrt{n+1}} \right) \right| 1_{D_{n,\xi_1}} J_n^* \right). \qquad (3.7)
\end{aligned}
$$

Conditioned on $\{X_\xi, d_i, O_i : i \ge 1\}$, by the Markov property, on the event D_{n,ξ_1}, we have

$$
\begin{aligned}
&\mathbf{E}_{X_v(n)} \left(J_n^* \middle| X_\xi, d_i, O_i : i \ge 1 \right) = \sum_{i:d_i\le 1} 1_{\{\min_{s\le d_i} X_\xi(s)>0\}} (O_i - 1) \\
&\quad \times \mathbb{E}_{X_\xi(d_i)} \left(\sum_{u\in N(z)} 1_{\{\min_{s\le z} X_u(s)>0\}} e^{\theta X_u(z)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left(\frac{X_u(z)}{\sqrt{n+1}} \right) \right| 1_{D_{n,u}} \right) \Big|_{z=1-d_i} \\
&\le \sum_{i:d_i\le 1} 1_{\{\min_{s\le d_i} X_\xi(s)>0\}} (O_i - 1) \\
&\quad \times \mathbb{E}_{X_\xi(d_i)} \left(\sum_{u\in N(z)} 1_{\{\min_{s\le z} X_u(s)>0\}} e^{\theta X_u(z)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left(\frac{X_u(z)}{\sqrt{n+1}} \right) \right| \right) \Big|_{z=1-d_i} \\
&\le e^{c_0 n} \sum_{i:d_i\le 1} 1_{\{\min_{s\le d_i} X_\xi(s)>0\}} e^{1-d_i} \\
&\quad \times \mathbf{E}_{X_\xi(d_i)} \left(1_{\{\min_{s\le z} X_\xi(s)>0\}} e^{\theta X_\xi(z)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left(\frac{X_\xi(z)}{\sqrt{n+1}} \right) \right| \right) \Big|_{z=1-d_i} \\
&=: e^{c_0 n} \sum_{i:d_i\le 1} 1_{\{\min_{s\le d_i} X_\xi(s)>0\}} e^{1-d_i} F_n(X_\xi(d_i), 1-d_i), \qquad (3.8)
\end{aligned}
$$

where in the last inequality we used the fact that $O_i - 1 \le e^{c_0 n}$ on D_{n,ξ_1} and Lemma 2.1. Using Lemma 2.5(ii) in the first inequality, Lemma 2.3 (i) in the first equality and an argument similar to that leading to (3.4) in the second inequality, we get

$$
\begin{aligned}
&F_n(X_\xi(d_i), 1-d_i) \lesssim \mathbf{E}_{X_\xi(d_i)} \left(1_{\{\min_{s\le z} X_\xi(s)>0\}} e^{\theta X_\xi(z)} X_\xi(z) \left((X_\xi(z))^{2k} + (n+1)^k \right) \right) \Big|_{z=1-d_i} \\
&= X_\xi(d_i) e^{\theta X_\xi(d_i)} e^{-\frac{\theta^2}{2}(1-d_i)} \Pi^{\uparrow}_{X_\xi(d_i)} \left((B_z)^{2k} + (n+1)^k \right) \Big|_{z=1-d_i} \\
&\lesssim X_\xi(d_i) e^{\theta X_\xi(d_i)} \left((X_\xi(d_i))^{2k} + n^k \right) \sup_{z\in(0,1)} \Pi_0^{\uparrow}((B_z)^{2k} + 1)
\end{aligned}
$$

$$\lesssim X_\xi(d_i)e^{\theta X_\xi(d_i)}\left((X_\xi(d_i))^{2k}+n^k\right). \tag{3.9}$$

Combining (3.8) and (3.9), we obtain that

$$\begin{aligned}
&\mathbf{E}_{X_v(n)}\left(J_n^*\big|X_\xi, d_i, O_i : i\geq 1\right)\\
&\lesssim e^{c_0 n}\sum_{i:d_i\leq 1} 1_{\{\min_{s\leq d_i} X_\xi(s)>0\}}e^{1-d_i}X_\xi(d_i)e^{\theta X_\xi(d_i)}\left((X_\xi(d_i))^{2k}+n^k\right)\\
&\lesssim e^{c_0 n}\sum_{i:d_i\leq 1} 1_{\{\min_{s\leq d_i} X_\xi(s)>0\}}X_\xi(d_i)e^{\theta X_\xi(d_i)}\left((X_\xi(d_i))^{2k}+n^k\right).
\end{aligned} \tag{3.10}$$

Plugging (3.10) into (3.7),

$$\begin{aligned}
\mathbb{E}_x\left((J_{n,v})^2\big|\mathcal{F}_n\right) &\lesssim e^{c_0 n}\mathbf{E}_{X_v(n)}\Bigg(1_{\{\min_{s\leq 1} X_\xi(s)>0\}}e^{\theta X_\xi(1)}(n+1)^{(2k+1)/2}\left|H_{2k+1}\left(\frac{X_\xi(1)}{\sqrt{n+1}}\right)\right|\\
&\quad\times\int_0^1 1_{\{\min_{s\leq r} X_\xi(s)>0\}}X_\xi(r)e^{\theta X_\xi(r)}\left((X_\xi(r))^{2k}+n^k\right)\mathrm{d}r\Bigg)\\
&\lesssim e^{c_0 n}\mathbf{E}_{X_v(n)}\Bigg(1_{\{\min_{s\leq 1} X_\xi(s)>0\}}e^{\theta X_\xi(1)}X_\xi(1)\left((X_\xi(1))^{2k}+(n+1)^k\right)\\
&\quad\times\int_0^1 1_{\{\min_{s\leq r} X_\xi(s)>0\}}X_\xi(r)e^{\theta X_\xi(r)}\left((X_\xi(r))^{2k}+n^k\right)\mathrm{d}r\Bigg),
\end{aligned} \tag{3.11}$$

where the last inequality follows by Lemma 2.5(ii). Using Lemma 2.3, we can continue the estimate (3.11) and get

$$\begin{aligned}
&\mathbb{E}_x\left((J_{n,v})^2\big|\mathcal{F}_n\right)\lesssim X_v(n)e^{\theta X_v(n)}e^{c_0 n}\Pi^{\uparrow}_{X_v(n)}\Bigg(\left((B_1)^{2k}+(n+1)^k\right)\int_0^1 B_r e^{\theta B_r}\left((B_r)^{2k}+n^k\right)\mathrm{d}r\Bigg)\\
&\leq X_v(n)e^{2\theta X_v(n)}e^{c_0 n}\Pi^{\uparrow}_0\Bigg(\left((B_1+y)^{2k}+(n+1)^k\right)\int_0^1 (B_r+y)e^{\theta B_r}\left((B_r+y)^{2k}+n^k\right)\mathrm{d}r\Bigg)\Big|_{y=X_v(n)}\\
&\lesssim e^{c_0 n}(X_v(n)+1)^2 e^{2\theta X_v(n)}\left((X_v(n))^{2k}+n^k\right)^2.
\end{aligned} \tag{3.12}$$

Combining (3.6) and (3.12), we have

$$\begin{aligned}
&\mathbb{E}_x\left(\left|M_{n+1}^{(2k+1,\theta,B)}-\mathbb{E}_x\left(M_{n+1}^{(2k+1,\theta,B)}\big|\mathcal{F}_n\right)\right|^\ell\big|\mathcal{F}_n\right)\\
&\lesssim e^{-\ell(1-\frac{\theta^2}{2})(n+1)}e^{c_0\ell n/2}\sum_{v\in N(n)}1_{\{X_v(n)>0\}}(X_v(n)+1)^\ell e^{\ell\theta X_v(n)}\left((X_v(n))^{2k}+n^k\right)^\ell\\
&\leq e^{-\ell(1-\frac{\theta^2}{2})n}e^{c_0\ell n/2}\sum_{v\in N(n)}(|X_v(n)|+1)^\ell e^{\ell\theta X_v(n)}\left((X_v(n))^{2k}+n^k\right)^\ell.
\end{aligned}$$

Taking expectation with respect to $\mathbb{P}_x$ and applying Lemma 2.1, we conclude that

$$\begin{aligned}
&\mathbb{E}_x\left(\left|M_{n+1}^{(2k+1,\theta,B)}-\mathbb{E}_x\left(M_{n+1}^{(2k+1,\theta,B)}\big|\mathcal{F}_n\right)\right|^\ell\right)\\
&\lesssim e^{-\ell(1-\frac{\theta^2}{2})n}e^{c_0\ell n/2}e^n\mathbf{E}_x\left((|X_\xi(n)|+1)^\ell e^{\ell\theta X_\xi(n)}\left((X_\xi(n))^{2k}+n^k\right)^\ell\right)\\
&= e^{-\ell(1-\frac{\theta^2}{2})n}e^{c_0\ell n/2}e^n\Pi_x^{-\theta}\left((|B_n|+1)^\ell e^{\ell\theta B_n}\left((B_n)^{2k}+n^k\right)^\ell\right)
\end{aligned}$$

$$
\begin{aligned}
&= e^{\theta x} e^{-(\ell-1)(1-\frac{\theta^2}{2})n} e^{c_0 \ell n/2} \Pi_x \left((|B_n|+1)^\ell e^{(\ell-1)\theta B_n} \left((B_n)^{2k} + n^k \right)^\ell \right) \\
&= e^{\theta \ell x} e^{-(\ell-1)(1-\frac{\theta^2}{2})n} e^{c_0 \ell n/2} e^{\frac{(\ell-1)^2\theta^2}{2} n} \Pi_x^{(\ell-1)\theta} \left((|B_n|+1)^\ell \left((B_n)^{2k} + n^k \right)^\ell \right) \\
&\lesssim e^{-(\ell-1)(1-\frac{\theta^2}{2})n} e^{c_0 \ell n/2} e^{\frac{(\ell-1)^2\theta^2}{2} n} n^{2k\ell+\ell} =: n^{2k\ell+\ell} e^{-c_1 \ell n}, \qquad (3.13)
\end{aligned}
$$

where in the second and third equalities we used the change-of-measure $\frac{\mathrm{d}\Pi_x^\eta}{\mathrm{d}\Pi_x}\big|_{\sigma(B_s:s\leq t)} = e^{\eta(B_t - x) - \frac{\eta^2}{2}t}$ for $\eta = -\theta$ and $\eta = (\ell-1)\theta$ respectively. Let $c_0 > 0$ be sufficiently small so that $c_0\ell/2 < (\ell-1)(1-\frac{\theta^2}{2}) - (\ell-1)^2\theta^2/2$, which implies that $c_1 > 0$. Thus, using the inequality:

$$
\begin{aligned}
&\mathrm{E}\left(|X - \mathrm{E}(X|\mathcal{F})|\right) \leq \mathrm{E}\left(|X-Y|\right) + \mathrm{E}\left(|Y - \mathrm{E}(Y|\mathcal{F})|\right) + \mathrm{E}\left(\left|\mathrm{E}\left(X - Y\middle|\mathcal{F}\right)\right|\right) \\
&\leq 2\mathrm{E}\left(|X-Y|\right) + \mathrm{E}\left(|Y - \mathrm{E}(Y|\mathcal{F})|^\ell\right)^{1/\ell}, \qquad (3.14)
\end{aligned}
$$

(3.5) and (3.13), we get

$$
\begin{aligned}
&\sum_{n=1}^\infty \mathbb{E}_x \left(\left| M_{n+1}^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \right) \\
&\leq 2 \sum_{n=1}^\infty \mathbb{E}_x \left(\left| M_{n+1}^{(2k+1,\theta)} - M_{n+1}^{(2k+1,\theta,B)} \right| \right) + \sum_{n=1}^\infty \mathbb{E}_x \left(\left| M_{n+1}^{(2k+1,\theta,B)} - \mathbb{E}_x \left(M_{n+1}^{(2k+1,\theta,B)} \middle| \mathcal{F}_n \right) \right|^\ell \right)^{1/\ell} \\
&\lesssim \sum_{n=1}^\infty \frac{n^k}{n^{1+\lambda}} + \sum_{n=1}^\infty n^{2k+1} e^{-c_1 n},
\end{aligned}
$$

which is finite since $\lambda > k$. Therefore, $M_n^{(2k+1,\theta)}$ converges to a limit $M_\infty^{(2k+1,\theta)}$ $\mathbb{P}_x$-a.s. and in $L^1(\mathbb{P}_x)$.

Step 3: In this step, we prove the second assertion of (ii). For any $\eta \in (0, \lambda - k)$, by (3.5) and (3.13),

$$
\sum_{n=1}^\infty n^{\lambda-k-\eta} \mathbb{E}_x \left(\left| M_{n+1}^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \right) \lesssim \sum_{n=1}^\infty n^{\lambda-k-\eta} \frac{n^k}{n^{1+\lambda}} + \sum_{n=1}^\infty n^{\lambda-k-\eta} n^{2k+1} e^{-c_1 n} < \infty.
$$

Thus, $n^{\lambda-k-\eta}\left(M_n^{(2k+1,\theta)} - M_\infty^{(2k+1,\theta)}\right) \overset{n\to\infty}{\longrightarrow} 0, \mathbb{P}_x$-a.s. (see for example [1, Lemma 2]). For $s \in [n, n+1]$, by Doob's inequality, for any $\varepsilon > 0$,

$$
\begin{aligned}
&\sum_{n=1}^\infty \mathbb{P}_x \left(n^{\lambda-k-\eta} \sup_{n\leq s\leq n+1} \left| M_s^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| > \varepsilon \right) \\
&\leq \frac{1}{\varepsilon} \sum_{n=1}^\infty n^{\lambda-k-\eta} \mathbb{E}_x \left(\left| M_{n+1}^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \right) < \infty,
\end{aligned}
$$

which implies that $n^{\lambda-k-\eta} \sup_{n\leq s\leq n+1} \left| M_s^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \overset{n\to\infty}{\longrightarrow} 0, \mathbb{P}_x$-a.s. Therefore, we have $\mathbb{P}_x$-almost surely,

$$
\begin{aligned}
\sup_{n\leq s\leq n+1} s^{\lambda-k-\eta} \left| M_s^{(2k+1,\theta)} - M_\infty^{(2k+1,\theta)} \right| &\leq (n+1)^{\lambda-k-\eta} \sup_{n\leq s\leq n+1} \left| M_s^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \\
&\quad + (n+1)^{\lambda-k-\eta} \left| M_n^{(2k+1,\theta)} - M_\infty^{(2k+1,\theta)} \right| \overset{n\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

which completes the proof of (ii). □

3.2 Asymptotic expansions along discrete time

Lemma 3.2 *Assume $x>0$ and $\theta\in[0,\sqrt{2})$. Let $A_n\subset(0,\infty)$ be a family of Borel sets such that*

$$\text{either}\quad \sup_n \sup\{y: y\in A_n\}<\infty \quad \text{or} \quad \inf_n \sup\{y: y\in A_n\}=\infty.$$

For any given $m\in\mathbb{N}$ and $\kappa>1$, if (1.3) holds for some $\lambda>2m+2\kappa+1$, then

$$r_n^m \frac{Z_{r_n}^{(0,\infty)}(A_n)-\mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n)\Big|\mathcal{F}_{\sqrt{r_n}}\right)}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}}=0, \quad \mathbb{P}_x\text{-a.s.}$$

where

$$b_\theta:=\begin{cases}3/2, & \theta=0 \text{ and } \sup_n\sup\{y:y\in A_n\}<\infty \text{ or } \theta\in(0,\sqrt{2});\\ 1/2, & \theta=0 \text{ and } \inf_n\sup\{y:y\in A_n\}=\infty.\end{cases}$$

In particular, for any Borel set $A\subset(0,\infty)$, as $n\to\infty$,

$$r_n^m \frac{Z_{r_n}^{(0,\infty)}(A)-\mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A)\Big|\mathcal{F}_{\sqrt{r_n}}\right)}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}}=0, \quad \mathbb{P}_x\text{-a.s.}$$

Proof: Suppose $m\in\mathbb{N}$, $\kappa>1$ and that (1.3) holds for some $\lambda>2m+2\kappa+1$. We divide the proof into three steps. In Step 1, we define a truncated process $Z_{r_n}^{(0,\infty),G}(A_n)$ and give a first-moment estimate for $Z_{r_n}^{(0,\infty)}(A_n)-Z_{r_n}^{(0,\infty),G}(A_n)$, see (3.21) below. In Step 2, we bound the ℓ-th moment of $Z_{r_n}^{(0,\infty),G}(A_n)-\mathbb{E}_x\left(Z_{r_n}^{(0,\infty),G}(A_n)\big|\mathcal{F}_{\sqrt{r_n}}\right)$ for appropriate $\ell\in(1,2)$, see (3.27) below; In Step 3, we combine the results obtained in Step 1 and Step 2 to get the assertion of the proposition.

Step 1: Recall that $v<u$ and $v\le u$ mean that v is an ancestor of u and that $v=u$ or $v<u$ respectively. For $u\in N(r_n)$, define $G_{n,u}$ to be the event that, for all $v<u$ with death time $d_v\in(\sqrt{r_n},r_n)$, it holds that $O_v\le e^{c_0\sqrt{r_n}}$, where $c_0>0$ is a small constant to be determined later. Define

$$Z_{r_n}^{(0,\infty),G}(A_n):=\sum_{u\in N(r_n)} 1_{\{\min_{s\le r_n}X_u(s)>0\}}1_{A_n}(X_u(r_n))1_{G_{n,u}}.$$

By the branching property, it holds that

$$Z_{r_n}^{(0,\infty),G}(A_n) = \sum_{v\in N(\sqrt{r_n})} 1_{\left\{\min_{s\le\sqrt{r_n}}X_v(s)>0\right\}}\sum_{u\in N(r_n):v\le u}1_{\left\{\min_{\sqrt{r_n}<s\le r_n}X_u(s)>0\right\}}1_{A_n}(X_u(r_n))1_{G_{n,u}}. \quad (3.15)$$

Therefore, by the Markov property,

$$\begin{aligned}&\frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}}\mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n)-Z_{r_n}^{(0,\infty),G}(A_n)\big|\mathcal{F}_{\sqrt{r_n}}\right)\\ &=\frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}}\sum_{v\in N(\sqrt{r_n})}1_{\left\{\min_{s\le\sqrt{r_n}}X_v(s)>0\right\}}\\ &\qquad\times\mathbb{E}_{X_v(\sqrt{r_n})}\left(\sum_{u\in N(r_n-\sqrt{r_n})}1_{\left\{\min_{s\le r_n-\sqrt{r_n}}X_u(s)>0\right\}}1_{A_n}\left(X_u(r_n-\sqrt{r_n})\right)1_{D^c_{n,u}}\right)\end{aligned}$$

$$=: \frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{v\in N(\sqrt{r_n})} 1_{\left\{\min_{s\le\sqrt{r_n}} X_v(s)>0\right\}} T_{A_n}(X_v(\sqrt{r_n}), r_n-\sqrt{r_n}), \tag{3.16}$$

where for $u \in N(r_n - \sqrt{r_n})$, $D_{n,u}$ denotes the event that, for all $w < u$ with $d_w < r_n - \sqrt{r_n}$, it holds that $O_w \le e^{c_0\sqrt{r_n}}$. Let d_i be the i-th splitting time of the spine and O_i be the number of children produced by the spine at time d_i. Define $D_{n,\xi_{r_n-\sqrt{r_n}}}$ to be the event that, for all i with $d_i < r_n - \sqrt{r_n}$, it holds that $O_i \le e^{c_0\sqrt{r_n}}$. By Lemma 2.1, we get that

$$\begin{aligned} &T_{A_n}(X_v(\sqrt{r_n}), r_n-\sqrt{r_n}) \\ &= e^{r_n-\sqrt{r_n}} \mathbf{E}_{X_v(\sqrt{r_n})}\left(1_{\left\{\min_{s\le r_n-\sqrt{r_n}} X_\xi(s)>0\right\}} 1_{A_n}(X_\xi(r_n-\sqrt{r_n})) 1_{D^c_{n,\xi_{r_n-\sqrt{r_n}}}}\right). \end{aligned} \tag{3.17}$$

Note that given X_ξ, $\{d_i : i \ge 1\}$ are the atoms for a Poisson point process with rate 2, $\{O_i : i \ge 1\}$ are iid with common law $\widehat{L}$ given by $\mathbf{P}_x(\widehat{L} = k) = kp_k/2$, and that $\{d_i : i \ge 1\}$ and $\{O_i : i \ge 1\}$ are independent. By (1.3), we conclude that

$$\begin{aligned} &\mathbf{E}_{X_v(\sqrt{r_n})}\left(1_{D^c_{n,\xi_{r_n-\sqrt{r_n}}}} \big| X_\xi(s) : s \ge 0\right) \le \mathbf{E}_{X_v(\sqrt{r_n})}\left(\sum_{i:d_i<r_n-\sqrt{r_n}} 1_{\left\{O_i > e^{c_0\sqrt{r_n}}\right\}} \big| X_\xi(s) : s\ge 0\right) \\ &= 2\int_0^{r_n-\sqrt{r_n}} \mathbf{P}_{X_v(\sqrt{r_n})}\left(\widehat{L} > e^{c_0\sqrt{r_n}}\right) \mathrm{d}s \le 2r_n \frac{\mathbf{E}_x\left(\log_+^{1+\lambda} L\right)}{(c_0\sqrt{r_n})^{1+\lambda}} \lesssim \frac{1}{r_n^{(\lambda-1)/2}}. \end{aligned} \tag{3.18}$$

Plugging (3.18) into (3.17), we get

$$T_{A_n}(X_v(\sqrt{r_n}), r_n-\sqrt{r_n}). \lesssim \frac{e^{r_n-\sqrt{r_n}}}{r_n^{(\lambda-1)/2}} \mathbf{P}_{X_v(\sqrt{r_n})}\left(\min_{s\le r_n-\sqrt{r_n}} X_\xi(s) > 0, X_{r_n-\sqrt{r_n}} \in A_n\right). \tag{3.19}$$

Since, under $\mathbf{P}_{X_v(\sqrt{r_n})}$, $X_\xi(t)$ is a standard Brownian motion with drift $-\theta$. By Lemma 2.4 (i) and (ii) with $B = A_n$, we get that

$$\begin{aligned} &\mathbf{P}_{X_v(\sqrt{r_n})}\left(\min_{s\le r_n-\sqrt{r_n}} X_\xi(s) > 0, X_{r_n-\sqrt{r_n}} \in A_n\right) \\ &\lesssim \frac{X_v(\sqrt{r_n})e^{\theta X_v(\sqrt{r_n})}}{(r_n-\sqrt{r_n})^{b_\theta} e^{\frac{\theta^2}{2}(r_n-\sqrt{r_n})}} \lesssim \frac{X_v(\sqrt{r_n})e^{\theta X_v(\sqrt{r_n})}}{r_n^{b_\theta} e^{\frac{\theta^2}{2}(r_n-\sqrt{r_n})}}. \end{aligned} \tag{3.20}$$

Combining (3.16), (3.19) and (3.20), we get that

$$\begin{aligned} &\frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n) - Z_{r_n}^{(0,\infty),G}(A_n) \big| \mathcal{F}_{\sqrt{r_n}}\right) \\ &\lesssim \frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{v\in N(\sqrt{r_n})} 1_{\left\{\min_{s\le\sqrt{r_n}} X_v(s)>0\right\}} \frac{e^{r_n-\sqrt{r_n}}}{r_n^{(\lambda-1)/2}} \frac{X_v(\sqrt{r_n})e^{\theta X_v(\sqrt{r_n})}}{r_n^{b_\theta} e^{\frac{\theta^2}{2}(r_n-\sqrt{r_n})}}. \\ &= \frac{1}{r_n^{(\lambda-1-2m)/2}} e^{-(1-\frac{\theta^2}{2})\sqrt{r_n}} \sum_{v\in N(\sqrt{r_n})} 1_{\left\{\min_{s\le\sqrt{r_n}} X_v(s)>0\right\}} X_v(\sqrt{r_n}) e^{\theta X_v(\sqrt{r_n})} = \frac{1}{r_n^{(\lambda-1-2m)/2}} M_{\sqrt{r_n}}^{(1,\theta)}, \end{aligned}$$

with $M_{\sqrt{r_n}}^{(1,\theta)}$ given in (1.4). Now taking expectation with respect to $\mathbb{P}_x$, we get that

$$\frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n) - Z_{r_n}^{(0,\infty),G}(A_n)\right) \lesssim \frac{1}{r_n^{(\lambda-1-2m)/2}}. \tag{3.21}$$

Step 2: By (3.15) and the branching property,

$$Z_{r_n}^{(0,\infty),G}(A_n)-\mathbb{E}_x\left(Z_{r_n}^{(0,\infty),G}(A_n)\big|\mathcal{F}_{\sqrt{r_n}}\right)=:\sum_{v\in N(\sqrt{r_n})}1_{\left\{\min_{s\le\sqrt{r_n}}X_v(s)>0\right\}}J_{n,u},$$

where conditioned on $\mathcal{F}_{\sqrt{r_n}}$, $\left\{J_{n,v}:v\in N(\sqrt{r_n})\right\}$ are centered independent random variables defined by

$$\begin{aligned}J_{n,v}=&\sum_{u\in N(r_n):v\le u}1_{\left\{\min_{\sqrt{r_n}<s\le r_n}X_u(s)>0\right\}}1_{A_n}(X_u(r_n))1_{G_{n,u}}\\&-\mathbb{E}_x\left(\sum_{u\in N(r_n):v\le u}1_{\left\{\min_{\sqrt{r_n}<s\le r_n}X_u(s)>0\right\}}1_{A_n}(X_u(r_n))1_{G_{n,u}}\Big|\mathcal{F}_{\sqrt{r_n}}\right).\end{aligned}$$

Thus, by Lemma 2.2, for any fixed ℓ with $1<\ell<\min\{2,2/\theta^2\}$ and $(\ell-1)\frac{\theta^2}{2}<1-\frac{\theta^2}{2}$,

$$\begin{aligned}&\mathbb{E}_x\left(\left|Z_{r_n}^{(0,\infty),G}(A_n)-\mathbb{E}_x\left(Z_{r_n}^{(0,\infty),G}(A_n)\big|\mathcal{F}_{\sqrt{r_n}}\right)\right|^\ell\Big|\mathcal{F}_{\sqrt{r_n}}\right)\\&\le 2\sum_{v\in N(\sqrt{r_n})}1_{\left\{\min_{s\le\sqrt{r_n}}X_v(s)>0\right\}}\mathbb{E}_x\left(|J_{n,v}|^\ell\,\big|\mathcal{F}_{\sqrt{r_n}}\right)\lesssim\sum_{v\in N(\sqrt{r_n})}M_{n,v},\end{aligned}\tag{3.22}$$

where for each $v\in N(\sqrt{r_n})$,

$$M_{n,v}:=\mathbb{E}_{X_v(\sqrt{r_n})}\left(\left(\sum_{u\in N(r_n-\sqrt{r_n})}1_{\left\{\min_{s\le r_n-\sqrt{r_n}}X_u(s)>0\right\}}1_{A_n}\left(X_u(r_n-\sqrt{r_n})\right)1_{D_{n,u}}\right)^\ell\right).$$

Set $V_n:=\sum_{u\in N(r_n-\sqrt{r_n})}1_{\left\{\min_{s\le r_n-\sqrt{r_n}}X_u(s)>0\right\}}1_{A_n}\left(X_u(r_n-\sqrt{r_n})\right)1_{D_{n,u}}$. By Lemma 2.1 and the fact that $A_n\subset(0,\infty),V_n\le\sum_{u\in N(r_n-\sqrt{r_n})}1_{(0,\infty)}(X_u(r_n-\sqrt{r_n}))$, we have

$$\begin{aligned}M_{n,v}&=e^{r_n-\sqrt{r_n}}\mathbf{E}_{X_v(\sqrt{r_n})}\left(1_{\left\{\min_{s\le r_n-\sqrt{r_n}}X_\xi(s)>0\right\}}1_{A_n}\left(X_\xi(r_n-\sqrt{r_n})\right)1_{D_{n,\xi_{r_n-\sqrt{r_n}}}}V_n^{\ell-1}\right)\\&\le e^{r_n-\sqrt{r_n}}\mathbf{E}_{X_v(\sqrt{r_n})}\left(1_{D_{n,\xi_{r_n-\sqrt{r_n}}}}1_{\{X_\xi(r_n-\sqrt{r_n})>0\}}\left(\sum_{u\in N(r_n-\sqrt{r_n})}1_{\{X_u(r_n-\sqrt{r_n})>0\}}\right)^{\ell-1}\right)\end{aligned}\tag{3.23}$$

Given X_ξ,d_i and O_i, by the Markov property and the inequality $(\sum_{i=1}^n x_i)^p\le\sum_{i=1}^n x_i^p$ for all $x_i\ge0$ and $p\in(0,1)$, it holds that

$$\begin{aligned}&1_{\left\{D_{n,\xi_{r_n-\sqrt{r_n}}}\right\}}\mathbf{E}_{X_v(\sqrt{r_n})}\left(\left(\sum_{u\in N(r_n-\sqrt{r_n})}1_{\{X_u(r_n-\sqrt{r_n})>0\}}\right)^{\ell-1}\big|X_\xi,d_i,O_i:i\ge1\right)\\&\le 1_{\left\{D_{n,\xi_{r_n-\sqrt{r_n}}}\right\}}\sum_{i:d_i<r_n-\sqrt{r_n}}(O_i-1)\mathbb{E}_{X_\xi(d_i)}\left(\left(\sum_{u\in N(z)}1_{\{X_u(r_n-\sqrt{r_n})>0\}}\right)^{\ell-1}\right)\Big|_{z=r_n-\sqrt{r_n}-d_i}\\&\le e^{c_0\sqrt{r_n}}\sum_{i:d_i<r_n-\sqrt{r_n}}\mathbb{E}_{X_\xi(d_i)}\left(\sum_{u\in N(z)}1_{\{X_u(r_n-\sqrt{r_n})>0\}}\right)^{\ell-1}\Big|_{z=r_n-\sqrt{r_n}-d_i},\end{aligned}\tag{3.24}$$

where in the last inequality, we used the fact that, on the event $D_{n,\xi_{r_n-\sqrt{r_n}}}$, $O_i - 1 \leq e^{c_0\sqrt{r_n}}$ and the fact that $\mathrm{E}(|X|^p) \leq \mathrm{E}(|X|)^p$ for $p \in (0,1)$. Note that by Lemma 2.1 and Lemma 2.4 (iii),

$$\mathbb{E}_x\left(\sum_{u\in N(z)} 1_{\{X_u(r_n-\sqrt{r_n})>0\}}\right) \leq e^{(1-\frac{\theta^2}{2})z}e^{\theta x}.$$

Using the fact that d_i are the atoms of a Poisson process with rate 2, taking expectation with respect to $\mathbf{P}_{X_v(\sqrt{r_n})}(\cdot|X_\xi)$ in (3.24), we get that

$$\begin{aligned}&\mathbf{E}_{X_v(\sqrt{r_n})}\left(1_{\left\{D_{n,\xi_{r_n-\sqrt{r_n}}}\right\}}\left(\sum_{u\in N(r_n-\sqrt{r_n})} 1_{\{X_u(r_n-\sqrt{r_n})>0\}}\right)^{\ell-1}\Big|X_\xi\right)\\ &\leq 2e^{c_0\sqrt{r_n}}\int_0^{r_n-\sqrt{r_n}} e^{\theta(\ell-1)X_\xi(s)}e^{(1-\frac{\theta^2}{2})(\ell-1)(r_n-\sqrt{r_n}-s)}\mathrm{d}s. \end{aligned}\tag{3.25}$$

Combining (3.23) and (3.25), noting that $X_\xi(s)$ under $\mathbf{P}_x$ is a standard Brownian motion with drift $-\theta$, and applying Lemma 2.4(iii), we conclude that

$$\begin{aligned}M_{n,v} &\lesssim e^{r_n-\sqrt{r_n}}\mathbf{E}_{X_v(\sqrt{r_n})}\left(e^{c_0\sqrt{r_n}}1_{\{X_\xi(r_n-\sqrt{r_n})>0\}}\int_0^{r_n-\sqrt{r_n}} e^{\theta(\ell-1)X_\xi(s)}e^{(1-\frac{\theta^2}{2})(\ell-1)(r_n-\sqrt{r_n}-s)}\mathrm{d}s\right)\\ &= e^{r_n-\sqrt{r_n}}e^{c_0\sqrt{r_n}}\int_0^{r_n-\sqrt{r_n}}\Pi^{-\theta}_{X_v(\sqrt{r_n})}\left(e^{\theta(\ell-1)B_s}\Pi^{-\theta}_{B_s}\left(B_{r_n-\sqrt{r_n}-s}>0\right)\right)e^{(1-\frac{\theta^2}{2})(\ell-1)(r_n-\sqrt{r_n}-s)}\mathrm{d}s\\ &\leq e^{r_n-\sqrt{r_n}}e^{c_0\sqrt{r_n}}\int_0^{r_n-\sqrt{r_n}}\Pi^{-\theta}_{X_v(\sqrt{r_n})}\left(e^{\ell\theta B_s}\right)e^{-\frac{\theta^2}{2}(r_n-\sqrt{r_n}-s)}e^{(1-\frac{\theta^2}{2})(\ell-1)(r_n-\sqrt{r_n}-s)}\mathrm{d}s.\end{aligned}$$

Using elementary calculus in the last integral, we get

$$\begin{aligned}M_{n,v} &\lesssim e^{\ell\theta X_v(\sqrt{r_n})}e^{\ell(1-\frac{\theta^2}{2})(r_n-\sqrt{r_n})}e^{c_0\sqrt{r_n}}\int_0^{r_n-\sqrt{r_n}} e^{-s(\ell-1)(1-\frac{\theta^2}{2}-(\ell-1)\frac{\theta^2}{2})}\mathrm{d}s\\ &\lesssim e^{\ell\theta X_v(\sqrt{r_n})}e^{\ell(1-\frac{\theta^2}{2})(r_n-\sqrt{r_n})}e^{c_0\sqrt{r_n}}.\end{aligned}$$

Plugging this upper-bound into (3.22), we obtain that

$$\begin{aligned}&\frac{r_n^{\ell(m+b_\theta)}}{e^{\ell(1-\frac{\theta^2}{2})r_n}}\mathbb{E}_x\left(\left|Z^{(0,\infty),G}_{r_n}(A_n)-\mathbb{E}_x\left(Z^{(0,\infty),G}_{r_n}(A_n)\big|\mathcal{F}_{\sqrt{r_n}}\right)\right|^\ell\Big|\mathcal{F}_{\sqrt{r_n}}\right)\\ &\lesssim \frac{r_n^{\ell(m+b_\theta)}}{e^{\ell(1-\frac{\theta^2}{2})r_n}}\sum_{v\in N(\sqrt{r_n})} e^{\theta\ell X_v(\sqrt{r_n})}e^{\ell(1-\frac{\theta^2}{2})(r_n-\sqrt{r_n})}e^{c_0\sqrt{r_n}}\\ &= \frac{r_n^{\ell(m+b_\theta)}e^{c_0\sqrt{r_n}}}{e^{\ell(1-\frac{\theta^2}{2})\sqrt{r_n}}}\sum_{v\in N(\sqrt{r_n})} e^{\theta\ell X_v(\sqrt{r_n})}.\end{aligned}\tag{3.26}$$

Taking expectation in (3.26) with respect to $\mathbb{P}_x$, and using Lemma 2.1, we get

$$\frac{r_n^{\ell(m+b_\theta)}}{e^{\ell(1-\frac{\theta^2}{2})r_n}}\mathbb{E}_x\left(\left|Z^{(0,\infty),G}_{r_n}(A_n)-\mathbb{E}_x\left(Z^{(0,\infty),G}_{r_n}(A_n)\big|\mathcal{F}_{\sqrt{r_n}}\right)\right|^\ell\right)$$

$$\lesssim \frac{r_n^{\ell(m+b_\theta)}e^{c_0\sqrt{r_n}}}{e^{\ell(1-\frac{\theta^2}{2})\sqrt{r_n}}}\mathbb{E}_x\left(\sum_{v\in N(\sqrt{r_n})} e^{\theta\ell X_v(\sqrt{r_n})}\right) = \frac{r_n^{\ell(m+b_\theta)}e^{c_0\sqrt{r_n}}}{e^{\ell(1-\frac{\theta^2}{2})\sqrt{r_n}}}e^{\sqrt{r_n}}\mathbf{E}_x\left(e^{\theta\ell X_\xi(\sqrt{r_n})}\right)$$
$$= r_n^{\ell(m+b_\theta)}e^{\theta\ell x}e^{-((\ell-1)(1-\frac{\theta^2\ell}{2})-c_0)\sqrt{r_n}}. \tag{3.27}$$

Step 3: Fix $c_0\in(0,(\ell-1)(1-\frac{\theta^2\ell}{2}))$ and set $c_1 := ((\ell-1)(1-\frac{\theta^2\ell}{2})-c_0)/\ell > 0$. Using (3.14) with $X = Z_{r_n}^{(0,\infty)}(A_n)$ and $Y = Z_{r_n}^{(0,\infty),G}(A_n)$, we have

$$\begin{aligned}
&\mathbb{E}_x\left(\frac{r_n^m}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}}\left|Z_{r_n}^{(0,\infty)}(A_n) - \mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n)\middle|\mathcal{F}_{\sqrt{r_n}}\right)\right|\right)\\
&\le 2\frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}}\mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n) - Z_{r_n}^{(0,\infty),G}(A_n)\right)\\
&\quad+\left(\frac{r_n^{\ell(m+b_\theta)}}{e^{\ell(1-\frac{\theta^2}{2})r_n}}\mathbb{E}_x\left(\left|Z_{r_n}^{(0,\infty),G}(A_n) - \mathbb{E}_x\left(Z_{r_n}^{(0,\infty),G}(A_n)\middle|\mathcal{F}_{\sqrt{r_n}}\right)\right|^\ell\right)\right)^{1/\ell}\\
&\lesssim \frac{1}{r_n^{(\lambda-1-2m)/2}} + r_n^{(m+b_\theta)}e^{-c_1\sqrt{r_n}} = \frac{1}{n^{(\lambda-1-2m)/(2\kappa)}} + n^{(m+b_\theta)/\kappa}e^{-c_1 n^{1/(2\kappa)}},
\end{aligned}$$

where in the last inequality, we used (3.21) and (3.27). Since $\lambda > 2m+2\kappa+1$, we conclude that for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^\infty \mathbb{P}_x\left(\frac{r_n^m}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}}\left|Z_{r_n}^{(0,\infty)}(A_n) - \mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n)\middle|\mathcal{F}_{\sqrt{r_n}}\right)\right| > \varepsilon\right)\\
&\lesssim \sum_{n=1}^\infty\left(\frac{1}{n^{(\lambda-1-2m)/(2\kappa)}} + n^{(m+b_\theta)/\kappa}e^{-c_1 n^{1/(2\kappa)}}\right) < \infty,
\end{aligned}$$

which completes the proof of the Lemma. □

Proposition 3.3 *Let $x > 0$. For any given $m\in\mathbb{N}$ and $\kappa > 2m+1$, if (1.3) holds for some $\lambda > 2m+2\kappa+2$, then (i) for any $\theta\in(0,\sqrt{2})$, $x > 0$ and Borel set $A\subset(0,\infty)$, $\mathbb{P}_x$-almost surely, as $n\to\infty$,*

$$\begin{aligned}
&\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2}e^{(1-\frac{\theta^2}{2})r_n}}\\
&= -\sqrt{\frac{2}{\pi}}\sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{r_n^\ell}\sum_{k=0}^\ell \frac{M_\infty^{(2k+1,\theta)}}{(2k+1)!(2\ell-2k+1)!}\int_A z^{2\ell-2k+1}e^{-\theta z}\mathrm{d}z + o(r_n^{-m});
\end{aligned}$$

(ii) for any bounded Borel set $A\subset(0,\infty)$, $\mathbb{P}_x$-almost surely, as $n\to\infty$,

$$\begin{aligned}
&\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2}e^{r_n}}\\
&= -\sqrt{\frac{2}{\pi}}\sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{r_n^\ell}\sum_{k=0}^\ell \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k+1)!}\int_A z^{2\ell-2k+1}\mathrm{d}z + o(r_n^{-m});
\end{aligned}$$

(iii) for any $a\ge 0$, $\mathbb{P}_x$-almost surely, as $n\to\infty$,

$$\frac{Z_{r_n}^{(0,\infty)}((a,\infty))}{r_n^{-1/2}e^{r_n}} = \sqrt{\frac{2}{\pi}}\sum_{\ell=0}^m \frac{H_{2\ell}(0)}{r_n^\ell}\sum_{k=0}^\ell \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k)!}a^{2\ell-2k} + o(r_n^{-m}).$$

Proof: Suppose $x>0, m\in\mathbb{N}, \kappa>2m+1$ and (1.3) holds with $\lambda>2m+2\kappa+2>2m+\kappa+1$.

First using Lemma 3.2 and then the Markov property, we get that for any Borel set $A\subset(0,\infty)$, $\mathbb{P}_x$-almost surely,

$$\begin{aligned}
\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}} &= o(r_n^{-m}) + \frac{\mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A)\middle|\mathcal{F}_{\sqrt{r_n}}\right)}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}} \\
&= o(r_n^{-m}) + \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_{X_v(\sqrt{r_n})}\left(Z_{r_n-\sqrt{r_n}}^{(0,\infty)}(A)\right) \\
&= o(r_n^{-m}) + \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}} \\
&\qquad \times e^{r_n-\sqrt{r_n}}\Pi_{X_v(\sqrt{r_n})}^{-\theta}\left(1_{\{\min_{s\le r_n-\sqrt{r_n}} B_s>0\}}1_A(B_{r_n-\sqrt{r_n}})\right),
\end{aligned}$$

where in the last equality we used Lemma 2.1. Let $K:=2m/\kappa+3$ and fix a sufficient small $\varepsilon>0$ such that

$$K(1-\varepsilon)>\frac{2m+1}{\kappa}+2. \tag{3.28}$$

Step 1: In this step, we prove that $\mathbb{P}_x$-almost surely,

$$\begin{aligned}
\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}} &= o(r_n^{-m}) + \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}} 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}} \\
&\quad \times e^{r_n-\sqrt{r_n}}\Pi_{X_v(\sqrt{r_n})}^{-\theta}\left(1_{\{\min_{s\le r_n-\sqrt{r_n}} B_s>0\}}1_A(B_{r_n-\sqrt{r_n}})\right).
\end{aligned} \tag{3.29}$$

Using Lemma 2.4 (i) (ii) first, and then Lemma 2.1, we get

$$\begin{aligned}
&\sum_{n=2}^\infty r_n^m \mathbb{E}_x\Bigg(\sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}} 1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}} \\
&\qquad \times e^{r_n-\sqrt{r_n}}\Pi_{X_v(\sqrt{r_n})}^{-\theta}\left(1_{\{\min_{s\le r_n-\sqrt{r_n}} B_s>0\}}1_A(B_{r_n-\sqrt{r_n}})\right)\Bigg) \\
&\lesssim \sum_{n=2}^\infty r_n^m \mathbb{E}_x\Bigg(\sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}} 1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}} \\
&\qquad \times X_v(\sqrt{r_n})e^{\theta X_v(\sqrt{r_n})}\frac{e^{(1-\frac{\theta^2}{2})(r_n-\sqrt{r_n})}}{(r_n-\sqrt{r_n})^{b_\theta}}\Bigg) \\
&= \sum_{n=2}^\infty \frac{r_n^m r_n^{b_\theta}e^{\frac{\theta^2}{2}\sqrt{r_n}}}{\left(r_n-\sqrt{r_n}\right)^{b_\theta}}\mathbf{E}_x\left(1_{\{\min_{s\le\sqrt{r_n}} X_\xi(s)>0\}}1_{\{X_\xi(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}}X_\xi(\sqrt{r_n})e^{\theta X_\xi(\sqrt{r_n})}\right).
\end{aligned} \tag{3.30}$$

Recall that $(X_\xi(t),\mathbf{P}_x)$ is equal in law to a standard Brownian motion with drift $-\theta$. By Lemma 2.3 (i), recalling the choice of ε in (3.28), the left-hand side of (3.30) is bounded from above by

$$\sum_{n=2}^\infty \frac{r_n^m r_n^{b_\theta}}{\left(r_n-\sqrt{r_n}\right)^{b_\theta}} x e^{\theta x}\Pi_x^\uparrow\left(B_{\sqrt{r_n}}>\sqrt{K\sqrt{r_n}\log n}\right)$$

$$\leq xe^{\theta x}\sum_{n=2}^{\infty}\frac{r_n^m r_n^{b_\theta}}{\left(r_n-\sqrt{r_n}\right)^{b_\theta}}\frac{1}{n^{K(1-\varepsilon)/2}}\Pi_x^{\uparrow}\left(e^{(1-\varepsilon)\left(B_{\sqrt{r_n}}\right)^2/(2\sqrt{r_n})}\right), \tag{3.31}$$

where in the last inequality we used the Markov inequality. Using Lemma 2.3 (ii), we get

$$\Pi_x^{\uparrow}\left(e^{(1-\varepsilon)\left(B_{\sqrt{r_n}}\right)^2/(2\sqrt{r_n})}\right)=\int_0^{\infty}e^{(1-\varepsilon)y^2/(2\sqrt{r_n})}p^{\uparrow}_{\sqrt{r_n}}(x,y)\mathrm{d}y$$
$$\lesssim\int_0^{\infty}e^{(1-\varepsilon)y^2/(2\sqrt{r_n})}\frac{y^2}{r_n^{3/4}}e^{-(x-y)^2/(2\sqrt{r_n})}\mathrm{d}y=\int_0^{\infty}e^{(1-\varepsilon)y^2/2}y^2e^{-(xr_n^{-1/4}-y)^2/2}\mathrm{d}y\lesssim 1. \tag{3.32}$$

Combining (3.31), (3.32) and the fact that $r_n(r_n-\sqrt{r_n})^{-1}\lesssim 1$, we obtain

$$\sum_{n=2}^{\infty}r_n^m\mathbb{E}_x\Bigg(\sum_{v\in N(\sqrt{r_n})}\frac{1_{\{\min_{s\leq\sqrt{r_n}}X_v(s)>0\}}}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}}1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}}$$
$$\times e^{r_n-\sqrt{r_n}}\Pi^{-\theta}_{X_v(\sqrt{r_n})}\left(1_{\{\min_{s\leq r_n-\sqrt{r_n}}B_s>0\}}1_A(B_{r_n-\sqrt{r_n}})\right)\Bigg)$$
$$\lesssim\sum_{n=2}^{\infty}\frac{r_n^m}{n^{K(1-\varepsilon)/2}}=\sum_{n=2}^{\infty}\frac{n^{m/\kappa}}{n^{K(1-\varepsilon)/2}}<\infty,$$

which implies that $\mathbb{P}_x$-almost surely,

$$r_n^m\sum_{v\in N(\sqrt{r_n})}\frac{1_{\{\min_{s\leq\sqrt{r_n}}X_v(s)>0\}}}{r_n^{-b_\theta}e^{(1-\frac{\theta^2}{2})r_n}}1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}}$$
$$\times e^{r_n-\sqrt{r_n}}\Pi^{-\theta}_{X_v(\sqrt{r_n})}\left(1_{\{\min_{s\leq r_n-\sqrt{r_n}}B_s>0\}}1_A(B_{r_n-\sqrt{r_n}})\right)\stackrel{n\to\infty}{\longrightarrow}0,$$

and thus (3.29) holds.

Step 2: In this step, we prove (i) and (ii). Recall that when $\theta\in(0,\sqrt{2})$, or $\theta=0$ and $\sup\{y:y\in A\}<\infty$, $b_\theta=3/2$. In this case, let $J:=2m+2\kappa+2>2m+\frac{K\kappa+1}{2}$. By using Lemma 2.3 (i) in the first two equalities below, Lemma 2.8 in the third, we get

$$\frac{Z^{(0,\infty)}_{r_n}(A)}{r_n^{-3/2}e^{(1-\frac{\theta^2}{2})r_n}}$$
$$=o(r_n^{-m})+\sum_{v\in N(\sqrt{r_n})}\frac{1_{\{\min_{s\leq\sqrt{r_n}}X_v(s)>0\}}}{r_n^{-3/2}e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}}X_v(\sqrt{r_n})e^{\theta X_v(\sqrt{r_n})}1_{\{X_v(\sqrt{r_n})\leq\sqrt{K\sqrt{r_n}\log n}\}}$$
$$\times\Pi^{\uparrow}_{X_v(\sqrt{r_n})}\left(\frac{1_A(B_{r_n-\sqrt{r_n}})}{B_{r_n-\sqrt{r_n}}e^{\theta B_{r_n-\sqrt{r_n}}}}\right)$$
$$=o(r_n^{-m})+\sum_{v\in N(\sqrt{r_n})}\frac{1_{\{\min_{s\leq\sqrt{r_n}}X_v(s)>0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}}e^{\theta X_v(\sqrt{r_n})}1_{\{X_v(\sqrt{r_n})\leq\sqrt{K\sqrt{r_n}\log n}\}}$$
$$\times\int_A\frac{r_n^{3/2}}{e^{\theta z}\sqrt{r_n-\sqrt{r_n}}}\left(\phi\left(\frac{z-X_v(\sqrt{r_n})}{\sqrt{r_n-\sqrt{r_n}}}\right)-\phi\left(\frac{z+X_v(\sqrt{r_n})}{\sqrt{r_n-\sqrt{r_n}}}\right)\right)\mathrm{d}z$$
$$=o(r_n^{-m})+\sum_{v\in N(\sqrt{r_n})}\frac{1_{\{\min_{s\leq\sqrt{r_n}}X_v(s)>0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}}e^{\theta X_v(\sqrt{r_n})}1_{\{X_v(\sqrt{r_n})\leq\sqrt{K\sqrt{r_n}\log n}\}}\int_A r_ne^{-\theta z}$$

$$\times\left(2\phi\left(\frac{z}{\sqrt{r_n}}\right)\sum_{k=0}^{J}\frac{1}{(2k+1)!}\frac{1}{r_n^{(2k+1)/2}}H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right)r_n^{(2k+1)/4}H_{2k+1}\left(\frac{X_v(\sqrt{r_n})}{r_n^{1/4}}\right)+\varepsilon_{m,v,z,n}\right)\mathrm{d}z,$$

where the error term $\varepsilon_{m,v,z,n}$ satisfies that

$$r_n^{m+1}\sup_{z>0}\sup_{v\in N(\sqrt{r_n})}|\varepsilon_{m,v,z,n}|\,1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}}\overset{n\to\infty}{\longrightarrow}0,\quad\mathbb{P}_x\text{-a.s.}$$

Noticing that $e^{-(1-\frac{\theta^2}{2})t}\sum_{v\in N(t)}e^{\theta X_v(t)}$ is a non-negative martingale, and that $\int_A e^{-\theta z}\mathrm{d}z<\infty$ when $\theta>0$, or $\theta=0$ and $\sup\{y:y\in A\}<\infty$, we get

$$\begin{aligned}
&r_n^m\sum_{v\in N(\sqrt{r_n})}\frac{1_{\{\min_{s\le\sqrt{r_n}}X_v(s)>0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}}e^{\theta X_v(\sqrt{r_n})}1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}}\int_A r_n e^{-\theta z}\,|\varepsilon_{m,v,z,n}|\,\mathrm{d}z\\
&\le\left(e^{-(1-\frac{\theta^2}{2})\sqrt{r_n}}\sum_{v\in N(\sqrt{r_n})}e^{\theta X_v(\sqrt{r_n})}\right)\left(\int_A e^{-\theta z}\mathrm{d}z\right)\\
&\qquad\times r_n^{m+1}\sup_{z>0}\sup_{v\in N(\sqrt{r_n})}|\varepsilon_{m,v,z,n}|\,1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}}\overset{n\to\infty}{\longrightarrow}0,\quad\text{a.s.}
\end{aligned}$$

Therefore, $\mathbb{P}_x$-almost surely,

$$\begin{aligned}
&\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2}e^{(1-\frac{\theta^2}{2})r_n}}\\
&=o(r_n^{-m})+\sum_{v\in N(\sqrt{r_n})}\frac{1_{\{\min_{s\le\sqrt{r_n}}X_v(s)>0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}}e^{\theta X_v(\sqrt{r_n})}1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}}\int_A r_n e^{-\theta z}\\
&\times 2\phi\left(\frac{z}{\sqrt{r_n}}\right)\sum_{k=0}^{J}\frac{1}{(2k+1)!}\frac{1}{r_n^{(2k+1)/2}}H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right)r_n^{(2k+1)/4}H_{2k+1}\left(\frac{X_v(\sqrt{r_n})}{r_n^{1/4}}\right)\mathrm{d}z\\
&=o(r_n^{-m})+2\sum_{k=0}^{J}\sum_{v\in N(\sqrt{r_n})}\frac{1_{\{\min_{s\le\sqrt{r_n}}X_v(s)>0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}}e^{\theta X_v(\sqrt{r_n})}r_n^{(2k+1)/4}H_{2k+1}\left(\frac{X_v(\sqrt{r_n})}{r_n^{1/4}}\right)\\
&\times 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}}\int_A e^{-\theta z}\phi\left(\frac{z}{\sqrt{r_n}}\right)\frac{1}{(2k+1)!}\frac{1}{r_n^{(2k-1)/2}}H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right)\mathrm{d}z.
\end{aligned}\tag{3.33}$$

Now we show that we can drop the indicator function from the last line above. Note that for all $0\le k\le J$, applying Lemma 2.5(ii) and the inequality

$$\int_A e^{-\theta z}\phi\left(\frac{z}{\sqrt{r_n}}\right)\left|H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right)\right|\mathrm{d}z\lesssim\int_A e^{-\theta z}\mathrm{d}z<\infty$$

first and then Lemma 2.1, we get that

$$\begin{aligned}
&\sum_{n=2}^{\infty}r_n^m\mathbb{E}_x\Bigg(\sum_{v\in N(\sqrt{r_n})}\frac{1_{\{\min_{s\le\sqrt{r_n}}X_v(s)>0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}}e^{\theta X_v(\sqrt{r_n})}\left|r_n^{(2k+1)/4}H_{2k+1}\left(\frac{X_v(\sqrt{r_n})}{r_n^{1/4}}\right)\right|\\
&\qquad\times 1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}}\int_A e^{-\theta z}\phi\left(\frac{z}{\sqrt{r_n}}\right)\frac{1}{(2k+1)!}\frac{1}{r_n^{(2k-1)/2}}\left|H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right)\right|)\mathrm{d}z\Bigg)
\end{aligned}$$

$$\lesssim \sum_{n=2}^{\infty} r_n^m \mathbb{E}_x \Bigg(\sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} r_n^{(2k+1)/4} \left(\left|\frac{X_v(\sqrt{r_n})}{r_n^{1/4}}\right|^{2k+1} + \left|\frac{X_v(\sqrt{r_n})}{r_n^{1/4}}\right| \right)$$
$$\times \frac{1}{r_n^{(2k-1)/2}} 1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}} \Bigg)$$
$$= \sum_{n=2}^{\infty} \frac{r_n^m e^{\frac{\theta^2}{2}\sqrt{r_n}}}{r_n^{(2k-1)/2}} \mathbf{E}_x \Bigg(1_{\{\min_{s\le\sqrt{r_n}} X_\xi(s)>0\}} X_\xi(\sqrt{r_n}) e^{\theta X_\xi(\sqrt{r_n})}$$
$$\times \left(X_\xi(\sqrt{r_n})^{2k} + r_n^{k/2} \right) 1_{\{X_\xi(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}} \Bigg). \tag{3.34}$$

Recalling the choice of ε in (3.28). By Lemma 2.3, the right-hand side of (3.34) is equal to

$$\sum_{n=2}^{\infty} \frac{r_n^m}{r_n^{(2k-1)/2}} x e^{\theta x} \Pi_x^{\uparrow} \left(\left(B_{\sqrt{r_n}}^{2k} + r_n^{k/2} \right) 1_{\{B_{\sqrt{r_n}}>\sqrt{K\sqrt{r_n}\log n}\}} \right)$$
$$\le \sum_{n=2}^{\infty} \frac{r_n^m}{r_n^{(2k-1)/2} n^{(1-\varepsilon)K/2}} x e^{\theta x} \Pi_x^{\uparrow} \left(\left(B_{\sqrt{r_n}}^{2k} + r_n^{k/2} \right) e^{(1-\varepsilon)(B_{\sqrt{r_n}})^2/(2\sqrt{r_n})} \right)$$
$$\lesssim \sum_{n=2}^{\infty} \frac{r_n^m}{r_n^{(2k-1)/2} n^{(1-\varepsilon)K/2}} \int_0^\infty \frac{y^2}{r_n^{3/4}} e^{-(x-y)^2/(2\sqrt{r_n})} \left(y^{2k} + r_n^{k/2} \right) e^{(1-\varepsilon)y^2/(2\sqrt{r_n})} \mathrm{d}y$$
$$= \sum_{n=2}^{\infty} \frac{r_n^m r_n^{k/2}}{r_n^{(2k-1)/2} n^{(1-\varepsilon)K/2}} \int_0^\infty y^2 e^{-(x r_n^{-1/4}-y)^2/2} \left(y^{2k} + 1 \right) e^{(1-\varepsilon)y^2/2} \mathrm{d}y$$
$$\le \sum_{n=2}^{\infty} \frac{n^{(2m+1)/(2\kappa)}}{n^{(1-\varepsilon)K/2}} \int_0^\infty y^2 e^{-(x r_n^{-1/4}-y)^2/2} \left(y^{2k} + 1 \right) e^{(1-\varepsilon)y^2/2} \mathrm{d}y, \tag{3.35}$$

which is summable. Hence, combining (3.34) and (3.35), we conclude that for all $0 \le k \le J$, almost surely,

$$r_n^m \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} \left| r_n^{(2k+1)/4} H_{2k+1}\left(\frac{X_v(\sqrt{r_n})}{r_n^{1/4}} \right) \right| 1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}}$$
$$\times \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}} \right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} \left| H_{2k+1}\left(\frac{z}{\sqrt{r_n}} \right) \right| \mathrm{d}z \overset{n\to\infty}{\longrightarrow} 0.$$

Plugging this back to (3.33) and recalling the definition of the martingales in (1.4), we obtain that

$$\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} = o(r_n^{-m})$$
$$+ 2\sum_{k=0}^{J} M_{\sqrt{r_n}}^{(2k+1,\theta)} \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}} \right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} H_{2k+1}\left(\frac{z}{\sqrt{r_n}} \right) \mathrm{d}z. \tag{3.36}$$

For all $m+1 \le k \le J = 2m+2\kappa+2$, we have $\lambda > J \ge k$. Therefore, for all $m+1 \le k \le J$, by Lemma 2.5 (ii), Proposition 3.1(ii) and the fact that $\phi(y)|H_{2k+1}(y)| \lesssim \phi(y)|y|(|y|^{2k}+1) \lesssim |y|$,

$$\left| M_{\sqrt{r_n}}^{(2k+1,\theta)} \right| \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}} \right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} \left| H_{2k+1}\left(\frac{z}{\sqrt{r_n}} \right) \right| \mathrm{d}z$$

$$\lesssim \left|M^{(2k+1,\theta)}_{\sqrt{r_n}}\right| \int_A e^{-\theta z} \frac{1}{r_n^{(2k-1)/2}} \frac{z}{\sqrt{r_n}} \mathrm{d}z = \frac{\left|M^{(2k+1,\theta)}_{\sqrt{r_n}}\right|}{r_n^k} \int_A z e^{-\theta z} \mathrm{d}z = o(r_n^{-m}). \tag{3.37}$$

Combining (3.36) and (3.37), we get that

$$\begin{aligned} \frac{Z^{(0,\infty)}_{r_n}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} &= o(r_n^{-m}) \\ &+ 2\sum_{k=0}^m M^{(2k+1,\theta)}_{\sqrt{r_n}} \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}}\right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) \mathrm{d}z. \end{aligned}$$

Noticing that $\lambda > 2m$, let $\eta := (\lambda - 2m)/2 < \lambda - 2m$. By Proposition 3.1(ii), similar to (3.37), for all $0 \le k \le m$,

$$\begin{aligned} &\left|M^{(2k+1,\theta)}_{\sqrt{r_n}} - M^{(2k+1,\theta)}_{\infty}\right| \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}}\right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} \left|H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right)\right| \mathrm{d}z \\ &\lesssim \frac{\left|M^{(2k+1,\theta)}_{\sqrt{r_n}} - M^{(2k+1,\theta)}_{\infty}\right|}{r_n^k} \int_A z e^{-\theta z} \mathrm{d}z = o(\sqrt{r_n}^{-(\lambda-k)+\eta} r_n^{-k}) = o(r_n^{-m} r_n^{-\frac{k}{2}}) = o(r_n^{-m}). \end{aligned} \tag{3.38}$$

Since $\lambda > 2m + 2\kappa + 2$, we have

$$\begin{aligned} \frac{Z^{(0,\infty)}_{r_n}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} &= o(r_n^{-m}) \\ &+ 2\sum_{k=0}^m M^{(2k+1,\theta)}_{\infty} \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}}\right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) \mathrm{d}z. \end{aligned}$$

By (2.5) and Lemma 2.5(i), for any $0 \le k \le m$ and $x > 0$, there exists $\xi \in (0, x)$ such that

$$\begin{aligned} \phi(x) H_{2k+1}(x) &= \sum_{j=0}^{2m+1} \frac{\mathrm{d}^j}{\mathrm{d}x^j} (\phi H_{2k+1})(0) \frac{x^j}{j!} + \frac{x^{2m+2}}{(2m+2)!} \frac{\mathrm{d}^{2m+2}}{\mathrm{d}x^{2m+2}} (\phi H_{2k+1})(\xi) \\ &= (-1)^{2k+1} \sum_{j=0}^m \frac{\mathrm{d}^{2k+1+j}\phi(0)}{\mathrm{d}x^{2k+1+j}} \frac{x^{2j+1}}{(2j+1)!} + O(x^{2m+2}) = -\sum_{j=0}^m \frac{H_{2k+2j+2}(0)}{\sqrt{2\pi}} \frac{x^{2j+1}}{(2j+1)!} + O(x^{2m+2}), \end{aligned}$$

where in the second equality we use the property that $H_{2\ell+1}(0) = 0$. Therefore,

$$\begin{aligned} \frac{Z^{(0,\infty)}_{r_n}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} &= o(r_n^{-m}) - \sum_{k=0}^m \frac{2}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} M^{(2k+1,\theta)}_{\infty} \\ &\quad \times \int_A e^{-\theta z} \left(\sum_{j=0}^m \frac{H_{2k+2j+2}(0)}{(2j+1)!\sqrt{2\pi}} \frac{z^{2j+1}}{r_n^{(2j+1)/2}} + O\left(\frac{z^{2m+2}}{r_n^{m+1}}\right) \right) \mathrm{d}z \\ &= o(r_n^{-m}) - \sqrt{\frac{2}{\pi}} \sum_{k=0}^m \sum_{j=0}^m \frac{1}{(2k+1)!(2j+1)!} \frac{H_{2k+2j+2}(0)}{r_n^{k+j}} M^{(2k+1,\theta)}_{\infty} \int_A z^{2j+1} e^{-\theta z} \mathrm{d}z \\ &= o(r_n^{-m}) - \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{r_n^\ell} \sum_{k=0}^\ell \frac{M^{(2k+1,\theta)}_{\infty}}{(2k+1)!(2\ell-2k+1)!} \int_A z^{2\ell-2k+1} e^{-\theta z} \mathrm{d}z, \end{aligned}$$

which completes the proof of (i) and (ii).

Step 3: In this step, we prove (iii). Recall that when $\theta = 0$ and $A = (a,\infty)$, $b_\theta = 1/2$. By (3.29) and Lemma 2.3 (i),

$$\begin{aligned}
\frac{Z_{r_n}^{(0,\infty)}((a,\infty))}{r_n^{-1/2}e^{r_n}} &= o(r_n^{-m}) + \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-1/2}e^{r_n}} 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}} \\
&\quad \times e^{r_n-\sqrt{r_n}}\Pi_{X_v(\sqrt{r_n})}\left(1_{\{\min_{s\le r_n-\sqrt{r_n}} B_s>0\}}1_{(a,\infty)}(B_{r_n-\sqrt{r_n}})\right) \\
&= o(r_n^{-m}) + \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-1/2}e^{r_n}} 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}} \\
&\quad \times \frac{e^{r_n-\sqrt{r_n}}}{\sqrt{r_n-\sqrt{r_n}}}\int_a^\infty \left(\phi\left(\frac{y-X_v(\sqrt{r_n})}{\sqrt{r_n-\sqrt{r_n}}}\right) - \phi\left(\frac{y+X_v(\sqrt{r_n})}{\sqrt{r_n-\sqrt{r_n}}}\right)\right)\mathrm{d}y \\
&= o(r_n^{-m}) + \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-1/2}e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}} \\
&\quad \times \left(\Phi\left(\frac{a+X_v(\sqrt{r_n})}{\sqrt{r_n-\sqrt{r_n}}}\right) - \Phi\left(\frac{a-X_v(\sqrt{r_n})}{\sqrt{r_n-\sqrt{r_n}}}\right)\right).
\end{aligned}$$

Put $J := 2m+2\kappa+1 > 2m + \frac{K\kappa-1}{2}$. By Lemma 2.7, it holds that

$$\begin{aligned}
\frac{Z_{r_n}^{(0,\infty)}((a,\infty))}{r_n^{-1/2}e^{r_n}} &= o(r_n^{-m}) + \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-1/2}e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}} \\
&\times \left(2\phi\left(\frac{a}{\sqrt{r_n}}\right)\sum_{k=0}^{J}\frac{1}{(2k+1)!}\frac{1}{r_n^{(2k+1)/2}}H_{2k}\left(\frac{a}{\sqrt{r_n}}\right)r_n^{(2k+1)/4}H_{2k+1}\left(\frac{X_u(\sqrt{r_n})}{r_n^{1/4}}\right) + \varepsilon_{m,v,a,n}\right).
\end{aligned}$$

where the error term $\varepsilon_{m,v,a,n}$ satisfies that

$$r_n^{(2m+1)/2}\sup_{a>0}\sup_{v\in N(\sqrt{r_n})}|\varepsilon_{m,v,a,n}|\, 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}} \overset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{P}_x\text{-a.s.}$$

Using the fact that $e^{-t}\sum_{v\in N(t)} 1$ is a non-negative martingale, we have

$$\begin{aligned}
&r_n^m \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{r_n^{-1/2}e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}}\, |\varepsilon_{m,v,a,n}| \\
&\le \left(e^{-\sqrt{r_n}}\sum_{v\in N(\sqrt{r_n})} 1\right) r_n^{(2m+1)/2}\sup_{a>0}\sup_{v\in N(\sqrt{r_n})}|\varepsilon_{m,v,a,n}|\, 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}} \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{Z_{r_n}^{(0,\infty)}((a,\infty))}{r_n^{-1/2}e^{r_n}} &= o(r_n^{-m}) + 2\sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n})\le\sqrt{K\sqrt{r_n}\log n}\}} \\
&\times \phi\left(\frac{a}{\sqrt{r_n}}\right)\sum_{k=0}^{J}\frac{1}{(2k+1)!}\frac{1}{r_n^k}H_{2k}\left(\frac{a}{\sqrt{r_n}}\right)r_n^{(2k+1)/4}H_{2k+1}\left(\frac{X_u(\sqrt{r_n})}{r_n^{1/4}}\right). \qquad (3.39)
\end{aligned}$$

Similar to the argument leading to (3.34), for each $0 \le k \le J$, applying Lemma 2.5 first, then Lemma 2.1 and Lemma 2.3(i) at last, we also have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} r_n^m \mathbb{E}_x\Bigg(\sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}} \\
&\quad \times \phi\left(\frac{a}{\sqrt{r_n}}\right) \frac{1}{r_n^k} \left| H_{2k}\left(\frac{a}{\sqrt{r_n}}\right)\right| \times \left| r_n^{(2k+1)/4} H_{2k+1}\left(\frac{X_u(\sqrt{r_n})}{r_n^{1/4}}\right)\right| \Bigg) \\
&\lesssim \sum_{n=1}^{\infty} r_n^m \mathbb{E}_x\Bigg(\sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}} \\
&\quad \times \frac{1}{r_n^k} X_u(\sqrt{r_n}) \left((X_u(\sqrt{r_n}))^{2k} + r_n^{k/2}\right) \Bigg) \\
&= \sum_{n=1}^{\infty} r_n^{m-k} \mathbf{E}_x\left(1_{\{\min_{s\le\sqrt{r_n}} X_\xi(s)>0\}} 1_{\{X_\xi(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}} X_\xi(\sqrt{r_n}) \left((X_\xi(\sqrt{r_n}))^{2k} + r_n^{k/2}\right) \right) \\
&= x \sum_{n=1}^{\infty} r_n^{m-k} \Pi_x^{\uparrow}\left(1_{\{B_{\sqrt{r_n}}>\sqrt{K\sqrt{r_n}\log n}\}} \left(\left(B_{\sqrt{r_n}}\right)^{2k} + r_n^{k/2}\right) \right) < \infty,
\end{aligned}$$

where the last inequality follows from (3.35). Therefore, for $0 \le k \le J$, $\mathbb{P}_x$-a.s.,

$$\begin{aligned}
& r_n^m \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n})>\sqrt{K\sqrt{r_n}\log n}\}} \\
&\quad \times \phi\left(\frac{a}{\sqrt{r_n}}\right) \frac{1}{r_n^k} \left| H_{2k}\left(\frac{a}{\sqrt{r_n}}\right)\right| \times \left| r_n^{(2k+1)/4} H_{2k+1}\left(\frac{X_u(\sqrt{r_n})}{r_n^{1/4}}\right)\right| \stackrel{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Plugging this back to (3.39), we finally get that

$$\begin{aligned}
\frac{Z_{r_n}^{(0,\infty)}((a,\infty))}{r_n^{-1/2} e^{r_n}} &= o(r_n^{-m}) + 2 \sum_{v\in N(\sqrt{r_n})} \frac{1_{\{\min_{s\le\sqrt{r_n}} X_v(s)>0\}}}{e^{\sqrt{r_n}}} \\
&\quad \times \phi\left(\frac{a}{\sqrt{r_n}}\right) \sum_{k=0}^{J} \frac{1}{(2k+1)!} \frac{1}{r_n^k} H_{2k}\left(\frac{a}{\sqrt{r_n}}\right) r_n^{(2k+1)/4} H_{2k+1}\left(\frac{X_u(\sqrt{r_n})}{r_n^{1/4}}\right) \\
&= o(r_n^{-m}) + 2\phi\left(\frac{a}{\sqrt{r_n}}\right) \sum_{k=0}^{J} \frac{1}{(2k+1)!} \frac{1}{r_n^k} H_{2k}\left(\frac{a}{\sqrt{r_n}}\right) M_{\sqrt{r_n}}^{(2k+1,0)}, \qquad (3.40)
\end{aligned}$$

with $M_t^{(2k+1,0)}$ given in (1.4). Using the same argument as (3.37) and (3.38), also noting that $\lambda > J = 2m + 2\kappa + 1$, Proposition 3.1 (ii) and (3.40) imply that

$$\frac{Z_{r_n}^{(0,\infty)}((a,\infty))}{r_n^{-1/2} e^{r_n}} = o(r_n^{-m}) + 2\phi\left(\frac{a}{\sqrt{r_n}}\right) \sum_{k=0}^{m} \frac{1}{(2k+1)!} \frac{1}{r_n^k} H_{2k}\left(\frac{a}{\sqrt{r_n}}\right) M_{\infty}^{(2k+1,0)}.$$

The remaining part is similar to the end of Step 2 we omit the details here. The proof of (iii) is complete. □

3.3 From discrete time to continuous time

Lemma 3.4 *Let $x > 0$ and $\theta \in [0, \sqrt{2})$. For any given $m \in \mathbb{N}$, if $\kappa > 2m+2$ and (1.3) holds for $\lambda > 2m + 2\kappa + 2$, then for any interval $J \subset (0, \infty)$, it holds that*

$$\liminf_{n\to\infty} r_n^m \inf_{t\in(r_n,r_{n+1})} \frac{Z_t^{(0,\infty)}(J) - Z_{r_n}^{(0,\infty)}(J)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \geq 0, \quad \mathbb{P}_x\text{-a.s.}$$

Proof: Suppose $x > 0$, $m \in \mathbb{N}$, $\kappa > 2m+2$ and that (1.3) holds for $\lambda > 2m + 2\kappa + 2$.

Step 1: Define $\varepsilon_n := \sqrt{r_{n+1} - r_n}$, by the mean value theorem,

$$\sqrt{r_n}\varepsilon_n = \sqrt{n^{1/\kappa}\left((n+1)^{1/\kappa} - n^{1/\kappa}\right)} \overset{\exists \xi\in[n,n+1]}{=} \sqrt{\frac{n^{1/\kappa}}{\kappa}\xi^{(-\kappa+1)/\kappa}} \lesssim n^{(-\kappa+2)/(2\kappa)} \to 0.$$

For any $\eta < |J|/2$, define

$$J_\eta := \{y \in J : \mathrm{dist}(y, J^c) \geq \eta\}.$$

For $u \in N(r_n)$, let G_u be the event that u does not split before r_{n+1} and that $\max_{s\in(r_n,r_{n+1})} |X_u(s) - X_u(r_n)| \leq \sqrt{r_n}\varepsilon_n$. When n is large enough so that $\eta_n := \sqrt{r_n}\varepsilon_n < |J|/2$, for $u \in N(r_n)$, on the event G_u, for $t \in (r_n, r_{n+1})$, it must hold that

$$\{X_u(r_n) \in J_{\eta_n}\} \subset \{X_u(s) \in J,\ \forall s \in (r_n, r_{n+1})\} \subset \{X_u(t) \in J\} \cap \{X_u(s) > 0, \forall s \in (r_n, t]\}.$$

Therefore, for $t \in (r_n, r_{n+1})$, by the branching property,

$$\begin{aligned}
& r_n^m \frac{Z_t^{(0,\infty)}(J)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\
&= \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u\in N(r_n)} 1_{\{\min_{s\leq r_n} X_u(s)>0\}} \sum_{v\in N(t):u\leq v} 1_{\{\min_{s<t-r_n} X_v(r_n+s)>0\}} 1_{\{X_v(t)\in J\}} \\
&\geq \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u\in N(r_n)} 1_{\{\min_{s\leq r_n} X_u(s)>0\}} 1_{\{X_u(r_n)\in J_{\eta_n}\}} 1_{G_u} =: I_n + II_n. \qquad (3.41)
\end{aligned}$$

Here I_n and II_n are given by

$$\begin{aligned}
I_n &:= \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u\in N(r_n)} 1_{\{\min_{s\leq r_n} X_u(s)>0\}} 1_{\{X_u(r_n)\in J_{\eta_n}\}} \left(1_{G_u} - \mathbb{P}_x\left(G_u \big| \mathcal{F}_{r_n}\right)\right), \\
II_n &:= \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u\in N(r_n)} 1_{\{\min_{s\leq r_n} X_u(s)>0\}} 1_{\{X_u(r_n)\in J_{\eta_n}\}} \mathbb{P}_x\left(G_u \big| \mathcal{F}_{r_n}\right).
\end{aligned}$$

We claim the following two limits hold:

$$\sup_{t\in(r_n,r_{n+1})} |I_n| = |I_n| \overset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{P}_x\text{-a.s.} \qquad (3.42)$$

and

$$\sup_{t\in(r_n,r_{n+1})} \left| II_n - r_n^{m+\frac{3}{2}} \frac{Z_{r_n}^{(0,\infty)}(J)}{e^{(1-\frac{\theta^2}{2})r_n}} \right| = \left| II_n - r_n^{m+\frac{3}{2}} \frac{Z_{r_n}^{(0,\infty)}(J)}{e^{(1-\frac{\theta^2}{2})r_n}} \right| \overset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{P}_x\text{-a.s.} \qquad (3.43)$$

If (3.42) and (3.43) hold, then we complete the proof of Lemma together with (3.41).

Step 2: In this step, we prove (3.43). Define

$$III_n := \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u\in N(r_n)} 1_{\{\min_{s\le r_n} X_u(s)>0\}} 1_{\{X_u(r_n)\in J\}} \mathbb{P}_x\left(G_u \middle| \mathcal{F}_{r_n}\right).$$

By Lemma 3.2 with $A_n := J \setminus J_{\eta_n}$, as $n\to\infty$,

$$|II_n - III_n| \le r_n^{m+\frac{3}{2}} \frac{Z_{r_n}^{(0,\infty)}(A_n)}{e^{(1-\frac{\theta^2}{2})r_n}} = o(1) + r_n^{m+\frac{3}{2}} \frac{\mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n)\middle|\mathcal{F}_{\sqrt{r_n}}\right)}{e^{(1-\frac{\theta^2}{2})r_n}}, \quad \mathbb{P}_x\text{-a.s.} \tag{3.44}$$

By the Markov property and Lemma 2.1, we get that

$$\begin{aligned} r_n^{m+\frac{3}{2}} \frac{\mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n)\middle|\mathcal{F}_{\sqrt{r_n}}\right)}{e^{(1-\frac{\theta^2}{2})r_n}} &= \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u\in N(\sqrt{r_n})} 1_{\{\min_{s\le\sqrt{r_n}} X_u(s)>0\}} \\ &\times e^{r_n-\sqrt{r_n}} \mathbf{E}_{X_u(\sqrt{r_n})}\left(1_{\{\min_{s\le r_n-\sqrt{r_n}} X_\xi(s)>0\}} 1_{A_n}(X_\xi(r_n-\sqrt{r_n}))\right). \end{aligned} \tag{3.45}$$

Since $(X_\xi(t), \mathbf{P}_x)$ is a standard Brownian motion with drift $-\theta$, by Lemma 2.3(i) and (ii),

$$\begin{aligned} &\mathbf{E}_{X_u(\sqrt{r_n})}\left(1_{\{\min_{s\le r_n-\sqrt{r_n}} X_\xi(s)>0\}} 1_{A_n}(X_\xi(r_n-\sqrt{r_n}))\right) \\ &= X_u(\sqrt{r_n}) e^{\theta X_u(\sqrt{r_n})} e^{-\frac{\theta^2}{2}(r_n-\sqrt{r_n})} \Pi^{\uparrow}_{X_u(\sqrt{r_n})}\left(\frac{e^{-\theta B_{r_b-\sqrt{r_n}}}}{B_{r_n-\sqrt{r_n}}} 1_{A_n}(B_{r_n-\sqrt{r_n}})\right) \\ &= X_u(\sqrt{r_n}) e^{\theta X_u(\sqrt{r_n})} e^{-\frac{\theta^2}{2}(r_n-\sqrt{r_n})} \int_{A_n} \frac{e^{-\theta y}}{y} p^{\uparrow}_{r_n-\sqrt{r_n}}(X_u(\sqrt{r_n}), y)\mathrm{d}y \\ &\lesssim X_u(\sqrt{r_n}) e^{\theta X_u(\sqrt{r_n})} e^{-\frac{\theta^2}{2}(r_n-\sqrt{r_n})} \int_{A_n} \frac{e^{-\theta y}}{y} \frac{y^2}{(r_n-\sqrt{r_n})^{3/2}}\mathrm{d}y. \end{aligned} \tag{3.46}$$

Combining (3.45), (3.46) and the definition of $M_t^{(1,\theta)}$ in (1.4), we see that

$$\begin{aligned} r_n^{m+\frac{3}{2}} \frac{\mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}(A_n)\middle|\mathcal{F}_{\sqrt{r_n}}\right)}{e^{(1-\frac{\theta^2}{2})r_n}} &\lesssim \frac{r_n^{m+\frac{3}{2}} M_{\sqrt{r_n}}^{(1,\theta)}}{(r_n-\sqrt{r_n})^{3/2}} \int_{A_n} y e^{-\theta y}\mathrm{d}y \\ &\lesssim r_n^m |A_n| M_{\sqrt{r_n}}^{(1,\theta)} \lesssim r_n^m \sqrt{r_n}\varepsilon_n = \sqrt{n^{(2m+1)/\kappa}\left((n+1)^{1/\kappa} - n^{1/\kappa}\right)}, \end{aligned} \tag{3.47}$$

where in the second inequality, we also used the fact that $\sup_n \sup\{y : y\in A_n\} < \infty$ and that $\int_{A_n} y e^{-\theta y}\mathrm{d}y \le \sup_n \sup\{y : y\in A_n\} \times |A_n|$. Since $\kappa > 2m+2$, the last term of (3.47) tends to 0. Combining (3.44), (3.45) and (3.47), it holds that

$$|II_n - III_n| \overset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{P}_x\text{-a.s.} \tag{3.48}$$

By the Markov property and the definition of G_u,

$$\mathbb{P}_x\left(G_u\middle|\mathcal{F}_{r_n}\right) = e^{-(r_{n+1}-r_n)} \Pi_0^{-\theta}\left(\max_{t<r_{n+1}-r_n} |B_t| \le \sqrt{r_n}\varepsilon_n\right).$$

Thus, we get that

$$\begin{aligned}
&\left|III_n - r_n^{m+\frac{3}{2}} \frac{Z_{r_n}^{(0,\infty)}(J)}{e^{(1-\frac{\theta^2}{2})r_n}}\right| \\
&\le \frac{Z_{r_n}^{(0,\infty)}((0,\infty))}{r_n^{-1/2} e^{(1-\frac{\theta^2}{2})r_n}} \cdot r_n^{m+1} \left(1 - e^{-(r_{n+1}-r_n)} \Pi_0^{-\theta}\left(\max_{t<r_{n+1}-r_n} |B_t| \le \sqrt{r_n}\varepsilon_n\right)\right) \\
&\lesssim r_n^{m+1} \left(1 - e^{-(r_{n+1}-r_n)} \Pi_0^{-\theta}\left(\max_{t<r_{n+1}-r_n} |B_t| \le \sqrt{r_n}\varepsilon_n\right)\right), \qquad (3.49)
\end{aligned}$$

where in the last inequality we used Proposition 3.3 (i) (iii) with $m = 0$. Note that under the assumption $\kappa > 2m+2$,

$$\begin{aligned}
&r_n^{m+1} \left(1 - e^{-(r_{n+1}-r_n)} \Pi_0^{-\theta}\left(\max_{t<r_{n+1}-r_n} |B_t| \le \sqrt{r_n}\varepsilon_n\right)\right) \\
&\le r_n^{m+1} O(r_{n+1}-r_n) + r_n^{m+1} \Pi_0^{-\theta}\left(\max_{t<r_{n+1}-r_n} |B_t| > \sqrt{r_n}\varepsilon_n\right) \\
&\le r_n^{m+1} O(n^{(-\kappa+1)/\kappa}) + r_n^{m+1} \Pi_0\left(\max_{t<r_{n+1}-r_n} |B_t| > \sqrt{r_n}\varepsilon_n - \theta(r_{n+1}-r_n)\right) \\
&= O(n^{-(\kappa-2-m)/\kappa}) + r_n^{m+1} \Pi_0\left(\max_{t<1} |B_t| > \sqrt{r_n} - \theta\sqrt{r_{n+1}-r_n}\right) = o(1). \qquad (3.50)
\end{aligned}$$

Hence, (3.43) follows according to (3.48), (3.49) and (3.50).

Step 3: In this step, we prove (3.42). Since, given $\mathcal{F}_{r_n}$, $\{G_u : u \in N(r_n)\}$ are independent, we have

$$\begin{aligned}
&\mathbb{E}_x\left(|I_n|^2 \,\middle|\, \mathcal{F}_{r_n}\right) = \frac{r_n^{2m+3}}{e^{2(1-\frac{\theta^2}{2})r_n}} \\
&\quad \times \sum_{u\in N(r_n)} 1_{\{\min_{s\le r_n} X_u(s)>0\}} 1_{\{X_u(r_n)\in J_{\eta n}\}} \mathbb{E}_x\left(\left(1_{G_u} - \mathbb{P}_x\left(G_u \middle| \mathcal{F}_n\right)\right)^2 \,\middle|\, \mathcal{F}_{r_n}\right) \\
&\le 4 \frac{r_n^{2m+3}}{e^{2(1-\frac{\theta^2}{2})r_n}} \sum_{u\in N(r_n)} 1_{\{\min_{s\le r_n} X_u(s)>0\}} 1_{\{X_u(r_n)\in(0,\infty)\}} \\
&= \frac{4 r_n^{2m+\frac{5}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \times \frac{Z_{r_n}^{(0,\infty)}((0,\infty))}{r_n^{-1/2} e^{(1-\frac{\theta^2}{2})r_n}}.
\end{aligned}$$

Now taking expectation with respect to $\mathbb{P}_x$, by Lemma 2.1 and Lemma 2.4(i)(ii), for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \mathbb{E}_x\left(|I_n| > \varepsilon\right) \le \frac{1}{\varepsilon^2} \sum_{n=2}^{\infty} \mathbb{E}_x\left(|I_n|^2\right) \lesssim \sum_{n=2}^{\infty} \frac{r_n^{2m+\frac{5}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \frac{\mathbb{E}_x\left(Z_{r_n}^{(0,\infty)}((0,\infty))\right)}{r_n^{-1/2} e^{(1-\frac{\theta^2}{2})r_n}} \\
&= \sum_{n=2}^{\infty} \frac{r_n^{2m+\frac{5}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \frac{e^{r_n} \mathbf{P}_x\left(\min_{s\le r_n} X_\xi(s) > 0\right)}{r_n^{-1/2} e^{(1-\frac{\theta^2}{2})r_n}} \lesssim \sum_{n=1}^{\infty} \frac{r_n^{2m+\frac{5}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} < \infty,
\end{aligned}$$

which implies (3.42). Thus, we complete the proof of the lemma. □

Lemma 3.5 *Let $x > 0$ and $\theta \in [0, \sqrt{2})$. For any $m \in \mathbb{N}$, suppose that $\kappa > 2m+2$ and (1.3) holds with $\lambda > 2m + 2\kappa + 2$, then for any interval $A \subset (0,\infty)$, $\mathbb{P}_x$-almost surely,*

$$r_n^m \sup_{t\in(r_n,r_{n+1})} \left| \frac{Z_t^{(0,\infty)}(A) - Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \right| \overset{n\to\infty}{\longrightarrow} 0.$$

Proof: Suppose $x > 0, m \in \mathbb{N}$, and that $\kappa > 2m+2$ and (1.3) holds with $\lambda > 2m+2\kappa+2$. Fix $\theta \in [0,\sqrt{2})$. Note that if

$$\liminf_{n\to\infty} x_n \geq 0, \quad \liminf_{n\to\infty} y_n \geq 0, \quad \liminf_{n\to\infty} z_n \geq 0 \text{ and } \limsup_{n\to\infty}(x_n+y_n+z_n) \leq 0,$$

then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$. Applying Lemma 3.4 with $J = (a,b)$, $[b,\infty)$ and $(0,a]$, we see that to prove Lemma 3.5, only need to prove that

$$\limsup_{n\to\infty} r_n^m \sup_{t\in(r_n,r_{n+1})} \frac{Z_t^{(0,\infty)}((0,\infty)) - Z_{r_n}^{(0,\infty)}((0,\infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \leq 0. \quad \text{a.s.}$$

For any $t \in (r_n, r_{n+1})$, by the branching property, we see that

$$\begin{aligned}
&r_n^m \sup_{t\in(r_n,r_{n+1})} \frac{Z_t^{(0,\infty)}((0,\infty)) - Z_{r_n}^{(0,\infty)}((0,\infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\
&\leq \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u\in N(r_n)} 1_{\{\min_{s\leq r_n} X_u(s)>0\}} \sup_{t\in(r_n,r_{n+1})} \Big(\sum_{v\in N(t):u\leq v} 1 - 1 \Big).
\end{aligned} \tag{3.51}$$

To drop the "sup" above, we modify the branching particle system for $t \in (r_n, r_{n+1})$ such that when a particle dies in (r_n, r_{n+1}) and it splits into L offspring, we modify the number of the offspring with $L+1$. For $t \in (r_n, r_{n+1})$, we use $\widetilde{N}(t)$ to denote the set of the particles alive at time t in the modified process. It is obvious that the mean of the number in the modified process is equal to $\sum_{k=0}^\infty (k+1)p_k = 3$ and that for each $u \in N(r_n)$,

$$\sup_{t\in(r_n,r_{n+1})} \Big(\sum_{v\in N(t):u\leq v} 1 - 1 \Big) \leq \sup_{t\in(r_n,r_{n+1})} \Big(\sum_{v\in \widetilde{N}(t):u\leq v} 1 - 1 \Big) = \Big(\sum_{v\in \widetilde{N}(r_{n+1}):u\leq v} 1 \Big) - 1.$$

Define

$$\widetilde{Z}_{n+1} := \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u\in N(r_n)} 1_{\{\min_{s\leq r_n} X_u(s)>0\}} \sum_{v\in \widetilde{N}(r_{n+1}):u\leq v} 1.$$

We claim that $\mathbb{P}_x$ almost surely,

$$\widetilde{Z}_{n+1} - \mathbb{E}_x \left(\widetilde{Z}_{n+1} \middle| \mathcal{F}_{r_n} \right) \to 0. \tag{3.52}$$

If the claim is true, then

$$\widetilde{Z}_{n+1} - \mathbb{E}_x \left(\widetilde{Z}_{n+1} \middle| \mathcal{F}_{r_n} \right) = \widetilde{Z}_{n+1} - e^{2(r_{n+1}-r_n)} r_n^m \frac{Z_{r_n}^{(0,\infty)}((0,\infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \to 0.$$

Using this and (3.51), we get

$$\begin{aligned}
&r_n^m \sup_{t\in(r_n,r_{n+1})} \frac{Z_t^{(0,\infty)}((0,\infty)) - Z_{r_n}^{(0,\infty)}((0,\infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \leq \widetilde{Z}_{n+1} - r_n^{m+\frac{3}{2}} \frac{Z_{r_n}^{(0,\infty)}((0,\infty))}{e^{(1-\frac{\theta^2}{2})r_n}} \\
&= \widetilde{Z}_{n+1} - \mathbb{E}_x \left(\widetilde{Z}_{n+1} \middle| \mathcal{F}_{r_n} \right) + r_n^{m+1} (e^{2(r_{n+1}-r_n)} - 1) \frac{Z_{r_n}^{(0,\infty)}((0,\infty))}{r_n^{-1/2} e^{(1-\frac{\theta^2}{2})r_n}} \to 0,
\end{aligned}$$

where for the last limit, we used Proposition 3.3 (i) (iii) with $m=0$ and the fact that $r_n^{m+1}(e^{2(r_{n+1}-r_n)}-1)=r_n^{m+1}O(r_{n+1}-r_n)=o(1)$ under the assumption $\kappa>2m+2$. Thus the assertion of the lemma is valid.

Now we prove the claim (3.52). We consider another branching Brownian motion with underlying motion according to a standard Brownian motion with drift $-\theta$, with branching rate equal to 1 and with offspring distribution according to $\mathbb{P}_x(\widetilde{L}=k+1)=p_k$ for all $k\in\mathbb{N}$, then we may define another change-of-measure

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}_x}{\mathrm{d}\mathbb{P}_x}\bigg|_{\widetilde{\mathcal{F}}_t}:=\frac{\sum_{u\in\widetilde{N}(t)}1}{e^{2t}},$$

then a similar formula as Lemma 2.1 can be established:

For any $t>0$ and $u\in\widetilde{N}(t)$, let $\Gamma(u,t)$ be a non-negative $\widetilde{\mathcal{F}}_t$-measurable random variable. Then

$$\mathbb{E}_x\Big(\sum_{u\in\widetilde{N}(t)}\Gamma(u,t)\Big)=e^{2t}\widetilde{\mathbf{E}}_x\left(\Gamma(\xi_t,t)\right). \tag{3.53}$$

For $w\in\widetilde{N}(t)$ with $t\in(r_n,r_{n+1})$, let $\widetilde{d}_w,\widetilde{O}_w$ denote the death time and the number of offspring of w respectively. For $v\in\widetilde{N}(r_{n+1})$, define $\widetilde{B}_{n,v}$ to be the event that, for all $w<v$ with $\widetilde{d}_w\in(r_n,r_{n+1})$, it holds that $\widetilde{O}_w\le e^{c_0 n}$, where $0<c_0<1-\frac{\theta^2}{2}$ is fixed. Define

$$\widetilde{Z}^B_{n+1}:=\frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}}\sum_{u\in N(r_n)}1_{\left\{\min_{s\le r_n}X_u(s)>0\right\}}\sum_{v\in\widetilde{N}(r_{n+1}):u\le v}1_{\widetilde{B}_{n,v}}.$$

Now for $v\in\widetilde{N}(r_{n+1}-r_n)$, $\widetilde{D}_{n,v}$ denotes the event that, for all $w<v$, it holds that $\widetilde{O}_w\le e^{c_0 n}$. Let $\widetilde{d}_i$ be the i-th splitting time of the spine and $\widetilde{O}_i$ be the number of children produced by the spine at time $\widetilde{d}_i$. Define $\widetilde{D}_{n,\xi_{r_{n+1}-r_n}}$ to be the event that, for all i with $\widetilde{d}_i<r_{n+1}-r_n$, it holds that $\widetilde{O}_i\le e^{c_0 n}$. Then by the branching property, the Markov property and (3.53), we have

$$\begin{aligned}
&\mathbb{E}_x\left(\left|\widetilde{Z}_{n+1}-\widetilde{Z}^B_{n+1}\right|\big|\mathcal{F}_{r_n}\right)\\
&=\frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}}\sum_{u\in N(r_n)}1_{\left\{\min_{s\le r_n}X_u(s)>0\right\}}\mathbb{E}_{X_u(r_n)}\Big(\sum_{v\in\widetilde{N}(r_{n+1}-r_n)}1_{\widetilde{D}^c_{n,v}}\Big)\\
&=e^{2(r_{n+1}-r_n)}\frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}}\sum_{u\in N(r_n)}1_{\left\{\min_{s\le r_n}X_u(s)>0\right\}}\widetilde{\mathbf{E}}_{X_u(r_n)}\Big(1_{\widetilde{D}^c_{n,\xi_{r_{n+1}-r_n}}}\Big).
\end{aligned}$$

Noticing that (1.3) implies $\widetilde{\mathbf{E}}_x(\log_+^{1+\lambda}\widetilde{O}_1)=\widetilde{\mathbf{E}}_0(\log_+^{1+\lambda}\widetilde{O}_1)<\infty$, we obtain

$$\begin{aligned}
&\mathbb{E}_x\left(\left|\widetilde{Z}_{n+1}-\widetilde{Z}^B_{n+1}\right|\big|\mathcal{F}_{r_n}\right)\\
&\lesssim\frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}}\sum_{u\in N(r_n)}1_{\left\{\min_{s\le r_n}X_u(s)>0\right\}}\widetilde{\mathbf{E}}_{X_u(r_n)}\Big(\sum_{i:\widetilde{d}_i<r_{n+1}-r_n}1_{\{\widetilde{O}_i>e^{c_0 n}\}}\Big)\\
&\lesssim\frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}}\sum_{u\in N(r_n)}1_{\left\{\min_{s\le r_n}X_u(s)>0\right\}}\frac{r_{n+1}-r_n}{n^{1+\lambda}}.
\end{aligned}$$

Now taking expectation with respect to $\mathbb{P}_x$, using Lemma 2.1 and Lemma 2.4(i) (ii), we get that

$$\mathbb{E}_x\left(\left|\widetilde{Z}_{n+1}-\widetilde{Z}^B_{n+1}\right|\right) \lesssim \frac{r_{n+1}-r_n}{n^{1+\lambda}}\frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}}e^{r_n}\mathbf{P}_x\left(\min_{s\le r_n}X_\xi(s)>0\right)$$
$$\lesssim xe^{\theta x}\frac{r_{n+1}-r_n}{n^{1+\lambda}}r_n^{m+1}. \tag{3.54}$$

Similarly, by the branching property and the Markov property,

$$\mathbb{E}_x\left(\left|\widetilde{Z}^B_{n+1}-\mathbb{E}_x\left(\widetilde{Z}^B_{n+1}\middle|\mathcal{F}_{r_n}\right)\right|^2\middle|\mathcal{F}_{r_n}\right) = \frac{r_n^{2m+3}}{e^{2(1-\frac{\theta^2}{2})r_n}}\sum_{u\in N(r_n)}1_{\{\min_{s\le r_n}X_u(s)>0\}}$$
$$\times\,\mathbb{E}_{X_u(r_n)}\Big(\Big(\sum_{v\in\widetilde{N}(r_{n+1}-r_n)}1_{\widetilde{D}_{n,v}}-\mathbb{E}_{X_u(r_n)}\Big(\sum_{v\in\widetilde{N}(r_{n+1}-r_n)}1_{\widetilde{D}_{n,v}}\Big)\Big)^2\Big)$$
$$\le \frac{r_n^{2m+3}}{e^{2(1-\frac{\theta^2}{2})r_n}}\sum_{u\in N(r_n)}1_{\{\min_{s\le r_n}X_u(s)>0\}}\mathbb{E}_{X_u(r_n)}\Big(\Big(\sum_{v\in\widetilde{N}(r_{n+1}-r_n)}1_{\widetilde{D}_{n,v}}\Big)^2\Big). \tag{3.55}$$

By (3.53), we see that

$$\mathbb{E}_{X_u(r_n)}\Big(\Big(\sum_{v\in\widetilde{N}(r_{n+1}-r_n)}1_{\widetilde{D}_{n,v}}\Big)^2\Big) = e^{2(r_{n+1}-r_n)}\widetilde{\mathbf{E}}_{X_u(r_n)}\Big(1_{\widetilde{D}_{n,\xi_{r_{n+1}-r_n}}}\sum_{v\in\widetilde{N}(r_{n+1}-r_n)}1_{\widetilde{D}_{n,v}}\Big)$$
$$\lesssim \widetilde{\mathbf{E}}_{X_u(r_n)}\Big(1_{\widetilde{D}_{n,\xi_{r_{n+1}-r_n}}}\sum_{v\in\widetilde{N}(r_{n+1}-r_n)}1\Big).$$

On the set $\widetilde{D}_{n,\xi_{r_{n+1}-r_n}}$, we have

$$\widetilde{\mathbf{E}}_{X_u(r_n)}\Big(\sum_{v\in\widetilde{N}(r_{n+1}-r_n)}1\Big|\widetilde{d}_i,\widetilde{O}_i:i\ge1\Big) = \sum_{i:\widetilde{d}_i<r_{n+1}-r_n}\left(\widetilde{O}_i-1\right)e^{2(r_{n+1}-r_n-d_i)}$$
$$\lesssim e^{c_0n}\sum_{i:\widetilde{d}_i<r_{n+1}-r_n}1,$$

which implies that

$$\widetilde{\mathbf{E}}_{X_u(r_n)}\Big(1_{\widetilde{D}_{n,\xi_{r_{n+1}-r_n}}}\sum_{v\in\widetilde{N}(r_{n+1}-r_n)}1\Big)\lesssim e^{c_0n}\widetilde{\mathbf{E}}_{X_u(r_n)}\Big(\sum_{i:\widetilde{d}_i<r_{n+1}-r_n}1\Big)\lesssim e^{c_0n}(r_{n+1}-r_n).$$

Therefore, plugging this upper-bound back to (3.55), we have

$$\mathbb{E}_x\left(\left|\widetilde{Z}^B_{n+1}-\mathbb{E}_x\left(\widetilde{Z}^B_{n+1}\middle|\mathcal{F}_{r_n}\right)\right|^2\middle|\mathcal{F}_{r_n}\right)\lesssim\frac{r_n^{2m+3}e^{c_0n}(r_{n+1}-r_n)}{e^{2(1-\frac{\theta^2}{2})r_n}}\sum_{u\in N(r_n)}1_{\{\min_{s\le r_n}X_u(s)>0\}}.$$

Taking expectation with respect to $\mathbb{P}_x$, using Lemma 2.1 and Lemma 2.4(i), we conclude that

$$\mathbb{E}_x\left(\left|\widetilde{Z}^B_{n+1}-\mathbb{E}_x\left(\widetilde{Z}^B_{n+1}\middle|\mathcal{F}_{r_n}\right)\right|^2\right)\lesssim\frac{r_n^{2m+3}e^{c_0n}(r_{n+1}-r_n)}{e^{2(1-\frac{\theta^2}{2})r_n}}e^{r_n}\mathbf{E}_x\left(1_{\{\min_{s\le r_n}X_\xi(s)>0\}}\right)$$
$$\lesssim\frac{r_n^{2m+3}e^{c_0n}(r_{n+1}-r_n)}{e^{(1-\frac{\theta^2}{2})r_n}}. \tag{3.56}$$

First using (3.14) with $X = \widetilde{Z}_{n+1}, Y = \widetilde{Z}^B_{n+1}$ and $\ell = 2$, and then (3.54) and (3.56), we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{E}_x\left(\left|\widetilde{Z}_{n+1} - \mathbb{E}_x\left(\widetilde{Z}_{n+1}\big|\mathcal{F}_{r_n}\right)\right|\right) \\
&\le 2\sum_{n=1}^{\infty} \mathbb{E}_x\left(\left|\widetilde{Z}_{n+1} - \widetilde{Z}^B_{n+1}\right|\right) + \sum_{n=1}^{\infty} \mathbb{E}_x\left(\left|\widetilde{Z}^B_{n+1} - \mathbb{E}_x\left(\widetilde{Z}^B_{n+1}\big|\mathcal{F}_{r_n}\right)\right|^2\right)^{1/2} \\
&\lesssim \sum_{n=1}^{\infty}\left(\frac{r_{n+1}-r_n}{n^{1+\lambda}} r_n^{m+1} + \left(\frac{r_n^{2m+3} e^{c_0 n}(r_{n+1}-r_n)}{e^{(1-\frac{\theta^2}{2})r_n}}\right)^{1/2}\right),
\end{aligned}$$

which is finite since for $\lambda > 0$ and $\kappa > 2m+2$, $(r_{n+1}-r_n) r_n^{m+1} = O(n^{(m-\kappa+2)/\kappa}) = o(1)$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1+\lambda}} < \infty$. Then we finished the proof of the claim (3.52). □

Proof of Theorem 1.1: For any $\lambda > 6m+6$, we can find an appropriate κ satisfying the conditions of Proposition 3.3 and Lemma 3.5. For instance, we can take $\kappa := 2m+2+(\lambda-6m-6)/4 > 2m+2$, then

$$\lambda = 4(\kappa - 2m - 2) + 6m + 6 > 6m + 6 + 2(\kappa - 2m - 2) = 2m + 2\kappa + 2.$$

For any $\ell \in [0, m]$, since $\kappa > 2m+2$, we have

$$r_{n+1}^{\ell} - r_n^{\ell} = \frac{\ell}{\kappa}\int_n^{n+1} y^{-(\kappa-\ell)/\kappa} dy \le \frac{m}{\kappa} n^{-(\kappa-m)/\kappa} = o(r_n^{-m}). \tag{3.57}$$

For $t \in (r_n, r_{n+1})$, by Lemma 3.5 and (3.57), we get that for any $a \ge 0$ and $t \in (r_n, r_{n+1})$,

$$\begin{aligned}
\frac{Z_t^{(0,\infty)}((a,\infty))}{t^{-3/2} e^{(1-\frac{\theta^2}{2})t}} &= \frac{r_n^{-3/2}}{t^{-3/2}} \cdot \frac{e^{(1-\frac{\theta^2}{2})r_n}}{e^{(1-\frac{\theta^2}{2})t}} \cdot \frac{Z_t^{(0,\infty)}((a,\infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\
&= \left(1 + o(r_n^{-m})\right) \cdot \left(o\left(r_n^{-m}\right) + \frac{Z_{r_n}^{(0,\infty)}((a,\infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}}\right) \\
&= o(r_n^{-m}) + \frac{Z_{r_n}^{(0,\infty)}((a,\infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}}.
\end{aligned}$$

By (3.57), we have

$$r_n^m \sup_{t \in (r_n, r_{n+1})} \left|\frac{1}{t^{\ell}} - \frac{1}{r_n^{\ell}}\right| = o(1).$$

Combining the above with Proposition 3.3(i), we get the assertion of Theorem 1.1. □

Proof of Theorem 1.2: The proof is similar to that of Theorem 1.1 and we omit the details. □

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