4

Random Walks

Let $X_1, X_2, \ldots$ be i.i.d. taking values in $\mathbf{R}^d$ and let $S_n = X_1 + \cdots + X_n$. S_n is a **random walk**. In the previous chapter, we were primarily concerned with the distribution of S_n. In this one, we will look at properties of the sequence $S_1(\omega), S_2(\omega), \ldots$ For example, does the last sequence return to (or near) 0 infinitely often? The first section introduces stopping times, a concept that will be very important in this and the next two chapters. After the first section is completed, the remaining three can be read in any order or skipped without much loss. The second section is not starred since it contains some basic facts about random walks.

4.1 Stopping Times

Most of the results in this section are valid for i.i.d. X's taking values in some nice measurable space $(S, \mathcal{S})$ and will be proved in that generality. For several reasons, it is convenient to use the special probability space from the proof of Kolmogorov's extension theorem:

$$\Omega = \{(\omega_1, \omega_2, \ldots) : \omega_i \in S\}$$

$$\mathcal{F} = \mathcal{S} \times \mathcal{S} \times \ldots$$

$$P = \mu \times \mu \times \ldots \qquad \mu \text{ is the distribution of } X_i$$

$$X_n(\omega) = \omega_n$$

So, throughout this section, we will suppose (without loss of generality) that our random variables are constructed on this special space.

Before taking up our main topic, we will prove a 0-1 law that, in the i.i.d. case, generalizes Kolmogorov's. To state the new 0-1 law, we need two definitions. A **finite permutation** of $\mathbf{N} = \{1, 2, \ldots\}$ is a map π from $\mathbf{N}$ onto $\mathbf{N}$ so that $\pi(i) \neq i$ for only finitely many i. If π is a finite permutation of $\mathbf{N}$ and $\omega \in S^{\mathbf{N}}$, we define $(\pi\omega)_i = \omega_{\pi(i)}$. In words, the coordinates of ω are rearranged according to π. Since $X_i(\omega) = \omega_i$, this is the same as rearranging the random variables. An event A is **permutable** if $\pi^{-1}A \equiv \{\omega : \pi\omega \in A\}$ is equal to A for any finite permutation π, or in other words, if its occurrence is not affected by rearranging finitely many of the random variables. The collection of permutable events is a σ-field. It is called the **exchangeable** σ-field and denoted by $\mathcal{E}$.

To see the reason for interest in permutable events, suppose $S = \mathbf{R}$ and let $S_n(\omega) = X_1(\omega) + \cdots + X_n(\omega)$. Two examples of permutable events are

(i) $\{\omega : S_n(\omega) \in B \text{ i.o.}\}$
(ii) $\{\omega : \limsup_{n\to\infty} S_n(\omega)/c_n \ge 1\}$

In each case, the event is permutable because $S_n(\omega) = S_n(\pi\omega)$ for large n. The list of examples can be enlarged considerably by observing:

(iii) All events in the tail σ-field $\mathcal{T}$ are permutable.

To see this, observe that if $A \in \sigma(X_{n+1}, X_{n+2}, \ldots)$, then the occurrence of A is unaffected by a permutation of $X_1, \ldots, X_n$. (i) shows that the converse of (iii) is false. The next result shows that for an i.i.d. sequence, there is no difference between $\mathcal{E}$ and $\mathcal{T}$. They are both trivial.

Theorem 4.1.1. Hewitt-Savage 0-1 law. *If $X_1, X_2, \ldots$ are i.i.d. and $A \in \mathcal{E}$ then $P(A) \in \{0, 1\}$.*

Proof. Let $A \in \mathcal{E}$. As in the proof of Kolmogorov's 0-1 law, we will show that A is independent of itself, that is, $P(A) = P(A \cap A) = P(A)P(A)$ so $P(A) \in \{0, 1\}$. Let $A_n \in \sigma(X_1, \ldots, X_n)$ so that

(a) $$P(A_n \Delta A) \to 0$$

Here $A \Delta B = (A - B) \cup (B - A)$ is the symmetric difference. The existence of the A_n's is proved in part ii of Lemma A.2.1. A_n can be written as $\{\omega : (\omega_1, \ldots, \omega_n) \in B_n\}$ with $B_n \in \mathcal{S}^n$. Let

$$\pi(j) = \begin{cases} j + n & \text{if } 1 \le j \le n \\ j - n & \text{if } n + 1 \le j \le 2n \\ j & \text{if } j \ge 2n + 1 \end{cases}$$

Observing that π^2 is the identity (so we don't have to worry about whether to write π or π^{-1}) and the coordinates are i.i.d. (so the permuted coordinates are) gives

(b) $$P(\omega : \omega \in A_n \Delta A) = P(\omega : \pi\omega \in A_n \Delta A)$$

Now $\{\omega : \pi\omega \in A\} = \{\omega : \omega \in A\}$, since A is permutable, and

$$\{\omega : \pi\omega \in A_n\} = \{\omega : (\omega_{n+1}, \ldots, \omega_{2n}) \in B_n\}$$

If we use A'_n to denote the last event then we have

(c) $$\{\omega : \pi\omega \in A_n \Delta A\} = \{\omega : \omega \in A'_n \Delta A\}$$

Combining (b) and (c) gives

(d) $$P(A_n \Delta A) = P(A'_n \Delta A)$$

It is easy to see that

$$|P(B) - P(C)| \le |P(B\Delta C)|$$

so (d) implies $P(A_n), P(A'_n) \to P(A)$. Now $A - C \subset (A - B) \cup (B - C)$ and, with a similar inequality for $C - A$, implies $A\Delta C \subset (A\Delta B) \cup (B\Delta C)$. The last inequality, (d), and (a) imply

$$P(A_n \Delta A'_n) \le P(A_n \Delta A) + P(A\Delta A'_n) \to 0$$

The last result implies

$$\begin{aligned} 0 &\le P(A_n) - P(A_n \cap A'_n) \\ &\le P(A_n \cup A'_n) - P(A_n \cap A'_n) = P(A_n \Delta A'_n) \to 0 \end{aligned}$$

so $P(A_n \cap A'_n) \to P(A)$. But A_n and A'_n are independent, so

$$P(A_n \cap A'_n) = P(A_n)P(A'_n) \to P(A)^2$$

This shows $P(A) = P(A)^2$ and proves Theorem 4.1.1. ■

A typical application of Theorem 4.1.1 is

Theorem 4.1.2. *For a random walk on* **R**, *there are only four possibilities, one of which has probability 1.*

(i) $S_n = 0$ *for all* n.
(ii) $S_n \to \infty$.
(iii) $S_n \to -\infty$.
(iv) $-\infty = \liminf S_n < \limsup S_n = \infty$.

Proof. Theorem 4.1.1 implies $\limsup S_n$ is a constant $c \in [-\infty, \infty]$. Let $S'_n = S_{n+1} - X_1$. Since S'_n has the same distribution as S_n, it follows that $c = c - X_1$. If c is finite, subtracting c from both sides we conclude $X_1 \equiv 0$ and (i) occurs. Turning the last statement around, we see that if $X_1 \not\equiv 0$, then $c = -\infty$ or ∞. The same analysis applies to the liminf. Discarding the impossible combination $\limsup S_n = -\infty$ and $\liminf S_n = +\infty$, we have proved the result. ■

Exercise 4.1.1. Symmetric random walk. Let $X_1, X_2, \ldots \in$ **R** be i.i.d. with a distribution that is symmetric about 0 and nondegenerate (i.e., $P(X_i = 0) < 1$). Show that we are in case (iv) of Theorem 4.1.2.

Exercise 4.1.2. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$. Use the central limit theorem to conclude that we are in case (iv) of Theorem 4.1.2. Later in Exercise 4.1.11 you will show that $EX_i = 0$ and $P(X_i = 0) < 1$ is sufficient.

The special case in which $P(X_i = 1) = P(X_i = -1) = 1/2$ is called **simple random walk**. Since a simple random walk cannot skip over any integers, it follows from either exercise above that with probability 1 it visits every integer infinitely many times.

Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n) =$ the information known at time n. A random variable N taking values in $\{1, 2, \ldots\} \cup \{\infty\}$ is said to be a **stopping time** or an **optional random variable** if for every $n < \infty$, $\{N = n\} \in \mathcal{F}_n$. If we think of S_n as giving the (logarithm of the) price of a stock at time n, and N as the time we sell it, then the last definition says that the decision to sell at time n must be based on the information known at that time. The last interpretation gives one explanation for the second name. N is a time at which we can exercise an option to buy a stock. Chung prefers the second name because N is "usually rather a momentary pause after which the process proceeds again: time marches on!"

The canonical example of a stopping time is $N = \inf\{n : S_n \in A\}$, the **hitting time of A**. To check that this is a stopping time, we observe that

$$\{N = n\} = \{S_1 \in A^c, \ldots, S_{n-1} \in A^c, S_n \in A\} \in \mathcal{F}_n$$

Two concrete examples of hitting times that have appeared above are

Example 4.1.1. $N = \inf\{k : |S_k| \geq x\}$ from the proof of Theorem 2.5.2.

Example 4.1.2. If the $X_i \geq 0$ and $N_t = \sup\{n : S_n \leq t\}$ is the random variable that first appeared in Example 2.4.1, then $N_t + 1 = \inf\{n : S_n > t\}$ is a stopping time.

The next result allows us to construct new examples from the old ones.

Exercise 4.1.3. If S and T are stopping times, then $S \wedge T$ and $S \vee T$ are stopping times. Since constant times are stopping times, it follows that $S \wedge n$ and $S \vee n$ are stopping times.

Exercise 4.1.4. Suppose S and T are stopping times. Is $S + T$ a stopping time? Give a proof or a counterexample.

Associated with each stopping time N is a σ-field $\mathcal{F}_N =$ the information known at time N. Formally, $\mathcal{F}_N$ is the collection of sets A that have $A \cap \{N = n\} \in \mathcal{F}_n$ for all $n < \infty$, that is, when $N = n$, A must be measurable with respect to the information known at time n. Trivial but important examples of sets in $\mathcal{F}_N$ are $\{N \leq n\}$, that is, N is measurable with respect to $\mathcal{F}_N$.

Exercise 4.1.5. Show that if $Y_n \in \mathcal{F}_n$ and N is a stopping time, $Y_N \in \mathcal{F}_N$. As a corollary of this result, we see that if $f : S \to \mathbf{R}$ is measurable, $T_n = \sum_{m \leq n} f(X_m)$, and $M_n = \max_{m \leq n} T_m$, then T_N and $M_N \in \mathcal{F}_N$. An important special case is $S = \mathbf{R}$, $f(x) = x$.

Exercise 4.1.6. Show that if $M \le N$ are stopping times, then $\mathcal{F}_M \subset \mathcal{F}_N$.

Exercise 4.1.7. Show that if $L \le M$ and $A \in \mathcal{F}_L$, then

$$N = \begin{cases} L & \text{on } A \\ M & \text{on } A^c \end{cases} \quad \text{is a stopping time}$$

Our first result about $\mathcal{F}_N$ is

Theorem 4.1.3. *Let $X_1, X_2, \dots$ be i.i.d., $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and N be a stopping time with $P(N < \infty) > 0$. Conditional on $\{N < \infty\}$, $\{X_{N+n}, n \ge 1\}$ is independent of $\mathcal{F}_N$ and has the same distribution as the original sequence.*

Proof. By Theorem A.1.5, it is enough to show that if $A \in \mathcal{F}_N$ and $B_j \in \mathcal{S}$ for $1 \le j \le k$, then

$$P(A, N < \infty, X_{N+j} \in B_j, 1 \le j \le k) = P(A \cap \{N < \infty\}) \prod_{j=1}^{k} \mu(B_j)$$

where $\mu(B) = P(X_i \in B)$. The method ("divide and conquer") is one that we will see many times below. We break things down according to the value of N in order to replace N by n and reduce to the case of a fixed time.

$$\begin{aligned} P(A, N = n, X_{N+j} \in B_j, 1 \le j \le k) &= P(A, N = n, X_{n+j} \in B_j, 1 \le j \le k) \\ &= P(A \cap \{N = n\}) \prod_{j=1}^{k} \mu(B_j) \end{aligned}$$

since $A \cap \{N = n\} \in \mathcal{F}_n$ and that σ-field is independent of $X_{n+1}, \dots, X_{n+k}$. Summing over n now gives the desired result. ∎

To delve further into properties of stopping times, we recall that we have supposed $\Omega = S^{\mathbf{N}}$ and define the **shift** $\theta : \Omega \to \Omega$ by

$$(\theta\omega)(n) = \omega(n+1) \qquad n = 1, 2, \dots$$

In words, we drop the first coordinate and shift the others one place to the left. The iterates of θ are defined by composition. Let $\theta^1 = \theta$, and for $k \ge 2$, let $\theta^k = \theta \circ \theta^{k-1}$. Clearly, $(\theta^k\omega)(n) = \omega(n+k)$, $n = 1, 2, \dots$ To extend the last definition to stopping times, we let

$$\theta^N \omega = \begin{cases} \theta^n \omega & \text{on } \{N = n\} \\ \Delta & \text{on } \{N = \infty\} \end{cases}$$

Here Δ is an extra point that we add to Ω. According to the only joke in Blumenthal and Getoor (1968), Δ is a "cemetery or heaven depending upon your point of view." Seriously, Δ is a convenience in making definitions like the next one.

Example 4.1.3. Returns to 0. For a concrete example of the use of θ, suppose $S = \mathbf{R}^d$ and let

$$\tau(\omega) = \inf\{n : \omega_1 + \cdots + \omega_n = 0\}$$

where $\inf \emptyset = \infty$, and we set $\tau(\Delta) = \infty$. If we let $\tau_2(\omega) = \tau(\omega) + \tau(\theta^\tau \omega)$, then on $\{\tau < \infty\}$,

$$\begin{aligned} \tau(\theta^\tau \omega) &= \inf\{n : (\theta^\tau \omega)_1 + \cdots + (\theta^\tau \omega)_n = 0\} \\ &= \inf\{n : \omega_{\tau+1} + \cdots + \omega_{\tau+n} = 0\} \\ \tau(\omega) + \tau(\theta^\tau \omega) &= \inf\{m > \tau : \omega_1 + \cdots + \omega_m = 0\} \end{aligned}$$

So τ_2 is the time of the second visit to 0 (and thanks to the conventions $\theta^\infty \omega = \Delta$ and $\tau(\Delta) = \infty$, this is true for all ω). The last computation generalizes easily to show that if we let

$$\tau_n(\omega) = \tau_{n-1}(\omega) + \tau(\theta^{\tau_{n-1}} \omega)$$

then τ_n is the time of the nth visit to 0.

If we have any stopping time T, we can define its iterates by $T_0 = 0$ and

$$T_n(\omega) = T_{n-1}(\omega) + T(\theta^{T_{n-1}} \omega) \quad \text{for } n \geq 1$$

If we assume $P = \mu \times \mu \times \ldots$ then

$$P(T_n < \infty) = P(T < \infty)^n \tag{4.1.1}$$

Proof. We will prove this by induction. The result is trivial when $n = 1$. Suppose now that it is valid for $n - 1$. Applying Theorem 4.1.3 to $N = T_{n-1}$, we see that $T(\theta^{T_{n-1}}) < \infty$ is independent of $T_{n-1} < \infty$ and has the same probability as $T < \infty$, so

$$\begin{aligned} P(T_n < \infty) &= P(T_{n-1} < \infty, T(\theta^{T_{n-1}} \omega) < \infty) \\ &= P(T_{n-1} < \infty) P(T < \infty) = P(T < \infty)^n \end{aligned}$$

by the induction hypothesis. ■

Letting $t_n = T(\theta^{T_{n-1}})$, we can extend Theorem 4.1.3 to

Theorem 4.1.4. *Suppose $P(T < \infty) = 1$. Then the "random vectors"*

$$V_n = (t_n, X_{T_{n-1}+1}, \ldots, X_{T_n})$$

are independent and identically distributed.

Proof. It is clear from Theorem 4.1.3 that V_n and V_1 have the same distribution. The independence follows from Theorem 4.1.3 and induction since $V_1, \ldots, V_{n-1} \in \mathcal{F}(T_{n-1})$. ■

Example 4.1.4. Ladder variables. Let $\alpha(\omega) = \inf\{n : \omega_1 + \cdots + \omega_n > 0\}$ where $\inf \emptyset = \infty$, and set $\alpha(\Delta) = \infty$. Let $\alpha_0 = 0$ and let

$$\alpha_k(\omega) = \alpha_{k-1}(\omega) + \alpha(\theta^{\alpha_{k-1}}\omega)$$

for $k \geq 1$. At time α_k, the random walk is at a record high value.

The next three exercises investigate these times.

Exercise 4.1.8. (i) If $P(\alpha < \infty) < 1$ then $P(\sup S_n < \infty) = 1$.
(ii) If $P(\alpha < \infty) = 1$, then $P(\sup S_n = \infty) = 1$.

Exercise 4.1.9. Let $\beta = \inf\{n : S_n < 0\}$. Prove that the four possibilities in Theorem 4.1.2 correspond to the four combinations of $P(\alpha < \infty) < 1$ or $= 1$, and $P(\beta < \infty) < 1$ or $= 1$.

Exercise 4.1.10. Let $S_0 = 0$, $\bar{\beta} = \inf\{n \geq 1 : S_n \leq 0\}$ and

$$A_m^n = \{0 \geq S_m, S_1 \geq S_m, \ldots, S_{m-1} \geq S_m, S_m < S_{m+1}, \ldots, S_m < S_n\}$$

(i) Show $1 = \sum_{m=0}^n P(A_m^n) = \sum_{m=0}^n P(\alpha > m)P(\bar{\beta} > n - m)$.
(ii) Let $n \to \infty$ and conclude $E\alpha = 1/P(\bar{\beta} = \infty)$.

Exercise 4.1.11. (i) Combine the last exercise with the proof of (ii) in Exercise 4.1.8 to conclude that if $EX_i = 0$, then $P(\bar{\beta} = \infty) = 0$. (ii) Show that if we assume in addition that $P(X_i = 0) < 1$, then $P(\beta = \infty) = 0$, and Exercise 4.1.9 implies we are in case (iv) of Theorem 4.1.2.

A famous result about stopping times for random walks is:

Theorem 4.1.5. Wald's equation. *Let $X_1, X_2, \ldots$ be i.i.d. with $E|X_i| < \infty$. If N is a stopping time with $EN < \infty$, then $ES_N = EX_1EN$.*

Proof. First suppose the $X_i \geq 0$.

$$ES_N = \int S_N dP = \sum_{n=1}^{\infty} \int S_n 1_{\{N=n\}} dP = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \int X_m 1_{\{N=n\}} dP$$

Since the $X_i \geq 0$, we can interchange the order of summation (i.e., use Fubini's theorem) to conclude that the last expression

$$= \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \int X_m 1_{\{N=n\}} dP = \sum_{m=1}^{\infty} \int X_m 1_{\{N \geq m\}} dP$$

Now $\{N \ge m\} = \{N \le m-1\}^c \in \mathcal{F}_{m-1}$ and is independent of X_m, so the last expression

$$= \sum_{m=1}^{\infty} EX_m P(N \ge m) = EX_1 EN$$

To prove the result in general, we run the last argument backwards. If we have $EN < \infty$ then

$$\infty > \sum_{m=1}^{\infty} E|X_m| P(N \ge m) = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \int |X_m| 1_{\{N=n\}} \, dP$$

The last formula shows that the double sum converges absolutely in one order, so Fubini's theorem gives

$$\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \int X_m 1_{\{N=n\}} dP = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \int X_m 1_{\{N=n\}} dP$$

Using the independence of $\{N \ge m\} \in \mathcal{F}_{m-1}$ and X_m, and rewriting the last identity, it follows that

$$\sum_{m=1}^{\infty} EX_m P(N \ge m) = ES_N$$

Since the left-hand side is $EN\, EX_1$, the proof is complete. ■

Exercise 4.1.12. Let $X_1, X_2, \ldots$ be i.i.d. uniform on (0,1), let $S_n = X_1 + \cdots + X_n$, and let $T = \inf\{n : S_n > 1\}$. Show that $P(T > n) = 1/n!$, so $ET = e$ and $ES_T = e/2$.

Example 4.1.5. Simple random walk. Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i = 1) = 1/2$ and $P(X_i = -1) = 1/2$. Let $a < 0 < b$ be integers and let $N = \inf\{n : S_n \notin (a,b)\}$. To apply Theorem 4.1.5, we have to check that $EN < \infty$. To do this, we observe that if $x \in (a,b)$, then

$$P(x + S_{b-a} \notin (a,b)) \ge 2^{-(b-a)}$$

since $b-a$ steps of size $+1$ in a row will take us out of the interval. Iterating the last inequality, it follows that

$$P(N > n(b-a)) \le \left(1 - 2^{-(b-a)}\right)^n$$

so $EN < \infty$. Applying Theorem 4.1.5 now gives $ES_N = 0$ or

$$bP(S_N = b) + aP(S_N = a) = 0$$

Since $P(S_N = b) + P(S_N = a) = 1$, it follows that $(b-a)P(S_N = b) = -a$, so

$$P(S_N = b) = \frac{-a}{b-a} \qquad P(S_N = a) = \frac{b}{b-a}$$

Letting $T_a = \inf\{n : S_n = a\}$, we can write the last conclusion as

$$P(T_a < T_b) = \frac{b}{b-a} \quad \text{for } a < 0 < b \tag{4.1.2}$$

Setting $b = M$ and letting $M \to \infty$ gives

$$P(T_a < \infty) \geq P(T_a < T_M) \to 1$$

for all $a < 0$. From symmetry (and the fact that $T_0 \equiv 0$), it follows that

$$P(T_x < \infty) = 1 \quad \text{for all } x \in \mathbf{Z} \tag{4.1.3}$$

Our final fact about T_x is that $ET_x = \infty$ for $x \neq 0$. To prove this, note that if $ET_x < \infty$ then Theorem 4.1.5 would imply

$$x = ES_{T_x} = EX_1 ET_x = 0$$

In Section 4.3, we will compute the distribution of T_1 and show that

$$P(T_1 > t) \sim C\, t^{-1/2}$$

Exercise 4.1.13. Asymmetric simple random walk. Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_1 = 1) = p > 1/2$ and $P(X_1 = -1) = 1 - p$, and let $S_n = X_1 + \cdots + X_n$. Let $\alpha = \inf\{m : S_m > 0\}$ and $\beta = \inf\{n : S_n < 0\}$.

(i) Use Exercise 4.1.9 to conclude that $P(\alpha < \infty) = 1$ and $P(\beta < \infty) < 1$.

(ii) If $Y = \inf S_n$, then $P(Y \leq -k) = P(\beta < \infty)^k$.

(iii) Apply Wald's equation to $\alpha \wedge n$ and let $n \to \infty$ to get $E\alpha = 1/EX_1 = 1/(2p-1)$. Comparing with Exercise 4.1.10 shows $P(\bar{\beta} = \infty) = 2p - 1$.

Exercise 4.1.14. An optimal stopping problem. Let X_n, $n \geq 1$ be i.i.d. with $EX_1^+ < \infty$ and let

$$Y_n = \max_{1 \leq m \leq n} X_m - cn$$

That is, we are looking for a large value of X, but we have to pay $c > 0$ for each observation. (i) Let $T = \inf\{n : X_n > a\}$, $p = P(X_n > a)$, and compute EY_T. (ii) Let α (possibly < 0) be the unique solution of $E(X_1 - \alpha)^+ = c$. Show that $EY_T = \alpha$ in this case and use the inequality

$$Y_n \leq \alpha + \sum_{m=1}^{n} ((X_m - \alpha)^+ - c)$$

for $n \geq 1$ to conclude that if $\tau \geq 1$ is a stopping time with $E\tau < \infty$, then $EY_\tau \leq \alpha$. The analysis above assumes that you have to play at least once. If the optimal $\alpha < 0$, then you shouldn't play at all.

Theorem 4.1.6. Wald's second equation. *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_n = 0$ and $EX_n^2 = \sigma^2 < \infty$. If T is a stopping time with $ET < \infty$, then $ES_T^2 = \sigma^2 ET$.*

Proof. Using the definitions and then taking expected value

$$S^2_{T\wedge n} = S^2_{T\wedge(n-1)} + (2X_nS_{n-1} + X_n^2)1_{(T\geq n)}$$
$$ES^2_{T\wedge n} = ES^2_{T\wedge(n-1)} + \sigma^2 P(T \geq n)$$

since $EX_n = 0$ and X_n is independent of S_{n-1} and $1_{(T\geq n)} \in \mathcal{F}_{n-1}$. [The expectation of $S_{n-1}X_n$ exists since both random variables are in L^2.] From the last equality and induction we get

$$ES^2_{T\wedge n} = \sigma^2 \sum_{m=1}^{n} P(T \geq m)$$
$$E(S_{T\wedge n} - S_{T\wedge m})^2 = \sigma^2 \sum_{k=m+1}^{n} P(T \geq n)$$

The second equality follows from the first applied to $X_{m+1}, X_{m+2}, \ldots$. The second equality implies that $S_{T\wedge n}$ is a Cauchy sequence in L^2, so letting $n \to \infty$ in the first, it follows that $ES_T^2 = \sigma^2 ET$. ■

Example 4.1.6. Simple random walk, II. Continuing Example 4.1.5 we investigate $N = \inf\{S_n \notin (a, b)\}$. We have shown that $EN < \infty$. Since $\sigma^2 = 1$, it follows from Theorem 4.1.6 and (4.1.2) that

$$EN = ES_N^2 = a^2\frac{b}{b-a} + b^2\frac{-a}{b-a} = -ab$$

If $b = L$ and $a = -L$, $EN = L^2$.

An amusing consequence of Theorem 4.1.6 is

Theorem 4.1.7. *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_n = 0$ and $EX_n^2 = 1$, and let $T_c = \inf\{n \geq 1 : |S_n| > cn^{1/2}\}$.*

$$ET_c \quad \begin{cases} < \infty & \text{for } c < 1 \\ = \infty & \text{for } c \geq 1 \end{cases}$$

Proof. One half of this is easy. If $ET_c < \infty$ then, the previous exercise implies $ET_c = E(S^2_{T_c}) > c^2 ET_c$, a contradiction if $c \geq 1$. To prove the other direction, we let $\tau = T_c \wedge n$ and observe $S^2_{\tau-1} \leq c^2(\tau - 1)$, so using the Cauchy-Schwarz inequality

$$E\tau = ES_\tau^2 = ES^2_{\tau-1} + 2E(S_{\tau-1}X_\tau) + EX_\tau^2 \quad \leq c^2E\tau + 2c(E\tau\, EX_\tau^2)^{1/2} + EX_\tau^2$$

To complete the proof now, we will show

Lemma 4.1.8. *If T is a stopping time with $ET = \infty$, then*

$$EX^2_{T\wedge n}/E(T \wedge n) \to 0$$

Theorem 4.1.7 follows, for if $\epsilon < 1 - c^2$ and n is large, we will have $E\tau \le (c^2 + \epsilon) E\tau$, a contradiction.

Proof. We begin by writing

$$E(X_{T\wedge n}^2) = E(X_{T\wedge n}^2; X_{T\wedge n}^2 \le \epsilon(T \wedge n)) + \sum_{j=1}^{n} E(X_j^2; T \wedge n = j, X_j^2 > \epsilon j)$$

The first term is $\le \epsilon E(T \wedge n)$. To bound the second, choose $N \ge 1$ so that for $n \ge N$

$$\sum_{j=1}^{n} E(X_j^2; X_j^2 > \epsilon j) < n\epsilon$$

This is possible since the dominated convergence theorem implies $E(X_j^2; X_j^2 > \epsilon j) \to 0$ as $j \to \infty$. For the first part of the sum, we use a trivial bound

$$\sum_{j=1}^{N} E(X_j^2; T \wedge n = j, X_j^2 > \epsilon j) \le NEX_1^2$$

To bound the remainder of the sum, we note (i) $X_j^2 \ge 0$; (ii) $\{T \wedge n \ge j\}$ is $\in \mathcal{F}_{j-1}$ and hence is independent of $X_j^2 1_{(X_j^2 > \epsilon j)}$, (iii) use some trivial arithmetic, (iv) use Fubini's theorem and enlarge the range of j, (v) use the choice of N and a trivial inequality

$$\sum_{j=N}^{n} E(X_j^2; T \wedge n = j, X_j^2 > \epsilon j) \le \sum_{j=N}^{n} E(X_j^2; T \wedge n \ge j, X_j^2 > \epsilon j)$$

$$= \sum_{j=N}^{n} P(T \wedge n \ge j) E(X_j^2; X_j^2 > \epsilon j) = \sum_{j=N}^{n} \sum_{k=j}^{\infty} P(T \wedge n = k) E(X_j^2; X_j^2 > \epsilon j)$$

$$\le \sum_{k=N}^{\infty} \sum_{j=1}^{k} P(T \wedge n = k) E(X_j^2; X_j^2 > \epsilon j) \le \sum_{k=N}^{\infty} \epsilon k P(T \wedge n = k) \le \epsilon E(T \wedge n)$$

Combining our estimates shows

$$EX_{T\wedge n}^2 \le 2\epsilon E(T \wedge n) + NEX_1^2$$

Letting $n \to \infty$ and noting $E(T \wedge n) \to \infty$, we have

$$\limsup_{n\to\infty} EX_{T\wedge n}^2 / E(T \wedge n) \le 2\epsilon$$

where ϵ is arbitrary. ∎

4.2 Recurrence

Throughout this section, S_n will be a random walk, that is, $S_n = X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are i.i.d., and we will investigate the question mentioned at the beginning of the chapter. Does the sequence $S_1(\omega), S_2(\omega), \dots$ return to (or near) 0 infinitely often? The answer to the last question is either Yes or No, and the random walk is called recurrent or transient accordingly. We begin with some definitions that formulate the question precisely and a result that establishes a dichotomy between the two cases.

The number $x \in \mathbf{R}^d$ is said to be a **recurrent value** for the random walk S_n if for every $\epsilon > 0$, $P(\|S_n - x\| < \epsilon \text{ i.o.}) = 1$. Here $\|x\| = \sup |x_i|$. The reader will see the reason for this choice of norm in the proof of Lemma 4.2.5. The Hewitt-Savage 0-1 law, Theorem 4.1.1, implies that if the last probability is < 1, it is 0. Our first result shows that to know the set of recurrent values, it is enough to check $x = 0$. A number x is said to be a **possible value** of the random walk if for any $\epsilon > 0$, there is an n so that $P(\|S_n - x\| < \epsilon) > 0$.

Theorem 4.2.1. *The set $\mathcal{V}$ of recurrent values is either $\emptyset$ or a closed subgroup of $\mathbf{R}^d$. In the second case, $\mathcal{V} = \mathcal{U}$, the set of possible values.*

Proof. Suppose $\mathcal{V} \neq \emptyset$. It is clear that $\mathcal{V}^c$ is open, so $\mathcal{V}$ is closed. To prove that $\mathcal{V}$ is a group, we will first show that

(∗) if $x \in \mathcal{U}$ and $y \in \mathcal{V}$ then $y - x \in \mathcal{V}$.

This statement has been formulated so that once it is established, the result follows easily. Let

$$p_{\delta,m}(z) = P(\|S_n - z\| \geq \delta \text{ for all } n \geq m)$$

If $y - x \notin \mathcal{V}$, there is an $\epsilon > 0$ and $m \geq 1$ so that $p_{2\epsilon,m}(y - x) > 0$. Since $x \in \mathcal{U}$, there is a k so that $P(\|S_k - x\| < \epsilon) > 0$. Since

$$P(\|S_n - S_k - (y - x)\| \geq 2\epsilon \text{ for all } n \geq k + m) = p_{2\epsilon,m}(y - x)$$

and is independent of $\{\|S_k - x\| < \epsilon\}$, it follows that

$$p_{\epsilon,m+k}(y) \geq P(\|S_k - x\| < \epsilon) p_{2\epsilon,m}(y - x) > 0$$

contradicting $y \in \mathcal{V}$, so $y - x \in \mathcal{V}$.

To conclude that $\mathcal{V}$ is a group when $\mathcal{V} \neq \emptyset$, let $q, r \in \mathcal{V}$, and observe: (i) taking $x = y = r$ in (∗) shows $0 \in \mathcal{V}$, (ii) taking $x = r$, $y = 0$ shows $-r \in \mathcal{V}$, and (iii) taking $x = -r$, $y = q$ shows $q + r \in \mathcal{V}$. To prove that $\mathcal{V} = \mathcal{U}$ now, observe that if $u \in \mathcal{U}$ taking $x = u$, $y = 0$ shows $-u \in \mathcal{V}$, and since $\mathcal{V}$ is a group, it follows that $u \in \mathcal{V}$. ∎

If $\mathcal{V} = \emptyset$, the random walk is said to be **transient**; otherwise it is called **recurrent**. Before plunging into the technicalities needed to treat a general random walk, we begin by analyzing the special case Polya considered in 1921. Legend has it that Polya thought of this problem while wandering around in a park near Zürich when he noticed that he kept encountering the same young couple. History does not record what the young couple thought.

Example 4.2.1. Simple random walk on $\mathbf{Z}^d$.

$$P(X_i = e_j) = P(X_i = -e_j) = 1/2d$$

for each of the d unit vectors e_j. To analyze this case, we begin with a result that is valid for any random walk. Let $\tau_0 = 0$ and $\tau_n = \inf\{m > \tau_{n-1} : S_m = 0\}$ be the time of the nth return to 0. From (4.1.1), it follows that

$$P(\tau_n < \infty) = P(\tau_1 < \infty)^n$$

a fact that leads easily to:

Theorem 4.2.2. *For any random walk, the following are equivalent:*
(i) $P(\tau_1 < \infty) = 1$, *(ii)* $P(S_m = 0 \text{ i.o.}) = 1$, *and (iii)* $\sum_{m=0}^{\infty} P(S_m = 0) = \infty$.

Proof. If $P(\tau_1 < \infty) = 1$, then $P(\tau_n < \infty) = 1$ for all n and $P(S_m = 0 \text{ i.o.}) = 1$. Let

$$V = \sum_{m=0}^{\infty} 1_{(S_m=0)} = \sum_{n=0}^{\infty} 1_{(\tau_n<\infty)}$$

be the number of visits to 0, counting the visit at time 0. Taking expected value and using Fubini's theorem to put the expected value inside the sum:

$$\begin{aligned} EV &= \sum_{m=0}^{\infty} P(S_m = 0) = \sum_{n=0}^{\infty} P(\tau_n < \infty) \\ &= \sum_{n=0}^{\infty} P(\tau_1 < \infty)^n = \frac{1}{1 - P(\tau_1 < \infty)} \end{aligned}$$

The second equality shows that (ii) implies (iii) and, in combination with the last two, shows that if (i) is false, then (iii) is false (i.e., (iii) implies (i)). ■

Theorem 4.2.3. *Simple random walk is recurrent in* $d \le 2$ *and transient in* $d \ge 3$.

To steal a joke from Kakutani (UCLA colloquium talk): "A drunk man will eventually find his way home, but a drunk bird may get lost forever."

Proof. Let $\rho_d(m) = P(S_m = 0)$. $\rho_d(m)$ is 0 if m is odd. From Theorem 3.1.3, we get $\rho_1(2n) \sim (\pi n)^{-1/2}$ as $n \to \infty$. This and Theorem 4.2.2 gives the result in one dimension. Our next step is

Simple random walk is recurrent in two dimensions. Note that in order for $S_{2n} = 0$, we must for some $0 \le m \le n$ have m up steps, m down steps, $n - m$ to the left, and $n-m$ to the right, so

$$\rho_2(2n) = 4^{-2n}\sum_{m=0}^{n}\frac{2n!}{m!\,m!\,(n-m)!\,(n-m)!}$$

$$= 4^{-2n}\binom{2n}{n}\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = 4^{-2n}\binom{2n}{n}^2 = \rho_1(2n)^2$$

To see the next-to-last equality, consider choosing n students from a class with n boys and n girls and observe that for some $0 \le m \le n$, you must choose m boys and $n-m$ girls. Using the asymptotic formula $\rho_1(2n) \sim (\pi n)^{-1/2}$, we get $\rho_2(2n) \sim (\pi n)^{-1}$. Since $\sum n^{-1} = \infty$, the result follows from Theorem 4.2.2.

Remark. For a direct proof of $\rho_2(2n) = \rho_1(2n)^2$, note that if T_n^1 and T_n^2 are independent, one-dimensional random walks, then T_n jumps from x to $x+(1,1)$, $x+(1,-1)$, $x+(-1,1)$, and $x+(-1,-1)$ with equal probability, so rotating T_n by 45 degrees and dividing by $\sqrt{2}$ gives S_n.

Simple random walk is transient in three dimensions. Intuitively, this holds since the probability of being back at 0 after $2n$ steps is $\sim cn^{-3/2}$, and this is summable. We will not compute the probability exactly but will get an upper bound of the right order of magnitude. Again, since the number of steps in the directions $\pm e_i$ must be equal for $i = 1, 2, 3$,

$$\rho_3(2n) = 6^{-2n}\sum_{j,k}\frac{(2n)!}{(j!k!(n-j-k)!)^2}$$

$$= 2^{-2n}\binom{2n}{n}\sum_{j,k}\left(3^{-n}\frac{n!}{j!k!(n-j-k)!}\right)^2$$

$$\le 2^{-2n}\binom{2n}{n}\max_{j,k} 3^{-n}\frac{n!}{j!k!(n-j-k)!}$$

where in the last inequality we have used the fact that if $a_{j,k}$ are ≥ 0 and sum to 1, then $\sum_{j,k} a_{j,k}^2 \le \max_{j,k} a_{j,k}$. Our last step is to show

$$\max_{j,k} 3^{-n}\frac{n!}{j!k!(n-j-k)!} \le Cn^{-1}$$

To do this, we note that (a) if any of the numbers j, k or $n-j-k$ is $< [n/3]$, increasing the smallest number and decreasing the largest number decreases the denominator (since $x(1-x)$ is maximized at 1/2), so the maximum occurs when all three numbers are as close as possible to $n/3$; (b) Stirling's formula implies

$$\frac{n!}{j!k!(n-j-k)!} \sim \frac{n^n}{j^j k^k (n-j-k)^{n-j-k}}\cdot\sqrt{\frac{n}{jk(n-j-k)}}\cdot\frac{1}{2\pi}$$

Taking j and k within 1 of $n/3$ the first term on the right is $\le C3^n$, and the desired result follows.

Simple random walk is transient in $d > 3$. Let $T_n = (S_n^1, S_n^2, S_n^3)$, $N(0) = 0$ and $N(n) = \inf\{m > N(n-1) : T_m \neq T_{N(n-1)}\}$. It is easy to see that $T_{N(n)}$ is a three-dimensional simple random walk. Since $T_{N(n)}$ returns infinitely often to 0 with probability 0 and the first three coordinates are constant in between the $N(n)$, S_n is transient. ■

Remark. Let $\pi_d = P(S_n = 0 \text{ for some } n \geq 1)$ be the probability that simple random walk on $\mathbf{Z}^d$ returns to 0. The last display in the proof of Theorem 4.2.2 implies

$$\sum_{n=0}^{\infty} P(S_{2n} = 0) = \frac{1}{1 - \pi_d} \tag{4.2.1}$$

In $d = 3$, $P(S_{2n} = 0) \sim Cn^{-3/2}$ so $\sum_{n=N}^{\infty} P(S_{2n} = 0) \sim C'N^{-1/2}$, and the series converges rather slowly. For example, if we want to compute the return probability to five decimal places, we would need 10^{10} terms. At the end of the section, we will give another formula that leads very easily to accurate results.

The rest of this section is devoted to proving the following facts about random walks:

- S_n is recurrent in $d = 1$ if $S_n/n \to 0$ in probability.
- S_n is recurrent in $d = 2$ if $S_n/n^{1/2} \Rightarrow$ a nondegenerate normal distribution.
- S_n is transient in $d \geq 3$ if it is "truly three-dimensional."

To prove the last result, we will give a necessary and sufficient condition for recurrence.

The first step in deriving these results is to generalize Theorem 4.2.2.

Lemma 4.2.4. *If $\sum_{n=1}^{\infty} P(\|S_n\| < \epsilon) < \infty$, then $P(\|S_n\| < \epsilon \text{ i.o.}) = 0$. If $\sum_{n=1}^{\infty} P(\|S_n\| < \epsilon) = \infty$ then $P(\|S_n\| < 2\epsilon \text{ i.o.}) = 1$.*

Proof. The first conclusion follows from the Borel-Cantelli lemma. To prove the second, let $F = \{\|S_n\| < \epsilon \text{ i.o.}\}^c$. Breaking things down according to the last time $\|S_n\| < \epsilon$,

$$\begin{aligned} P(F) &= \sum_{m=0}^{\infty} P(\|S_m\| < \epsilon, \|S_n\| \geq \epsilon \text{ for all } n \geq m+1) \\ &\geq \sum_{m=0}^{\infty} P(\|S_m\| < \epsilon, \|S_n - S_m\| \geq 2\epsilon \text{ for all } n \geq m+1) \\ &= \sum_{m=0}^{\infty} P(\|S_m\| < \epsilon)\rho_{2\epsilon,1} \end{aligned}$$

where $\rho_{\delta,k} = P(\|S_n\| \ge \delta \text{ for all } n \ge k)$. Since $P(F) \le 1$, and

$$\sum_{m=0}^{\infty} P(\|S_m\| < \epsilon) = \infty$$

it follows that $\rho_{2\epsilon,1} = 0$. To extend this conclusion to $\rho_{2\epsilon,k}$ with $k \ge 2$, let

$$A_m = \{\|S_m\| < \epsilon, \|S_n\| \ge \epsilon \text{ for all } n \ge m+k\}$$

Since any ω can be in at most k of the A_m, repeating the argument above gives

$$k \ge \sum_{m=0}^{\infty} P(A_m) \ge \sum_{m=0}^{\infty} P(\|S_m\| < \epsilon)\rho_{2\epsilon,k}$$

So $\rho_{2\epsilon,k} = P(\|S_n\| \ge 2\epsilon \text{ for all } j \ge k) = 0$, and since k is arbitrary, the desired conclusion follows. ■

Our second step is to show that the convergence or divergence of the sums in Lemma 4.2.4 is independent of ϵ. The previous proof works for any norm. For the next one, we need $\|x\| = \sup_i |x_i|$.

Lemma 4.2.5. *Let m be an integer ≥ 2.*

$$\sum_{n=0}^{\infty} P(\|S_n\| < m\epsilon) \le (2m)^d \sum_{n=0}^{\infty} P(\|S_n\| < \epsilon)$$

Proof. We begin by observing

$$\sum_{n=0}^{\infty} P(\|S_n\| < m\epsilon) \le \sum_{n=0}^{\infty} \sum_{k} P(S_n \in k\epsilon + [0,\epsilon)^d)$$

where the inner sum is over $k \in \{-m, \dots, m-1\}^d$. If we let

$$T_k = \inf\{\ell \ge 0 : S_\ell \in k\epsilon + [0,\epsilon)^d\}$$

then breaking things down according to the value of T_k and using Fubini's theorem gives

$$\begin{aligned}\sum_{n=0}^{\infty} P(S_n \in k\epsilon + [0,\epsilon)^d) &= \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} P(S_n \in k\epsilon + [0,\epsilon)^d, T_k = \ell) \\ &\le \sum_{\ell=0}^{\infty} \sum_{n=\ell}^{\infty} P(\|S_n - S_\ell\| < \epsilon, T_k = \ell)\end{aligned}$$

Since $\{T_k = \ell\}$ and $\{\|S_n - S_\ell\| < \epsilon\}$ are independent, the last sum

$$= \sum_{m=0}^{\infty} P(T_k = m) \sum_{j=0}^{\infty} P(\|S_j\| < \epsilon) \le \sum_{j=0}^{\infty} P(\|S_j\| < \epsilon)$$

Since there are $(2m)^d$ values of k in $\{-m, \dots, m-1\}^d$, the proof is complete. ■

Combining Lemmas 4.2.4 and 4.2.5 gives:

Theorem 4.2.6. *The convergence (resp. divergence) of* $\sum_n P(\|S_n\| < \epsilon)$ *for a single value of* $\epsilon > 0$ *is sufficient for transience (resp. recurrence).*

In $d = 1$, if $EX_i = \mu \neq 0$, then the strong law of large numbers implies $S_n/n \to \mu$, so $|S_n| \to \infty$ and S_n is transient. As a converse, we have

Theorem 4.2.7. Chung-Fuchs theorem. *Suppose* $d = 1$. *If the weak law of large numbers holds in the form* $S_n/n \to 0$ *in probability, then* S_n *is recurrent.*

Proof. Let $u_n(x) = P(|S_n| < x)$ for $x > 0$. Lemma 4.2.5 implies

$$\sum_{n=0}^{\infty} u_n(1) \geq \frac{1}{2m} \sum_{n=0}^{\infty} u_n(m) \geq \frac{1}{2m} \sum_{n=0}^{Am} u_n(n/A)$$

for any $A < \infty$ since $u_n(x) \geq 0$ and is increasing in x. By hypothesis $u_n(n/A) \to 1$, so letting $m \to \infty$ and noticing the right-hand side is $A/2$ times the average of the first Am terms

$$\sum_{n=0}^{\infty} u_n(1) \geq A/2$$

Since A is arbitrary, the sum must be ∞, and the desired conclusion follows from Theorem 4.2.6. ■

Theorem 4.2.8. *If* S_n *is a random walk in* $\mathbf{R}^2$ *and* $S_n/n^{1/2} \Rightarrow$ *a nondegenerate normal distribution, then* S_n *is recurrent.*

Remark. The conclusion is also true if the limit is degenerate, but in that case the random walk is essentially one- (or zero)-dimensional, and the result follows from the Chung-Fuchs theorem.

Proof. Let $u(n, m) = P(\|S_n\| < m)$. Lemma 4.2.5 implies

$$\sum_{n=0}^{\infty} u(n, 1) \geq (4m^2)^{-1} \sum_{n=0}^{\infty} u(n, m)$$

If $m/\sqrt{n} \to c$, then

$$u(n, m) \to \int_{[-c,c]^2} n(x)\, dx$$

where $n(x)$ is the density of the limiting normal distribution. If we use $\rho(c)$ to denote the right-hand side and let $n = [\theta m^2]$, it follows that $u([\theta m^2], m) \to \rho(\theta^{-1/2})$. If we write

$$m^{-2}\sum_{n=0}^{\infty} u(n,m) = \int_0^{\infty} u([\theta m^2], m)\, d\theta$$

let $m \to \infty$, and use Fatou's lemma, we get

$$\liminf_{m\to\infty} (4m^2)^{-1}\sum_{n=0}^{\infty} u(n,m) \geq 4^{-1}\int_0^{\infty} \rho(\theta^{-1/2})\, d\theta$$

Since the normal density is positive and continuous at 0,

$$\rho(c) = \int_{[-c,c]^2} n(x)\, dx \sim n(0)(2c)^2$$

as $c \to 0$. So $\rho(\theta^{-1/2}) \sim 4n(0)/\theta$ as $\theta \to \infty$, the integral diverges, and backtracking to the first inequality in the proof, it follows that $\sum_{n=0}^{\infty} u(n,1) = \infty$, proving the result. ∎

We come now to the promised necessary and sufficient condition for recurrence. Here $\phi = E\exp(it \cdot X_j)$ is the ch.f. of one step of the random walk.

Theorem 4.2.9. *Let $\delta > 0$. S_n is recurrent if and only if*

$$\int_{(-\delta,\delta)^d} Re\,\frac{1}{1-\varphi(y)}\, dy = \infty$$

We will prove a weaker result:

Theorem 4.2.10. *Let $\delta > 0$. S_n is recurrent if and only if*

$$\sup_{r<1}\int_{(-\delta,\delta)^d} Re\,\frac{1}{1-r\varphi(y)}\, dy = \infty$$

Remark. Half of the work needed to get the first result from the second is trivial.

$$0 \leq \text{Re}\,\frac{1}{1-r\varphi(y)} \to \text{Re}\,\frac{1}{1-\varphi(y)} \quad \text{as } r \to 1$$

so Fatou's lemma shows that if the integral is infinite, the walk is recurrent. The other direction is rather difficult: the second result is in Chung and Fuchs (1951), but a proof of the first result had to wait for Ornstein (1969) and Stone (1969) to solve the problem independently. Their proofs use a trick to reduce to the case where the increments have a density and then a second trick to deal with that case, so we will not give the details here. The reader can consult either of the sources cited or Port and Stone (1969), where the result is demonstrated for random walks on Abelian groups.

Proof. The first ingredient in the solution is the

Lemma 4.2.11. Parseval relation. *Let μ and ν be probability measures on $\mathbf{R}^d$ with ch.f.'s φ and ψ.*

$$\int \psi(t)\,\mu(dt) = \int \varphi(x)\,\nu(dx)$$

Proof. Since $e^{it\cdot x}$ is bounded, Fubini's theorem implies

$$\int \psi(t)\mu(dt) = \iint e^{itx}\nu(dx)\mu(dt) = \iint e^{itx}\mu(dt)\nu(dx) = \int \varphi(x)\nu(dx) \quad \blacksquare$$

Our second ingredient is a little calculus.

Lemma 4.2.12. *If $|x| \le \pi/3$ then $1 - \cos x \ge x^2/4$.*

Proof. It suffices to prove the result for $x > 0$. If $z \le \pi/3$, then $\cos z \ge 1/2$,

$$\sin y = \int_0^y \cos z\,dz \ge \frac{y}{2}$$

$$1 - \cos x = \int_0^x \sin y\,dy \ge \int_0^x \frac{y}{2}\,dy = \frac{x^2}{4}$$

which proves the desired result. ■

From Example 3.3.5, we see that the density

$$\frac{\delta - |x|}{\delta^2} \quad \text{when} \quad |x| \le \delta, \qquad 0 \quad \text{otherwise}$$

has ch.f. $2(1 - \cos \delta t)/(\delta t)^2$. Let μ_n denote the distribution of S_n. Using Lemma 4.2.12 (note $\pi/3 \ge 1$) and then Lemma 4.2.11, we have

$$\begin{aligned} P(\|S_n\| < 1/\delta) &\le 4^d \int \prod_{i=1}^d \frac{1 - \cos(\delta t_i)}{(\delta t_i)^2}\mu_n(dt) \\ &= 2^d \int_{(-\delta,\delta)^d} \prod_{i=1}^d \frac{\delta - |x_i|}{\delta^2}\varphi^n(x)\,dx \end{aligned}$$

Our next step is to sum from 0 to ∞. To be able to interchange the sum and the integral, we first multiply by r^n, where $r < 1$:

$$\sum_{n=0}^{\infty} r^n P(\|S_n\| < 1/\delta) \le 2^d \int_{(-\delta,\delta)^d} \prod_{i=1}^d \frac{\delta - |x_i|}{\delta^2}\frac{1}{1 - r\varphi(x)}\,dx$$

Symmetry dictates that the integral on the right is real, so we can take the real part without affecting its value. Letting $r \uparrow 1$ and using $(\delta - |x|)/\delta \le 1$

$$\sum_{n=0}^{\infty} P(\|S_n\| < 1/\delta) \le \left(\frac{2}{\delta}\right)^d \sup_{r<1} \int_{(-\delta,\delta)^d} \text{Re}\,\frac{1}{1 - r\varphi(x)}\,dx$$

and using Theorem 4.2.6 gives half of Theorem 4.2.10.

To prove the other direction, we begin by noting that Example 3.3.8 shows that the density $(1 - \cos(x/\delta))/\pi x^2/\delta$ has ch.f. $1 - |\delta t|$ when $|t| \le 1/\delta$, 0 otherwise. Using $1 \ge \prod_{i=1}^d (1 - |\delta x_i|)$ and then Lemma 4.2.11,

$$\begin{aligned} P(\|S_n\| < 1/\delta) &\ge \int_{(-1/\delta,1/\delta)^d} \prod_{i=1}^d (1 - |\delta x_i|)\,\mu_n(dx) \\ &= \int \prod_{i=1}^d \frac{1 - \cos(t_i/\delta)}{\pi t_i^2/\delta}\,\varphi^n(t)\,dt \end{aligned}$$

Multiplying by r^n and summing gives

$$\sum_{n=0}^{\infty} r^n P(\|S_n\| < 1/\delta) \ge \int \prod_{i=1}^d \frac{1 - \cos(t_i/\delta)}{\pi t_i^2/\delta}\,\frac{1}{1 - r\varphi(t)}\,dt$$

The last integral is real, so its value is unaffected if we integrate only the real part of the integrand. If we do this and apply Lemma 4.2.12, we get

$$\sum_{n=0}^{\infty} r^n P(\|S_n\| < 1/\delta) \ge (4\pi\delta)^{-d} \int_{(-\delta,\delta)^d} \text{Re}\,\frac{1}{1 - r\varphi(t)}\,dt$$

Letting $r \uparrow 1$ and using Theorem 4.2.6 now completes the proof of Theorem 4.2.10. ■

We will now consider some examples. Our goal in $d = 1$ and $d = 2$ is to convince you that the conditions in Theorems 4.2.7 and 4.2.8 are close to the best possible.

$d = 1$. Consider the symmetric stable laws that have ch.f. $\varphi(t) = \exp(-|t|^\alpha)$. To avoid using facts that we have not proved, we will obtain our conclusions from Theorem 4.2.10. It is not hard to use that form of the criterion in this case since

$$\begin{aligned} 1 - r\varphi(t) &\downarrow 1 - \exp(-|t|^\alpha) \qquad &\text{as } r \uparrow 1 \\ 1 - \exp(-|t|^\alpha) &\sim |t|^\alpha \qquad &\text{as } t \to 0 \end{aligned}$$

From this, it follows that the corresponding random walk is transient for $\alpha < 1$ and recurrent for $\alpha \ge 1$. The case $\alpha > 1$ is covered by Theorem 4.2.7 since these random walks have mean 0. The result for $\alpha = 1$ is new because the Cauchy distribution does not satisfy $S_n/n \to 0$ in probability. The random walks with $\alpha < 1$ are interesting because Theorem 4.1.2 implies (see Exercise 4.1.1)

$$-\infty = \liminf S_n < \limsup S_n = \infty$$

but $P(|S_n| < M \text{ i.o.}) = 0$ for any $M < \infty$.

Remark. The stable law examples are misleading in one respect. Shepp (1964) proved that recurrent random walks may have arbitrarily large tails. To be precise, given a function $\epsilon(x) \downarrow 0$ as $x \uparrow \infty$, there is a recurrent random walk with $P(|X_1| \geq x) \geq \epsilon(x)$ for large x.

d = 2. Let $\alpha < 2$, and let $\varphi(t) = \exp(-|t|^\alpha)$ where $|t| = (t_1^2 + t_2^2)^{1/2}$. φ is the characteristic function of a random vector (X_1, X_2) that has two nice properties:

(i) the distribution of (X_1, X_2) is invariant under rotations,
(ii) X_1 and X_2 have symmetric stable laws with index α.

Again, $1 - r\varphi(t) \downarrow 1 - \exp(-|t|^\alpha)$ as $r \uparrow 1$ and $1 - \exp(-|t|^\alpha) \sim |t|^\alpha$ as $t \to 0$. Changing to polar coordinates and noticing

$$2\pi \int_0^\delta dx\, x\, x^{-\alpha} < \infty$$

when $1 - \alpha > -1$ shows that the random walks with ch.f. $\exp(-|t|^\alpha)$, $\alpha < 2$ are transient. When $p < \alpha$, we have $E|X_1|^p < \infty$ by Exercise 3.7.5, so these examples show that Theorem 4.2.8 is reasonably sharp.

d ≥ 3. The integral $\int_0^\delta dx\, x^{d-1} x^{-2} < \infty$, so if a random walk is recurrent in $d \geq 3$, its ch.f. must $\to 1$ faster than t^2. In Exercise 3.3.19, we observed that (in one dimension) if $\varphi(r) = 1 + o(r^2)$, then $\varphi(r) \equiv 1$. By considering $\varphi(r\theta)$ where r is real and θ is a fixed vector, the last conclusion generalizes easily to $\mathbf{R}^d$, $d > 1$, and suggests that once we exclude walks that stay on a plane through 0, no three-dimensional random walks are recurrent.

A random walk in $\mathbf{R}^3$ is **truly three-dimensional** if the distribution of X_1 has $P(X_1 \cdot \theta \neq 0) > 0$ for all $\theta \neq 0$.

Theorem 4.2.13. *No truly three-dimensional random walk is recurrent.*

Proof. We will deduce the result from Theorem 4.2.10. We begin with some arithmetic. If z is complex, the conjugate of $1 - z$ is $1 - \bar{z}$, so

$$\frac{1}{1-z} = \frac{1-\bar{z}}{|1-z|^2} \qquad \text{and} \qquad \text{Re}\,\frac{1}{1-z} = \frac{\text{Re}\,(1-z)}{|1-z|^2}$$

If $z = a + bi$ with $a \leq 1$, then using the previous formula and dropping the b^2 from the denominator,

$$\text{Re}\,\frac{1}{1-z} = \frac{1-a}{(1-a)^2 + b^2} \leq \frac{1}{1-a}$$

Taking $z = r\phi(t)$ and supposing for the second inequality that $0 \leq \text{Re}\,\phi(t) \leq 1$, we have

$$\text{(a)} \qquad \text{Re}\,\frac{1}{1 - r\varphi(t)} \leq \frac{1}{\text{Re}\,(1 - r\varphi(t))} \leq \frac{1}{\text{Re}\,(1 - \varphi(t))}$$

The last calculation shows that it is enough to estimate

$$\operatorname{Re}(1-\varphi(t)) = \int \{1-\cos(x\cdot t)\}\mu(dx) \geq \int_{|x\cdot t|<\pi/3} \frac{|x\cdot t|^2}{4}\,\mu(dx)$$

by Lemma 4.2.12. Writing $t = \rho\theta$ where $\theta \in S = \{x : |x| = 1\}$ gives

(b) $$\operatorname{Re}(1-\varphi(\rho\theta)) \geq \frac{\rho^2}{4}\int_{|x\cdot\theta|<\pi/3\rho} |x\cdot\theta|^2\mu(dx)$$

Fatou's lemma implies that if we let $\rho \to 0$ and $\theta(\rho) \to \theta$, then

(c) $$\liminf_{\rho\to 0} \int_{|x\cdot\theta(\rho)|<\pi/3\rho} |x\cdot\theta(\rho)|^2\mu(dx) \geq \int |x\cdot\theta|^2\mu(dx) > 0$$

I claim that this implies that for $\rho < \rho_0$

(d) $$\inf_{\theta\in S} \int_{|x\cdot\theta|<\pi/3\rho} |x\cdot\theta|^2\mu(dx) = C > 0$$

To get the last conclusion, observe that if it is false, then for $\rho = 1/n$ there is a θ_n so that

$$\int_{|x\cdot\theta_n|<n\pi/3} |x\cdot\theta_n|^2\mu(dx) \leq 1/n$$

All the θ_n lie in S, a compact set, so if we pick a convergent subsequence, we contradict (c). Combining (b) and (d) gives

$$\operatorname{Re}(1-\varphi(\rho\theta)) \geq C\rho^2/4$$

Using the last result and (a) then changing to polar coordinates, we see that if δ is small (so $\operatorname{Re}\phi(y) \geq 0$ on $(-\delta,\delta)^d$)

$$\int_{(-\delta,\delta)^d} \operatorname{Re}\frac{1}{1-r\phi(y)}\,dy \leq \int_0^{\delta\sqrt{d}} d\rho\,\rho^{d-1}\int d\theta\,\frac{1}{\operatorname{Re}(1-\phi(\rho\theta))}$$
$$\leq C'\int_0^1 d\rho\,\rho^{d-3} < \infty$$

when $d > 2$, so the desired result follows from Theorem 4.2.10. ■

Remark. The analysis becomes much simpler when we consider random walks on $\mathbf{Z}^d$. The inversion formula given in Exercise 3.3.2 implies

$$P(S_n = 0) = (2\pi)^{-d}\int_{(-\pi,\pi)^d} \varphi^n(t)\,dt$$

Multiplying by r^n and summing gives

$$\sum_{n=0}^{\infty} r^n P(S_n = 0) = (2\pi)^{-d}\int_{(-\pi,\pi)^d} \frac{1}{1-r\varphi(t)}\,dt$$

In the case of simple random walk in $d = 3$, $\phi(t) = \frac{1}{3}\sum_{j=1}^{3} \cos t_j$ is real.

$$\frac{1}{1 - r\phi(t)} \uparrow \frac{1}{1 - \phi(t)} \quad \text{when } \phi(t) > 0$$

$$0 \le \frac{1}{1 - r\phi(t)} \le 1 \quad \text{when } \phi(t) \le 0$$

So, using the monotone and bounded convergence theorems

$$\sum_{n=0}^{\infty} P(S_n = 0) = (2\pi)^{-3} \int_{(-\pi,\pi)^3} \left(1 - \frac{1}{3}\sum_{i=1}^{3} \cos x_i\right)^{-1} dx$$

This integral was first evaluated by Watson in 1939 in terms of elliptic integrals, which could be found in tables. Glasser and Zucker (1977) showed that it was

$$(\sqrt{6}/32\pi^3)\Gamma(1/24)\Gamma(5/24)\Gamma(7/24)\Gamma(11/24) = 1.516386059137\ldots$$

so it follows from (4.2.1) that

$$\pi_3 = 0.340537329544 \ldots$$

For numerical results in $4 \le d \le 9$, see Kondo and Hara (1987).

4.3 Visits to 0, Arcsine Laws*

In the last section, we took a broad look at the recurrence of random walks. In this section, we will take a deep look at one example: simple random walk (on **Z**). To steal a line from Chung, "We shall treat this by combinatorial methods as an antidote to the analytic skulduggery above." The developments here follow Chapter III of Feller, vol. I. To facilitate discussion, we will think of the sequence $S_1, S_2, \ldots, S_n$ as being represented by a polygonal line with segments $(k-1, S_{k-1}) \to (k, S_k)$. A **path** is a polygonal line that is a possible outcome of simple random walk. To count the number of paths from (0,0) to (n, x), it is convenient to introduce a and b defined as follows: $a = (n+x)/2$ is the number of positive steps in the path and $b = (n-x)/2$ is the number of negative steps. Notice that $n = a + b$ and $x = a - b$. If $-n \le x \le n$ and $n - x$ is even, the a and b defined above are nonnegative integers, and the number of paths from (0,0) to (n, x) is

$$N_{n,x} = \binom{n}{a} \tag{4.3.1}$$

Otherwise, the number of paths is 0.

Theorem 4.3.1. Reflection principle. *If $x, y > 0$, then the number of paths from $(0, x)$ to (n, y) that are 0 at some time is equal to the number of paths from $(0, -x)$ to (n, y).*

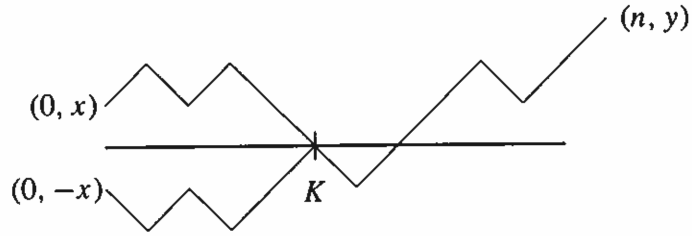

Figure 4.1. Reflection principle.

Proof. Suppose $(0, s_0), (1, s_1), \ldots, (n, s_n)$ is a path from $(0, x)$ to (n, y). Let $K = \inf\{k : s_k = 0\}$. Let $s'_k = -s_k$ for $k \le K$, $s'_k = s_k$ for $K \le k \le n$. Then (k, s'_k), $0 \le k \le n$, is a path from $(0, -x)$ to (n, y). Conversely, if $(0, t_0), (1, t_1), \ldots, (n, t_n)$ is a path from $(0, -x)$ to (n, y), then it must cross 0. Let $K = \inf\{k : t_k = 0\}$. Let $t'_k = -t_k$ for $k \le K$, $t'_k = t_k$ for $K \le k \le n$. Then (k, t'_k), $0 \le k \le n$, is a path from $(0, -x)$ to (n, y) that is 0 at time K. The last two observations set up a one-to-one correspondence between the two classes of paths, so their numbers must be equal. ■

From Theorem 4.3.1 we get a result first proved in 1878.

Theorem 4.3.2. Ballot theorem. *Suppose that in an election candidate A gets α votes, and candidate B gets β votes where $\beta < \alpha$. The probability that throughout the counting A always leads B is $(\alpha - \beta)/(\alpha + \beta)$.*

Proof. Let $x = \alpha - \beta$, $n = \alpha + \beta$. Clearly, there are as many such outcomes as there are paths from (1,1) to (n, x) that are never 0. The reflection principle implies that the number of paths from (1,1) to (n, x) that are 0 at some time the number of paths from (1,−1) to (n, x), so by (4.3.1) the number of paths from (1,1) to (n, x) that are never 0 is

$$\begin{aligned} N_{n-1,x-1} - N_{n-1,x+1} &= \binom{n-1}{\alpha-1} - \binom{n-1}{\alpha} \\ &= \frac{(n-1)!}{(\alpha-1)!(n-\alpha)!} - \frac{(n-1)!}{\alpha!(n-\alpha-1)!} \\ &= \frac{\alpha-(n-\alpha)}{n} \cdot \frac{n!}{\alpha!(n-\alpha)!} = \frac{\alpha-\beta}{\alpha+\beta} N_{n,x} \end{aligned}$$

since $n = \alpha + \beta$, this proves the desired result. ■

Using the ballot theorem, we can compute the distribution of the time to hit 0 for simple random walk.

Lemma 4.3.3. $P(S_1 \neq 0, \ldots, S_{2n} \neq 0) = P(S_{2n} = 0)$.