10th Homework Set — Solutions

Chapter 7

Problem 7.50 We have

$$f_Y(y) = \int_0^\infty \frac{e^{-\frac{x}{y}-y}}{y} dx = e^{-y}$$

for $y > 0$, so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{e^{-\frac{x}{y}}}{y} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Now, we have

$$E\left[X^2|Y\right] = \int_0^\infty \frac{x^2}{y} e^{-\frac{x}{y}} dx = 2y^2.$$

Problem 7.51 We have

$$f_Y(y) = \int_0^y \frac{e^{-y}}{y} dx = e^{-y},$$

so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0, y) \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that

$$E\left[X^3|Y = y\right] = \int_0^y \frac{x^3}{y} dx = \frac{y^3}{4}.$$

Problem 7.56 Let Y_i be one if the elevator stops at the i-th floor, for $i = 1, \ldots, N$. Let $Y = Y_1 + \cdots + Y_N$. Let X be the number of passengers, i.e., X is Poisson with parameter 10. We have $E\left[Y_i = 1|X = k\right] = 1 - \left(\frac{N-1}{N}\right)^k$, so that

$$E\left[Y|X = k\right] = N\left(1 - \left(\frac{N-1}{N}\right)^k\right).$$

We have

$$\begin{aligned} E\left[Y\right] = E\left[E\left[Y|X\right]\right] &= E\left[N\left(1 - \left(\frac{N-1}{N}\right)^X\right)\right] \\ &= N - N\sum_{k=0}^\infty \left(\frac{N-1}{N}\right)^k \frac{10^k}{k!} e^{-10} \\ &= N(1 - e^{-\frac{10}{N}}). \end{aligned}$$

Problem 7.57 By Example 5d in Section 7.5, we have

$$E\left[\sum_{i=1}^{N} X_i\right] = E\left[N\right] E\left[X_1\right] = 12.5.$$

Problem 7.75 X is a random variable with moment generating function $M_X(t) = \exp\{2e^t - 2\} = \exp\{2(e^t - 1)\}$, i.e., X is Poisson with parameter $\lambda = 2$.

Y is a random variable with moment generating function $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$, i.e., Y is binomial with parameters $(10, \frac{3}{4})$.

(a)

$$\begin{aligned} P\left\{X+Y=2\right\} &= P\left\{X=0\right\} P\left\{Y=2\right\} + P\left\{X=1\right\} P\left\{Y=1\right\} \\ &\quad + P\left\{X=2\right\} P\left\{Y=0\right\} \\ &= e^{-2} \cdot \binom{10}{2} \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^8 + 2e^{-2} \cdot 10\frac{3}{4}\left(\frac{1}{4}\right)^9 \\ &\quad + 2e^{-2} \cdot \left(\frac{1}{4}\right)^{10} \\ &= e^{-2}\left(\frac{1}{4}\right)^{10}(405+60+2) = \frac{467}{4^{10}e^2}. \end{aligned}$$

(b)

$$\begin{aligned} P\left\{XY=0\right\} &= P\left\{X=0\right\} + P\left\{Y=0\right\} - P\left\{X=0\right\} P\left\{Y=0\right\} \\ &= e^{-2} + \frac{1}{4^{10}} - e^{-2}\frac{1}{4^{10}} = \frac{4^{10} + e^2 - 1}{4^{10}e^2}. \end{aligned}$$

(c)

$$\begin{aligned} E[XY] &= E[X] \cdot E[Y] \quad \text{by independence} \\ &= 2 \cdot 7.5 \\ &= 15. \end{aligned}$$

Chapter 8

Problem 8.1 $P(0 < X < 40) = 1 - P(|X-20| \geq 20) \geq 1 - \frac{20}{400} = \frac{19}{20}$.

Problem 8.2 (a) $P(X \geq 85) \leq \frac{75}{85} = \frac{15}{17}$.

(b) $P(65 \leq X \leq 85) = 1 - P(|X - 75| > 10) \geq 1 - \frac{25}{100} = \frac{3}{4}$.

(c) Since

$$P\left(|\sum_{i=1}^{n} \frac{X_i}{n} - 75| > 5\right) \leq \frac{25}{25n},$$

we need $n = 10$.