Sixth Homework Set — Solutions

Chapter 5

Problem 5.6 (a)

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{4}\int_0^{\infty} x^2 e^{-\frac{x}{2}}dx$$
$$= \frac{1}{4}\left(-2x^2 - 8x - 16\right) e^{-\frac{x}{2}}|_0^{\infty} = 4$$

(b) $E[X] = \int_{-1}^{1} c(1-x^2)xdx = 0$ by symmetry

(c) $E[X] = \int_5^{\infty} x\frac{5}{x^2}dx = \int_5^{\infty} \frac{5}{x} = \infty$

Problem 5.10 (a) Let X be uniform on $[0, 60]$. Then

$$\begin{aligned}&P\,(\text{passenger goes to } A)\\ &= P\{5 \le X < 15\} + P\{20 \le X < 30\}\, P\{35 \le X < 45\}\\ &\quad + P\{50 \le X < 60\}\\ &= \frac{2}{3}.\end{aligned}$$

(b) Same as above.

Problem 5.12 If service stations are located in A, B, and the center, then the distance between two service stations is 50 miles, so that the expected distance from a service station at the time of a breakdown is

$$\frac{1}{50}\left(\int_0^{25} xdx + \int_{25}^{50}(50-x)dx\right)$$
$$= \frac{1}{50}\left(\frac{25^2}{2} + 25\cdot 50 - \frac{50^2}{2} + \frac{25^2}{2}\right) = 12.5.$$

If the service stations are located at mile 25, 50, and 75, then the expected distance from a station at the time of a breakdown is

$$\frac{1}{50}\left(\int_0^{25} xdx + \int_{25}^{37.5}(x-25)xdx + \int_{37.5}^{50}(50-x)dx\right)$$
$$= \frac{1}{50}\left(\frac{25^2}{2} + 2\frac{12.5^2}{2}\right) = 9.375.$$

The second strategy is more efficient.

Problem 5.13 (a) $P\{X > 10\} = \frac{2}{3}$

(b) $P\{X > 25|X > 15\} = \frac{P\{X>25\}}{P\{X>15\}} = \frac{\frac{5}{30}}{\frac{15}{30}} = \frac{1}{3}$.

Problem 5.15 (a) $P\{X > 5\} = P\left\{\frac{X-10}{6} > \frac{5-10}{6}\right\} = 1 - \Phi\left(-\frac{5}{6}\right) = \Phi\left(\frac{5}{6}\right) = 0.7977$

(b)

$$P\{4 < X < 16\} = P\left\{-1 < \frac{X-10}{6} < 1\right\} = \Phi(1) - \Phi(-1)$$
$$= 2\Phi(1) - 1 = 0.6827$$

(c)

$$P\{X < 8\} = P\left\{\frac{X-10}{6} < -\frac{1}{3}\right\}$$
$$= \Phi\left(-\frac{1}{3}\right) = 1 - \Phi\left(\frac{1}{3}\right) = 0.3695$$

(d) $P\{X < 20\} = P\left\{\frac{X-10}{6} < -\frac{10}{6}\right\} = \Phi\left(\frac{5}{3}\right) = 0.9522$

(e) $P\{X > 16\} = P\left\{\frac{X-10}{6} > 1\right\} = 1 - \Phi(1) = 0.1587$

Problem 5.18 We have $P\{X > 9\} = P\left\{\frac{X-5}{\sigma} > \frac{4}{\sigma}\right\} = 1 - \Phi\left(\frac{4}{\sigma}\right) = 0.2$, so that $\Phi\left(\frac{4}{\sigma}\right) = 0.8$, hence $\frac{4}{\sigma} = 0.85$. This implies that $\sigma = 4.7059$, so that the variance is $\sigma^2 = 22.145$.

Problem 5.21 Let X be a normal random variable with $\mu = 71$ and $\sigma^2 = 6.25$. Then $P\{X > 74\} = P\left\{\frac{X-71}{2.5} > \frac{3}{2.5}\right\} = 1 - \Phi\left(\frac{6}{5}\right) = 0.1151$. Moreover, $P\{X > 77|X \geq 72\} = \frac{P\left\{\frac{X-71}{2.5} > \frac{6}{2.5}\right\}}{P\left\{\frac{X-71}{2.5} \geq \frac{1}{2.5}\right\}} = \frac{1-\Phi\left(\frac{12}{5}\right)}{1-\Phi\left(\frac{2}{5}\right)} = 0.024$.

Problem 5.22 Let X be normal with $\mu = 0.9$ and $\sigma = 0.003$.

(a) $P\{|X - 0.9| > 0.005\} = P\left\{\frac{|X-0.9|}{0.003} > \frac{5}{3}\right\} = 2 - 2\Phi\left(\frac{5}{3}\right) = 0.095$.

(b) We want $P\left\{\frac{|X-0.9|}{\sigma} > 0.005\right\} = 2 - 2\Phi\left(\frac{0.005}{\sigma}\right) \leq 0.01$, hence $\Phi\left(\frac{0.005}{\sigma}\right) \geq 0.995$, so that $\frac{0.005}{\sigma} \geq 2.58$, hence $\sigma = 0.0019$.

Problem 5.23 Let X be the number of times the number six appears.

$$\begin{aligned}
&P\{149.5 < X < 200.5\} \\
&= P\left\{\frac{149.5 - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}} < \frac{X - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}} < \frac{200.5 - \frac{5000}{36}}{\sqrt{\frac{5000}{36}}}\right\} \\
&= \Phi(2.87) + \Phi(1.46) - 1 = 0.9258.
\end{aligned}$$

$$P\{X < 149.5\} = P\left\{\frac{X - \frac{800}{5}}{\sqrt{\frac{3200}{25}}} < \frac{149.5 - \frac{800}{5}}{\sqrt{320025}}\right\} = 1 - \Phi(0.92) = 0.1762.$$

Problem 5.25 Let X be a binomial random variable with $p = 0.05$ and $n = 150$. Then $P\{X \leq 10\} = P\{X \leq 10.5\} = P\left\{\frac{X-7.5}{\sqrt{7.125}} \leq \frac{10.5-7.5}{\sqrt{7.125}}\right\} = \Phi(1.1239) = 0.8695$, using DeMoivre-Laplace.

Problem 5.28 Let X be the number of lefthanders. Then X is binomial with $p = 0.12$ and $n = 200$. Then

$$\begin{aligned}
&P\{X \geq 20\} = P\{X > 19.5\} \\
&= P\left\{\frac{X - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}} > \frac{19.5 - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}}\right\} \\
&= 1 - \Phi(-0.9792) = \Phi(0.9792) = 0.8363.
\end{aligned}$$

Problem 5.32 Let X be exponential with parameter $\lambda = \frac{1}{2}$.

(a) $P\{X > 2\} = 1 - F(2) = e^{-1}$

(b) $P\{X > 10 | X > 9\} = P\{X > 1\} = 1 - F(1) = e^{-\frac{1}{2}}$ because X is memoryless.

Problem 5.33 Let X be an exponential random variable with parameter $\lambda = \frac{1}{8}$. Since X is memoryless, we have $P\{X > t + 8 | X > t\} = P\{X > 8\} = e^{-1}$.

Problem 5.34 Let X be an exponential random variable with parameter $\lambda = \frac{1}{20}$. Since X is memoryless, we have $P\{X > 30 | X > 10\} = P\{X > 20\} = e^{-1}$.

Let Y be a uniform random variable on $[0, 40]$. Then

$$P\{X > 30 | X > 10\} = \frac{P\{X > 30\}}{P\{X > 10\}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$