Math 561 Final Exam, Spring 2024

Due 05/06/2024

You should finish this final on your own. No collaboration is allowed.

1. (20 points) Let $Y_1, Y_2, \ldots$ be iid positive random variables with $E(Y_1) = \mu$ and $\mathrm{Var}(Y_1) = \sigma^2 \in (0, \infty)$. Let $S_n = Y_1 + \cdots + Y_n$ and $N_t = \sup\{m : S_m \leq t\}$. Using Exercise 3.4.6 to to prove that, as $t \uparrow \infty$,

$$\frac{\mu N_t - t}{\sigma\sqrt{t/\mu}}$$

converges weakly to a standard normal random variable.

2. (20 points) Suppose that $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables, and $S_n = \xi_1 + \cdots + \xi_n$ for all $n \geq 1$. Assume that ξ_1 is not constant and that

$$\varphi(\theta) = E\exp(\theta\xi_1) < \infty, \qquad \theta \in (-\delta, \delta)$$

for some $\delta > 0$. Let $\psi(\theta) = \log \varphi(\theta)$. Show that (i) $X_n^\theta = \exp(\theta S_n - n\psi(\theta))$ is martingale; (ii) ψ is strictly convex on $(-\delta, \delta)$ and (iii) $E\sqrt{X_n^\theta} \to 0$ and $X_n^\theta \to 0$ a.s.

3. (20 points) Let S_n be an asymmetric simple random walk with $p \geq \frac{1}{2}$. Let $T_1 = \inf\{n : S_n = 1\}$. (i) Use the martingale from the previous problem to prove that if $\theta > 0$, then $1 = e^\theta E(\varphi(\theta)^{-T_1})$, where $\varphi(\theta) = pe^\theta + qe^{-\theta}$ and $q = 1 - p$. (ii) Set $pe^\theta + qe^{-\theta} = \frac{1}{s}$ and then solve for $x = e^{-\theta}$ to get

$$E(s^{T_1}) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs}.$$

4. (20 points) Suppose that $X_1, X_2, \cdots$ are independent non-negative random variables with $E[X_n] = 1$ (and $P(X_n = 1) < 1$) for all $n \geq 1$. Define $M_0 = 1$ and for $n \geq 1$, let

$$M_n = X_1 X_2 \cdots X_n$$

Show that M_n is a non-negative martingale, and so $M_\infty = \lim_n M_n$ exists almost surely. Let $a_n = E[X_n^{1/2}]$. Prove the following statements are equivalent:

(i) $E[M_\infty] = 1$;

(ii) $M_n \to M_\infty$ in L^1;

(iii) $\{M_n, n \geq 1\}$ is uniformly integrable;

(iv) $\prod_{n=1}^\infty a_n > 0$;

(v) $\sum_{n=1}^\infty (1 - a_n) < \infty$.

Prove also that if one of the 5 statements above fails to hold, then $P(M_\infty = 0) = 1$.

5. (20 points) Suppose that $X_1, X_2, \cdots$ are independent standard normal random variables and that, for each $n > 1$, $S_n = \sum_{k=1}^n X_k$. (a) Show that for any number θ, $e^{\theta S_n}$ is a submartingale and then use this to show that, for any $c > 0$ and positive integer n,

$$P(\sup_{1 \le k \le n} S_k > c) \le e^{-\frac{c^2}{2n}}.$$

(b) Use (a) and the Borel-Cantelli lemma to show that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} \le 1$$

almost surely.
(c) Use the second Borel-Cantelli lemma to show that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} \ge 1$$

almost surely.