7th Homework Set — Solutions

Chapter 5

Problem 5.37 Let X be uniformly distributed over $(-1, 1)$.

(a) $P\left\{|X| > \frac{1}{2}\right\} = P\left\{X > \frac{1}{2}\right\} + P\left\{X < -\frac{1}{2}\right\} = \frac{1}{2}$

(b) Let $Y = |X|$. If $y \in (0, 1)$, then

$$F_Y(y) = P\{Y \leq y\} = P\{-y \leq Y \leq y\} = y,$$

so that

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5.39 Let X be exponential with $\lambda = 1$, and let $Y = \log X$. Then $F_Y(y) = P\{Y \leq y\} = P\{\log X \leq y\} = P\{X \leq e^y\} = 1 - e^{-e^y}$, so that

$$f_Y(y) = e^{y - e^y}.$$

Problem 5.40 Let X be uniform on $(0, 1)$, and $Y = e^X$. Then, for $1 < y < e$, $F_Y(y) = P\{Y \leq y\} = P\left\{e^X \leq y\right\} = P\{X \leq \log Y\} = \log Y$, so that

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

Problem 5.41 For any $r \in (-A, A)$, we have $F_R r = P\{R \leq r\} = P\{A \sin\theta \leq r\} = P\left\{\theta \leq \arcsin\frac{r}{A}\right\} = \frac{1}{\pi}\arcsin\frac{r}{A}$, so that

$$f_R(r) = \begin{cases} \frac{1}{\pi\sqrt{A^2 - r^2}} & r \in (-A, A) \\ 0 & \text{otherwise} \end{cases}$$

Chapter 6

Problem 6.2 (a)

$P\{X_1 = i, X_2 = j\}$	$j = 0$	$j = 1$	$P\{X_1 = i\}$
$i = 0$	$\frac{8}{13}\frac{7}{12} = \frac{14}{39}$	$\frac{8}{13}\frac{5}{12} = \frac{10}{39}$	$\frac{24}{39}$
$i = 1$	$\frac{5}{13}\frac{8}{12} = \frac{10}{39}$	$\frac{5}{13}\frac{4}{12} = \frac{5}{39}$	$\frac{15}{39}$
$P\{X_2 = j\}$	$\frac{24}{39}$	$\frac{15}{39}$	1

(b)

$$
\begin{aligned}
P\{X_1=0,X_2=0,X_3=0\} &= \frac{8}{13}\frac{7}{12}\frac{6}{11} = \frac{28}{143}\\
P\{X_1=0,X_2=0,X_3=1\} &= \frac{8}{13}\frac{7}{12}\frac{5}{11} = \frac{70}{429}\\
P\{X_1=0,X_2=1,X_3=0\} &= \frac{8}{13}\frac{5}{12}\frac{7}{11} = \frac{70}{429}\\
P\{X_1=1,X_2=0,X_3=0\} &= \frac{5}{13}\frac{8}{12}\frac{7}{11} = \frac{70}{429}\\
P\{X_1=0,X_2=1,X_3=1\} &= \frac{8}{13}\frac{5}{12}\frac{4}{11} = \frac{40}{429}\\
P\{X_1=1,X_2=0,X_3=1\} &= \frac{5}{13}\frac{8}{12}\frac{4}{11} = \frac{40}{429}\\
P\{X_1=1,X_2=1,X_3=0\} &= \frac{5}{13}\frac{4}{12}\frac{8}{11} = \frac{40}{429}\\
P\{X_1=1,X_2=1,X_3=1\} &= \frac{5}{13}\frac{4}{12}\frac{3}{11} = \frac{5}{143}
\end{aligned}
$$

Problem 6.7 $P\{X_1=i,X_2=j\}=p^2(1-p)^{i+j}$

Problem 6.8 X, Y are jointly continuous with probability density function

$$
f(x,y)=\begin{cases} c(y^2-x^2)e^{-y} & -y\le x\le y, 0<y<\infty \\ 0 & \text{otherwise.}\end{cases}
$$

(a) Note that

$$
\int\int_{R^2} f(x,y) = \int_0^\infty \int_{-y}^{y} c(y^2-x^2)e^{-y}dxdy = 8c,
$$

so that $c=\frac{1}{8}$.

(b)

$$
f_X(x)=\frac{1}{8}\int_{|x|}^{\infty}(y^2-x^2)e^{-y}dy = \frac{(|x|+1)e^{-|x|}}{4}
$$

$$
f_Y(y)=\frac{1}{8}\int_{-y}^{y}(y^2-x^2)e^{-y}dx=\frac{1}{6}y^3e^{-y} \qquad \text{for } y>0
$$

(c) $E[X] = 0$ by symmetry.

Problem 6.9 Let X, Y be jointly continuous with joint density function $f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$ for $0 < x < 1$, $0 < y < 2$.

(a)

$$\int_0^1 \int_0^2 x^2 + \frac{xy}{2} dydx = \int_0^1 2x^2 + xdx = \frac{7}{6}$$

(b)

$$f_X(x) = \frac{6}{7}x(2x+1) \qquad \text{for } 0 < x < 1$$

(c)

$$P\{X > Y\} = \int_0^1 \int_0^x f(x,y)dydx = \frac{15}{56}$$

(d)

$$P\left\{Y > \frac{1}{2} | X < \frac{1}{2}\right\} = \frac{P\left\{X < \frac{1}{2}, Y > \frac{1}{2}\right\}}{P\left\{X < \frac{1}{2}\right\}}$$

$$= \frac{\int_{\frac{1}{2}}^2 \int_0^{\frac{1}{2}} f(x,y)dxdy}{\int_0^{\frac{1}{2}} f_X(x)dx} = 0.8625$$

(e)

$$E[X] = \int_0^1 xf_X(x)dx = \frac{5}{7}$$

(f)

$$E[Y] = \int_0^2 y \int_0^1 f(x,y)dxdy = \frac{8}{7}$$

Problem 6.10 Let X, Y be jointly distributed with density function $f(x,y) = e^{-(x+y)}$ for $0 \le x < \infty$, $0 \le y < \infty$.

(a) $P\{X < Y\} = \frac{1}{2}$ by symmetry

(b) $P\{X < a\} = \int_0^a \int_0^\infty e^{-(x+y)}dydx = 1 - e^{-a}$