Math 561 Midterm Test

1. (25 points) Suppose that $X_1, X_2, \ldots$ are uncorrelated random variables with $\mathbb{E}(X_j) = \mu_j$ and $\mathrm{Var}(X_j)/j \to 0$ as $j \to \infty$. Let $S_n = X_1 + \cdots + X_n$ and $\nu_n = \mathbb{E}(S_n)/n$. Show that as $n \to \infty$,

$$\frac{S_n}{n} - \nu_n \to 0, \qquad \text{in probability.}$$

2. (25 points) Suppose that $X_1, X_2, \ldots$ are independent random variables. For all positive integers $m < n$, let $S_{m,n} = X_{m+1} + \cdots + X_n$. Show that, for any $a > 0$,

$$\mathbb{P}(\max_{m<j\leq n} |S_{m,j}| > 2a) \min_{m<k\leq n} P(|S_{k,n}| \leq a) \leq P(|S_{m,n}| > a).$$

3. (25 points) Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables with

$$\mathbb{P}(X_1 = 2^j) = 2^{-j}, \qquad j = 1, 2, \ldots$$

Use the second Borel-Cantelli lemma to show that

$$\limsup_{n\to\infty} \frac{X_n}{n \log_2 n} = \infty$$

almost surely.

4. (25 points) Suppose that $X_1, X_2, \ldots$ are uncorrelated and identically distributed random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^4) < \infty$. Let $S_n = X_1 + \cdots + X_n$. Use the Borel-Cantelli lemma to prove that S_n/n converges to 0 almost surely.