Second Homework Set — Solutions
Chapter 2

Problem 17 There are $64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57$ ways of arranging 8 castles on a chess board. Of these, there are $64 \cdot 49 \cdot 36 \cdot 25 \cdot 16 \cdot 9 \cdot 4 \cdot 1 = \prod_{i=1}^{8} i^2$ in which none of the rooks can capture any of the others. So the answer is

$$\frac{\prod_{i=1}^{8} i^2}{64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57}.$$

Problem 18

$$\frac{2 \cdot 4 \cdot 16}{52 \cdot 51}.$$

Problem 20 Let A be the event that you are dealt a blackjack, and let B be the event that the dealer is dealt a blackjack.

Then

$$\begin{aligned} P(A) = P(B) &= \frac{2 \cdot 4 \cdot 16}{52 \cdot 51} \\ P(AB) &= \frac{4 \cdot 4 \cdot 16 \cdot 3 \cdot 15}{52 \cdot 51 \cdot 50 \cdot 49} \\ P(A \cup B) &= P(A) + P(B) - P(AB) = 0.0948. \end{aligned}$$

Hence, then probability that neither you nor the dealer is dealt a blackjack is $1 - P(A \cup B) = 0.9052$.

Problem 21 (a) $P(1) = \frac{4}{20} = \frac{1}{5}$, $P(2) = \frac{8}{20} = \frac{2}{5}$, $P(3) = \frac{5}{20} = \frac{1}{4}$, $P(4) = \frac{2}{20} = \frac{1}{10}$, and $P(5) = \frac{1}{20}$.

(b) There are 48 children altogether, so that $P(1) = \frac{4}{48} = \frac{1}{12}$, $P(2) = \frac{2 \cdot 8}{48} = \frac{1}{3}$, $P(3) = \frac{3 \cdot 5}{48} = \frac{5}{16}$, $P(4) = \frac{4 \cdot 2}{48} = \frac{1}{6}$, and $P(5) = \frac{5}{48}$.

Problem 25 Let E_n be the event that a sum of 5 occurs on the nth roll, and no sum of 5 or 7 occurs on the first $n-1$ rolls. There are 36 outcomes of a single roll, and four of them give a sum of 5, while 6 of them give a sum of 7. Hence,

$$P(E_n) = \left(\frac{26}{36}\right)^{n-1} \frac{4}{36} = \left(\frac{13}{18}\right)^{n-1} \frac{1}{9}.$$

A sum of 5 occurs before a sum of 7 precisely if the events E_n occurs for some n. Since E_n and E_m are disjoint if $n \neq m$, the desired probability is

$$\sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1} \cdot \frac{1}{9} = \frac{1}{9} \cdot \frac{1}{1-\frac{13}{18}} = \frac{1}{9}\frac{18}{5} = \frac{2}{5}.$$

Problem 27

$$\begin{aligned}
P(A \text{ wins in one move}) &= \frac{3}{10} \\
P(A \text{ wins in three moves}) &= \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} = \frac{7}{40} \\
P(A \text{ wins in five moves}) &= \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{1}{12} \\
P(A \text{ wins in seven moves}) &= \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{1}{40} \\
P(A \text{ wins}) &= \frac{3}{10} + \frac{7}{40} + \frac{1}{12} + \frac{1}{40} = \frac{7}{12}
\end{aligned}$$

Problem 28 (a) Without replacement:

$$P(\text{all three balls are the same color}) = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$$

With replacement:

$$P(\text{all three balls are the same color}) = \left(\frac{5}{19}\right)^3 + \left(\frac{6}{19}\right)^3 + \left(\frac{8}{19}\right)^3$$

(b) Without replacement:

$$P(\text{all three balls are of different colors}) = \frac{\binom{5}{1} \cdot \binom{6}{1} \cdot \binom{8}{1}}{\binom{19}{3}}$$

With replacement:

$$P(\text{all three balls are of different colors}) = 3! \cdot \frac{5}{19} \cdot \frac{6}{19} \cdot \frac{8}{19}$$

Problem 32 There are $(b+g)!$ ways to line up the children. There are $g \cdot (b+g-1)!$ arrangements with a girl in the ith position. The desired probability is $\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$.

Problem 37 (a) There are $\binom{10}{5}$ selections for the final exam. The number of selections that allow the student to solve all problems is $\binom{7}{5}$, so that the desired probability is $\frac{\binom{7}{5}}{\binom{10}{5}} = 0.08333$.

(b) There are $\binom{7}{4} \cdot \binom{3}{1}$ selections that'll let the student solve exactly four problems, so that the probability of solving at least four problems is $\frac{\binom{7}{5}+\binom{7}{4}\cdot\binom{3}{1}}{\binom{10}{5}} = \frac{1}{2}$.

Problem 43 (a) There are $n!$ ways to arrange n people in a line. There are $2(n-1)!$ ways to arrange them such that A and B are next to each other. Hence, the probability of A and B being next to each other is $\frac{2(n-1)!}{n!} = \frac{2}{n}$.

(b) If $n = 2$, then A and B will always be next to each other. Now, assume that $n > 3$. After A picks a seat, there are $n-1$ seats left, two of which are next to A, so that the desired probability is $\frac{2}{n-1}$.

Problem 50 The probability that you have five spades and your partner has the remaining eight spades is

$$\frac{\binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5}}{\binom{52}{13,13,26}} = 2.6084 \cdot 10^{-6}.$$

Problem 53 Let E_i be the event that the i-th couple sit together, for $j = 1, \ldots, 4$. Then $P(E_i) = \frac{2}{8} = \frac{1}{4}$ (Problem 43(a)). Moreover, if $i < j$, then $P(E_iE_j) = \frac{2^2 \cdot 6!}{8!}$. Similarly, if $i < j < k$, then $P(E_iE_jE_k) = \frac{2^3 \cdot 5!}{8!}$. Finally, we have $P(E_1E_2E_3E_4) = \frac{2^4 \cdot 4!}{8!}$. Using inclusion-exclusion, we obtain

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3 \cup E_4) &= \sum_{i=1}^{4} P(E_i) - \sum_{i<j} P(E_iE_j) + \sum_{i<j<k} P(E_iE_jE_k) - P(E_1E_2E_3E_4) \\ &= 4 \cdot \frac{1}{4} - \binom{4}{2}\frac{2^2 \cdot 6!}{8!} + \binom{4}{3}\frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!} \\ &= 1 - \frac{3}{7} + \frac{2}{21} - \frac{1}{105} = \frac{23}{35}. \end{aligned}$$

Hence, the probability that no husband sits next to his wife is $1 - \frac{23}{35} = \frac{12}{35}$.

Problem 54 Let S, H, C, and D be the event that spades are missing, hearts are missing, etc. Then

$$\begin{aligned}P(S \cup H \cup C \cup D) =& P(S) + P(H) + P(C) + P(D)\\ &- P(SH) - P(SC) - P(SD) - P(HC) - P(HD) - P(CD)\\ &+ P(SHC) + P(SHD) + P(SCD) + P(HCD)\\ &- P(SHCD)\\ =&4 \cdot \frac{\binom{39}{13}}{\binom{52}{13}} - 6 \cdot \frac{\binom{26}{13}}{\binom{52}{13}} + 4 \cdot \frac{1}{\binom{52}{13}} - 0\\ =&0.0511.\end{aligned}$$

Chapter 3

Problem 1 Let E be the event that at least one die lands on six, and let F be the event that the dice land of different numbers. Then $P(EF) = 2 \cdot \frac{1}{6} \cdot 56 = \frac{5}{18}$ and $P(F) = \frac{30}{36} = \frac{5}{6}$. Hence,

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{5}{18}}{\frac{5}{6}} = \frac{1}{3}.$$

Problem 5

$$\frac{6 \cdot 5 \cdot 9 \cdot 8}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{6}{91}$$

Problem 6 Let A be the event that the sample drawn contains exactly three white balls. Let B be the event that the first and third ball drawn are white.

Without replacement

$P(A) = 4\frac{8 \cdot 7 \cdot 6 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 9}$ and $P(AB) = 2\frac{8 \cdot 7 \cdot 6 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 9}$, hence $P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2}$.

With replacement

$P(A) = \binom{4}{3}\left(\frac{2}{3}\right)^3 \frac{1}{3}$ and $P(AB) = 2\left(\frac{2}{3}\right)^3 \frac{1}{3}$, hence $P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2}$.

Problem 9 Let E_i be the event that the ball drawn from the i-th urn is white, for $i = 1, 2, 3$. Let F be the event that exactly two white balls were drawn.

Then

$$\begin{aligned}
P(E_1|F) &= \frac{E_1F}{P(F)} \\
&= \frac{P(E_1E_2E_3^c) + P(E_1E_2^cE_3)}{P(E_1E_2E_3^c) + P(E_1E_2^cE_3) + P(E_1^cE_2E_3)} \\
&= \frac{\frac{2\cdot 8\cdot 3}{6\cdot 12\cdot 4} + \frac{2\cdot 4\cdot 1}{6\cdot 12\cdot 4}}{\frac{2\cdot 8\cdot 3}{6\cdot 12\cdot 4} + \frac{2\cdot 4\cdot 1}{6\cdot 12\cdot 4} + \frac{4\cdot 8\cdot 1}{6\cdot 12\cdot 4}} \\
&= \frac{7}{11}.
\end{aligned}$$

Problem 10 For $i = 1, 2, 3$, let E_i be the event that the i-th card is a spade. Then

$$P(E_1E_2E_3) = \frac{13}{52}\frac{12}{51}\frac{11}{50}$$

and

$$P(E_2E_3) = P(E_1E_2E_3) + P(E_1^cE_2E_3) = \frac{13}{52}\frac{12}{51}\frac{11}{50} + \frac{39}{52}\frac{13}{51}\frac{12}{50}.$$

Thus

$$P(E_1|E_2E_3) = \frac{P(E_1E_2E_3)}{P(E_2E_3)} = \frac{11}{50}.$$

Problem 23 (a) Let W be the event that the ball selected urn II is white, E be the event that the transfered ball is white and F be the event that the transfered ball is red, then

$$P(W) = P(E)P(W|E) + P(F)P(W|F) = \frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3} = \frac{4}{9}.$$

(b)

$$P(E|W) = \frac{P(EW)}{P(W)} = \frac{P(E)P(W|E)}{P(W)} = \frac{1}{2}.$$

Problem 30 Let B and W be the events that the marble is black and white respectively, and let B_i be the event that box i is chosen. Then

$$P(B) = P(B_1)P(B|B_1) + P(B_2)P(B|B_2) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{2}{3} = \frac{7}{12},$$

$$P(B_1|W) = \frac{P(B_1W)}{P(W)} = \frac{P(B_1)P(W|B_1)}{P(W)} = \frac{\frac{1}{2}\frac{1}{2}}{\frac{5}{12}} = \frac{3}{5}.$$

Problem 47 (a)

$$P(\text{all white}) = \frac{1}{6}(\frac{5}{15}+\frac{5}{15}\frac{4}{14}+\frac{5}{15}\frac{4}{14}\frac{3}{13}+\frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}+\frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}\frac{1}{11}).$$

(b)

$$P(3|\text{all white}) = \frac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P(\text{all white})}.$$

Problem 51 Let R be the event that she receives a job offer, S be the event that event of a strong recommendation, M the event of a moderate recommendation and W the event of a weak recommendation.

(a)

$$P(R) = P(S)P(R|S) + P(M)P(R|M) + P(W)P(R|W)$$
$$= (.8)(.7) + (.4)(.2) + (.1)(.1) = .65.$$

(b)

$$P(S|R) = \frac{P(SR)}{P(R)} = \frac{P(S)P(R|S)}{P(R)} = \frac{(.8)(.7)}{.65} = \frac{56}{65}.$$

Similarly

$$P(M|R) = \frac{8}{65}, \quad P(W|R) = \frac{1}{65}.$$

Problem 56

$$P(\text{new}) = \sum_{i=1}^{m} p_i P(\text{new}|\text{the } n\text{-th coupon is of type i}) = \sum_{i=1}^{m} p_i(1-p_i)^{n-1}.$$