First Homework Set — Solutions
Chapter 1

Problem 4 If each of the boys can play any instrument, then there are $4! = 24$ possible arrangements. If Jay and Jack can only play piano and drums, then there are two ways to assign instruments to Jay and Jack, which leaves two possible assignments for John and Jim, so that there are $2 \cdot 2 = 4$ possibilities in this case.

Problem 5 Number of possible area codes: $8 \cdot 2 \cdot 9 = 144$ (Generalized counting principle: There are eight choices for the first digit, two for the second digit, and nine for the third digit.)

Number of area codes starting with a 4: $1 \cdot 2 \cdot 9 = 18$

Problem 7

(a) There are six children altogether, so that there are $6! = 720$ ways to arrange them in a row.

(b) There are $3! = 6$ ways to arrange the girls (resp. boys) in a row, and there are two ways to arrange the two blocks of boys/girls (i.e., girls first or boys first), so that there are $3! \cdot 3! \cdot 2 = 72$ arrangements altogether.

(c) There are $4! = 24$ ways to arrange the girls and the block of boys in a row, and there are $3! = 6$ ways to arrange the boys within their block, so that there are $4! \cdot 3! = 144$ arrangements altogether.

(d) If no two people of the same sex are allowed to sit together, then either the girls are in the even seats, and the boys are in the odd seats, or vice versa. In total, there are $3! \cdot 3! \cdot 2 = 72$ possibilities.

Problem 8

(a) $5! = 120$ (five distinct letters)

(b) $\frac{7!}{2! \cdot 2!} = 1260$ (seven letters, including two Ps and two Os)

(c) $\frac{11!}{4! \cdot 4! \cdot 2!} = 34,650$ (eleven letters, including four Is, four Ss, and two Ps)

(d) $\frac{7!}{2! \cdot 2!} = 1260$ (seven letters, two As, two Rs)

Problem 9 There are $\frac{12!}{6! \cdot 4!} = 27,720$ possibilities.

Problem 11 This problem is similar to Problem 7.

(a) There are six books altogether, so that there are $6! = 720$ ways to arrange them.

(b) The number of possible arrangements is $3! \cdot 2! \cdot 3! = 72$.

(c) There are $3! \cdot 4! = 144$ possible arrangements.

Problem 12 (a) Each award can go to any one of 30 students, so that there are $30^5 = 24,300,000$ possibilities if a student can receive any number of awards.

(b) The first award can go to any one of 30 students, for the second award there are 29 choices left, etc. In total, there are $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 = 17,100,720$ possibilities.

Problem 18 The number of possibilities is $\binom{5}{2} \cdot \binom{6}{2} \cdot \binom{4}{3} = 600$.

Problem 19 There are $\binom{8}{3} \cdot \binom{6}{3}$ possible committees if there are no constraints.

(a) Let's find the number of bad choices first. If the two finicky men serve together, there is room for one more man on the committee, and there are four men left to take this spot, i.e., there are $4 \cdot \binom{6}{3}$ bad choices, so that there are $\binom{8}{3} \cdot \left(\binom{6}{3} - 4\right) = 896$ good choices.

(b) Same reasoning as in (a): There are $6 \cdot \binom{6}{3}$ bad choices, so that there are $\left(\binom{8}{3} - 6\right)\binom{6}{3} = 1000$ good choices.

(c) Here, there are $\binom{7}{2} \cdot \binom{5}{2}$ bad choices, so that there are $\binom{8}{3} \cdot \binom{6}{3} - \binom{7}{2} \cdot \binom{5}{2} = 910$ good choices.

Problem 20 (a) There are $\binom{6}{3}$ bad choices, so that there are $\binom{8}{5} - \binom{6}{3} = 36$ good choices left.

(b) There are $\binom{6}{3}$ choices involving the inseparable pair of friends, and $\binom{6}{5}$ other choices. In total, there are $\binom{6}{3} + \binom{6}{5} = 26$ possibilities in this case.

Problem 21 You need to take seven steps to get from A to B, four steps to the right and three steps upward. The order of the steps doesn't matter, i.e., any four of your seven steps may be steps to the right, resulting in $\binom{7}{3} = 35$ possible paths.

Problem 24 Binomial theorem:

$$\begin{aligned}(3x^2+y)^5 &= \sum_{k=0}^{k=5} \binom{5}{k} (3x^2)^k y^{5-k} \\ &= y^5 + 15x^2y^4 + 90x^4y^3 + 270x^6y^2 + 405x^8y + 243x^{10}\end{aligned}$$

Problem 27 Multinomial coefficient: $\binom{12}{3,4,5} = 27,720$

Chapter 2

Problem 2 Let $X = \{1,2,3,4,5\}$. The sample space S consists of all finite sequences $(x_1,\ldots,x_n)$, where $x_1,\ldots,x_{n-1} \in X$ and $x_n = 6$, as well as infinite sequences with values in X.

E_n is the set of all sequences of length n in S, and $(\cup_{n=1}^{\infty} E_n)^c$ is the set of all infinite sequences in S, i.e., the set of infinite sequences with values in X.

Problem 3

$$\begin{aligned}E =& \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),\\ &(3,2),(3,4),(3,6),(4,1),(4,3),(4,5),\\ &(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}\\ F =& \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(3,1),(4,1),(5,1),(6,1)\}\\ G =& \{(1,4),(2,3),(3,2),(4,1)\}\end{aligned}$$

- Meaning of EF: The sum is odd *and* at least one of the dice lands on 1.
- Meaning of $E \cup F$: The sum is odd *or* at least one the dice lands on 1.
- Meaning of FG: At least one of the dice lands on 1 and the sum equals 5, i.e., $FG = \{(1,4),(4,1)\}$.
- Meaning of EF^c: The sum is odd and none of the dice lands on 1.

– Meaning of EFG: The sum is odd and one of the dice lands on 1 and the sum is five, i.e., $EFG = FG$.

Problem 5 (a) There are 2^5 possible outcomes (two outcomes per component, use generalized counting principle).

(b) Represent an outcome as a binary number, e.g., 00101 means that components 3 and 5 work, and components 1, 2, and 4 do not work. Then

$$W = \{11000, 11001, 11010, 11011, 11100, 11101, 11110, 11111, \\ 00110, 00111, 01110, 01111, 10110, 10111, 10101\}.$$

(c) There are $2^3 = 8$ outcomes in A.

(d) $AW = \{11000, 11100\}$.

Problem 6 (a) $S = \{(0, g), (0, f), (0, s), (1, g), (1, f), (1, s)\}$.

(b) $A = \{(0, s), (1, s)\}$.

(c) $B = \{(0, g), (0, f), (0, s)\}$.

(d) $B^c \cup A = \{(1, g), (1, f), (1, s), (0, s)\}$.

Problem 7 (a) There are 6^{15} outcomes in the sample space.

(b) There are 3^{15} outcomes without any blue-collar workers, so that there are $6^{15} - 3^{15}$ outcomes with at least one blue-collar worker.

(c) If there are no independents, then for each player, there are 4 outcomes, so that there are 4^{15} outcomes altogether.

Problem 8 Suppose that A, B are mutually exclusive events for which $P(A) = 0.3$ and $P(B) = 0.5$.

(a) $P(\text{either } A \text{ or } B) = P(A) + P(B) = 0.8$ because A and B are mutually exclusive.

(b) $P(A \text{ occurs but } B \text{ does not}) = 0.3$.

(c) $P(A \cap B) = 0$.

Problem 9 Let A be the event that a randomly chosen customer carries an Amex card, and let V be the event that a randomly chosen customer carries a Visa card. Then $P\{(\}A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(AB) = 0.24 + 0.61 - 0.11 = 0.74$. Hence, 74 percent of all customers carry an acceptable credit card.

Problem 12 Let E be the event that a randomly chosen student is taking Spanish, let F be the event that a randomly chosen student is taking French, and let D be the event that a randomly chosen student is taking German. Note that

$$\begin{aligned} P(E) &= \frac{28}{100} = \frac{7}{25} \\ P(F) &= \frac{26}{100} = \frac{13}{50} \\ P(D) &= \frac{16}{100} = \frac{4}{25} \\ P(EF) &= \frac{12}{100} = \frac{3}{25} \\ P(ED) &= \frac{4}{100} = \frac{1}{25} \\ P(FD) &= \frac{6}{100} = \frac{3}{50} \\ P(EFD) &= \frac{2}{100} = \frac{1}{50} \end{aligned}$$

(a) The probability of a student not being in any of these classes is

$$\begin{aligned} 1 - P(E \cup F \cup D) =& 1 - (P(E) + P(F) + P(D) \\ & - P(EF) - P(ED) - P(FD) + P(EFD)) \\ =& 1 - \frac{28 + 26 + 16 - 12 - 4 - 6 + 2}{100} = 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

(b) We have

$$\begin{aligned} P(\text{only } E) &= P(E) - P(EF) - P(ED) + P(EFD) = 0.14 \\ P(\text{only } F) &= P(F) - P(EF) - P(FD) + P(EFD) = 0.1 \\ P(\text{only } D) &= P(D) - P(ED) - P(FD) + P(EFD) = 0.08 \end{aligned}$$

Hence, $P(\text{exactly one language class}) = 0.14 + 0.1 + 0.08 = 0.32$.

(c) Since fifty students are not taking any of the courses, the probability that neither of two randomly picked students is $\frac{\binom{50}{2}}{\binom{100}{2}} = \frac{49}{198}$, so that the probability of at least one of them taking a language class is $1 - \frac{49}{198} = \frac{149}{198}$.