Fourth Homework Set — Solutions

Chapter 4

Problem 21 (a) $E[X]$ is larger than $E[Y]$ because the random selection of students favors larger busloads.

(b) $E[X] = \frac{40\cdot40+33\cdot33+25\cdot25+50\cdot50}{40+33+25+50} = \frac{5814}{148} = 39.3$, $E[Y] = \frac{148}{4} = 37$.

Problem 23 (a) Suppose that that you use x dollars to buy $x/2$ ounces of the commodity and keep the rest $1000 - x$ dollars as cash, and then sell your commodity at the end of the week. Then the expected amount of money you have at the end of the week is

$$\frac{1}{2}\frac{x}{2} + \frac{1}{2}2x + 1000 - x = 1000 + \frac{x}{4}$$

which is an increasing function of x. Therefore the best strategy is to use all your money to buy 500 ounces of the commodity and then sell at the end of the week.

(b) Suppose that you use x dollars to buy $x/2$ ounces of the commodity at the beginning of the first week and use the remaining $1000 - x$ dollars to buy the commodity after one week, then the expected ounces of the commodity that you own after one week is

$$\frac{x}{2} + \frac{1}{2}(1000 - x) + \frac{1}{2}\frac{1000 - x}{4} == 625 - \frac{x}{8}$$

which is a decreasing function of x. Therefore the best strategy in this case is that you do not immediately buy anything but use all your money after one week to buy the commodity.

Problem 32 Let X be the number of tests needed for a group of ten people. Then $X = 1$ or $X = 11$, and $P(X = 1) = 0.9^{10} = 0.3487$ and $P(X = 11) = 1 - 0.9^{10} = 0.6513$. Hence $E[X] = 7.5132$.

Problem 35 Let X be the win/loss after one game. Then $P(X = 1.1) = \frac{2\binom{5}{2}}{\binom{10}{2}} = \frac{20}{45} = \frac{4}{9}$, and $P(X = -1) = \frac{5}{9}$.

(a) $E[X] = 1.1 \cdot \frac{4}{9} - \frac{5}{9} = -\frac{1}{15}$.

(b) $\mathrm{Var}(X) = E[X^2] - E[X]^2 = 1.21 \cdot \frac{4}{9} + \frac{5}{9} - \frac{1}{225} = 1.0889$.

Problem 37

$$\begin{aligned}\mathrm{Var}(X) &= E\left[X^2\right] - E[X]^2 \\ &= \frac{40^3+33^2+25^3+50^3}{148} - \left(\frac{40^2+33^2+25^2+50^2}{148}\right)^2 = 82.2 \\ \mathrm{Var}(Y) &= \frac{40^2+33^2+25^2+50^2}{4} - 37^2 = 84.5\end{aligned}$$

Problem 38 Note that $E[X^2] = \mathrm{Var}(X) + E[X]^2 = 5 + 1 = 6$.

(a) $E[(2+X)^2] = E[4+4X+X^2] = 4 + 4E[X] + E[X^2] = 14$.

(b) $\mathrm{Var}(4+3X) = 9\mathrm{Var}(X) = 45$.

Problem 40 Let X be the number of correct answers. Then

$$P(X \geq 4) = P(X=4) + P(X=5) = \binom{5}{4}\frac{1}{3^4}\cdot\frac{2}{3} + \frac{1}{3^5} = \frac{11}{243}.$$

Problem 42 See part (a) of Example 6f in the book.

Problem 45 Let A be the event that the student has an 'on' day, and let E_3, E_5 be the event that a majority of a panel of three (resp. five) examiners passes him. Then

$$\begin{aligned}P(A) &= \frac{1}{3}, P(A^c) = \frac{2}{3} \\ P(E_3|A) &= \binom{3}{2}0.8^2\cdot 0.2 + 0.8^3 = 0.896 \\ P(E_3|A^c) &= \binom{3}{2}0.4^2\cdot 0.6 + 0.4^3 = 0.352 \\ P(E_5|A) &= \binom{5}{3}0.8^3\cdot 0.2^2 + \binom{5}{4}0.8^4\cdot 0.2 + 0.8^5 = 0.9421 \\ P(E_5|A^c) &= \binom{5}{3}0.4^3\cdot 0.6^2 + \binom{5}{4}0.4^4\cdot 0.6 + 0.4^5 = 0.3174 \\ P(E_3) &= P(E_3|A)P(A) + P(E_3|A^c)P(A^c) = 0.5333 \\ P(E_5) &= P(E_5|A)P(A) + P(E_5|A^c)P(A^c) = 0.5256\end{aligned}$$

The student would be marginally better off with three examiners.

Problem 48 Let p be the probability that a single package contains more than one defective diskette. Then $p = 1-0.99^{10}-10\cdot0.99^9\cdot0.01 = 0.0043$, and the probability of returning exactly one of three packages is $\binom{3}{1}p(1-p)^2 = 0.0127$.

Problem 50 Let E be the event that six of the first ten coin tosses come up heads.

(a) $P(H,T,T|E) = \frac{P(H,T,T \text{ and } E)}{P(E)} = \frac{p(1-p)^2\binom{7}{5}p^5(1-p)^2}{\binom{10}{6}p^6(1-p)^4} = \frac{\binom{7}{5}}{\binom{10}{6}} = \frac{1}{10}$

(b) $P(T,H,T|E) = P(H,T,T|E) = \frac{1}{10}$

Problem 55

$$\begin{aligned} P(\text{no errors}) = & \quad P(\text{no errors}|\text{first typist})\, P(\text{first typist}) \\ & +P(\text{no errors}|\text{second typist})\, P(\text{second typist}) \\ = & \quad \frac{1}{2}\left(\frac{3^0}{0!}e^{-3} + \frac{4.2^0}{0!}e^{-4.2}\right) \\ = & \quad \frac{1}{2}\left(e^{-3} + e^{-4.2}\right). \end{aligned}$$

Problem 57 X is Poisson with parameter $\lambda = 3$.

(a) $P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2) = 1 - e^{-3}\left(1+3+\frac{9}{2}\right) = 0.5768$.

(b) $P(X \geq 3|X \geq 1) = \frac{P(X\geq3)}{P(X\geq1)} = \frac{P(X\geq3)}{1-e^{-3}} = 0.6070$.

Problem 59 Let X be the number of times you win a prize. Then X is binomial with $n = 50$ and $p = \frac{1}{100}$, i.e., we can use the Poisson approximation with $\lambda = 50 \cdot \frac{1}{100} = \frac{1}{2}$.

(a) $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-\frac{1}{2}} = 0.3935$

(b) $P(X = 1) = \frac{1}{2}e^{-\frac{1}{2}} = 0.3033$

(c) $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - e^{-\frac{1}{2}}\left(1+\frac{1}{2}\right) = 0.0902$

Problem 61 Let X be Poisson with parameter $\lambda = 1000 \cdot 0.0014 = 1.4$. Then $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - e^{-1.4}(1+1.4) = 0.4082$.

Problem 63 Let X be a Poisson random variable with parameter $\lambda = \frac{5}{2}$. Then X gives a reasonable description of the number of people entering the casino between 12 and 12:05.

(a) $P(X = 0) = e^{-\frac{5}{2}} = 0.0821$

(b) $P(X \geq 4) = 1 - e^{-\frac{5}{2}}\left(1 + \frac{5}{2} + \frac{25}{8} + \frac{125}{48}\right) = 0.2424$