Fifth Homework Set — Solutions
Chapter 4

Problem 72 Let A be the stronger team. $P\left(A \text{ wins in } i \text{ games}\right) = \binom{i-1}{i-4} 0.6^i 0.4^{i-4}$, for $i = 4, \ldots, 7$. Hence

$$P\left(A \text{ wins best-of-seven series}\right) = \sum_{i=4}^{7} \binom{i-1}{i-4} 0.6^4 0.4^{i-4} = 0.7102.$$

Similarly,

$$P\left(A \text{ wins best-of-three series}\right) = \sum_{i=2}^{3} \binom{i-1}{i-2} 0.6^4 0.4^{i-2} = 0.6480.$$

Problem 73 Let X be the number of games played in a match. Then $P\left\{X = i\right\} = 2\binom{i-1}{i-4}\left(\frac{1}{2}\right)^i$ for $i = 4, \ldots, 7$. Hence, $E[X] = 2\sum_{i=4}^{7} i\binom{i-1}{i-4}\left(\frac{1}{2}\right)^i = 5.8125$.

Problem 77 Let E be the event that right-hand box is emptied while the left-hand box still contains k matches. Then, using a negative binomial random variable with $p = \frac{1}{2}$, $r = N$, and $n = 2N - k$, we see that $P\left(E\right) = \binom{2N-k-1}{N-1}\left(\frac{1}{2}\right)^{2N-k}$. Now the desired probability is $2P\left(E\right)$.

Problem 78 Let E be the event that a single drawing results in two white and two black balls. Then $P\left(E\right) = \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{18}{35}$.

Let X be the number of selections until E occurs. Then

$$P\left(X = n\right) = \frac{17^{n-1} \cdot 18}{35^n}.$$

Problem 79 (a) $P(X = 0) = \frac{\binom{94}{10}}{\binom{100}{10}} = 0.5223$

(b)

$$\begin{aligned} P(X > 2) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ &= \frac{\binom{100}{10} - \binom{94}{10} - \binom{6}{1}\binom{94}{9} - \binom{6}{2}\binom{94}{8}}{\binom{100}{10}} = 0.0126 \end{aligned}$$

Problem 84 (a) For $i = 1, \ldots, 5$, let $X_i = 1$ if the i-th box is empty and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_5$ is the number of empty boxes. For $i = 1, \ldots, 5$,

$$E[X_i] = P(X_i = 1) = (1 - p_i)^{10}.$$

Thus

$$E[X] = E[X_1] + \cdots + E[X_5] = \sum_{i=1}^{5} (1 - p_i)^{10}.$$

(b) For $i = 1, \ldots, 5$, let $Y_i = 1$ if the i-th box has exactly 1 ball and $Y_i = 0$ otherwise. Then $Y = Y_1 + \cdots + Y_5$ is the number of boxes that have exactly 1 ball. For $i = 1, \ldots, 5$,

$$E[Y_i] = P(Y_i = 1) = 10p_i (1 - p_i)^9.$$

Thus

$$E[Y] = E[Y_1] + \cdots + E[Y_5] = \sum_{i=1}^{5} 10p_i (1 - p_i)^9.$$

Problem 85 For $i = 1, \ldots, k$, let $X_i = 1$ if the i-th type appear at least once in the set of n coupons. Then $X = X_1 + \cdots + X_k$ is the number of distinct types that appear in this set. For $i = 1, \ldots, k$,

$$E[X_i] = P(X_i = 1) = 1 - P(X_i = 0) = 1 - (1 - p_i)^n.$$

Thus

$$E[X] = E[X_1] + \cdots + E[X_k] = k - \sum_{i=1}^{k} (1 - p_i)^n.$$

Chapter 5

Problem 1 (a) We have $1 = \int_{-1}^{1} c(1 - x^2)dx = cx\left(1 - \frac{x^2}{3}\right)\Big|_{-1}^{1} = \frac{4}{3}c$, so that $c = \frac{3}{4}$.

(b) We have $\int_{-1}^{x} f(y)dy = \frac{3}{4}y\left(1 - \frac{y^2}{3}\right)\Big|_{-1}^{x} = \frac{1}{2} + \frac{3}{4}x\left(1 - \frac{x^2}{3}\right)$ if $-1 \le x \le 1$. Hence,

$$F(x) = \begin{cases} 0 & x < -1, \\ \frac{1}{2} + \frac{3}{4}x\left(1 - \frac{x^2}{3}\right) & -1 \le x \le 1, \\ 1 & x > 1. \end{cases}$$

Problem 2 Determine C: $\int_0^\infty xe^{-\frac{x}{2}}dx = -2xe^{-\frac{x}{2}}|_0^\infty + \int_0^\infty 2e^{-\frac{x}{2}}dx = (-2x-4)e^{-\frac{x}{2}}|_0^\infty = 4$, so that $C = \frac{1}{4}$.

Now, we have $P(X \geq 5) = \int_5^\infty \frac{1}{4}xe^{-\frac{x}{2}} = -(\frac{x}{2}+1)e^{-\frac{x}{2}}|_5^\infty = \frac{7}{2}e^{-\frac{5}{2}}$

Problem 4 (a) $P(X > 20) = \int_{20}^\infty \frac{10}{x^2}dx = -\frac{10}{x}|_{20}^\infty = \frac{1}{2}$.

(b)

$$F(x) = \begin{cases} 0 & x < 10 \\ 1 - \frac{10}{x} & x \geq 10 \end{cases}$$

(c) Let's assume that lifetimes of the six devices are independent of each other. Let $p = 1 - F(15)$. Then the desired probability is

$$\sum_{i=3}^{6} \binom{6}{i} p^i (1-p)^{6-i}.$$

Problem 5 We want to find C such that $F(C) \geq 0.99$. We have $F(C) = \int_0^C 5(1-x)^4 dx = -(1-x)^5|_0^C = 1-(1-C)^5$. We want $1-(1-C)^5 \geq 0.99$, i.e., $(1-C)^5 \leq 0.01$, hence $C \geq 1-(0.01)^{0.2}$.