Math 461 Test 2, Spring 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (10 points) A machine produces screws, 1 percent of which are defective. Use Poisson approximation to find the probability that in a box of 100 screws there is at most 1 defective.

2. (10 points) A certain basketball player knows that on average he will make 80 percent of his free throw attempts. Use normal approximation to find the probability that in 100 attempts he will be successful at least 90 times.

3. (10 points) Teams A and B play a series of games; the series will end when one of the teams wins 4 games. Suppose that team A wins each game with probability $\frac{2}{3}$, independent of the outcomes of the other games. Find the probability that a total of 6 games are played.

4. (20 points) Let X and Y be independent random variables each uniformly distributed on $(0,1)$. Find $P(Y \geq X | Y \geq \frac{1}{2})$.

5. (10 points) The time X it takes John to solve a problem is an exponential random variable with parameter $\lambda = 1$, and the time Y it take Mike to solve the same problem is also an exponential random variable with parameter $\lambda = 2$. Suppose that X and Y are independent. Find $P(X \leq Y)$.

6. (10 points) Suppose that X is uniformly distributed over the interval $(-1,1)$. Find the density of $Y = -\ln(1 - |X|)$.

7. (20 points) The joint density of X and Y is given by

$$f(x,y) = \begin{cases} x+y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} . \end{cases}$$

(a) Find the marginal density of X. (b) Find EX and $\mathrm{Var}(X)$.

8. (10 points) Suppose that X and Y are independent random variables, X is Poisson with parameter λ_1 and Y is Poisson with parameter λ_2. Find $P(X = k | X + Y = 100)$ for $k = 0, 1, \ldots, 100$.

1. Let X be the number of defective ones in the box. Then X is approximately a Poisson random variable with parameter $\lambda = 1$. Thus

$$P(X \leq 1) = P(X = 0) + P(X = 1) \approx 2e^{-1}.$$

2. Let X be the number of times that he will be successful in 100 attempts. Then X is a binomial random variable with parameter $n = 100$ and $p = .8$. Thus, by using normal approximation, we get

$$P(X \geq 90) = P(X \geq 89.5) = P(\frac{X - 80}{4} \geq \frac{9.5}{4}) \approx 1 - \Phi(2.37) = .0089.$$

3. Let A be the event that team A wins in 6 games, B be the event that B wins in 6 games, and C be the event that a total of 6 games are played. Then

$$P(C) = P(A) + P(B) = \binom{5}{3}(\frac{2}{3})^4(\frac{1}{3})^2 + \left(5 \right)(\frac{1}{3})^4(\frac{2}{3})^2.$$

4. By geometrical considerations, we can easily see that $P(Y \geq \frac{1}{2}) = \frac{1}{2}$ and $P(Y \geq X, Y \geq \frac{1}{2}) = \frac{3}{8}$. Thus

$$P(Y \geq X | Y \geq \frac{1}{2}) = \frac{P(Y \geq X, Y \geq \frac{1}{2})}{P(Y \geq \frac{1}{2})} = \frac{3}{4}.$$

5.

$$P(X \leq Y) = \int_0^\infty (\int_x^\infty 2e^{-x}e^{-2y}dy)dx = \int_0^\infty e^{-x}(\int_x^\infty 2e^{-2y}dy)dx = \int_0^\infty e^{-3x}dx = \frac{1}{3}.$$

6. Y is a positive random variable. For any $y > 0$,

$$P(Y \leq y) = P(-\ln(1 - |X|) \leq y) = P(1 - |X| \geq e^{-y}) = P(|X| \leq 1 - e^{-y}) = 1 - e^{-y}.$$

Thus the density of Y is given

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

7. (a). For any $x \in (0, 1)$,

$$f_X(x) = \int_0^1 (x + y)dy = x + \frac{1}{2}.$$

Thus the marginal density of X is

$$f_X(x) = \begin{cases} x + \frac{1}{2}, & x \in (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

(b) $EX = \int_0^1 x(x + \frac{1}{2})dx = \frac{7}{12}$, $E(X^2) = \int_0^1 x^2(x + \frac{1}{2})dx = \frac{5}{12}$. Thus

$$\text{Var}(X) = E(X^2) - (EX)^2 = \frac{11}{144}.$$

8. Since X and Y are independent Poisson random variables with parameters λ_1 and λ_2 respectively, we know that $X+Y$ is a Poisson random variable with parameter $\lambda_1+\lambda_2$. Thus, for any $k=0,1,\ldots,100$,

$$P(X=k|X+Y=100)=\frac{P(X=k,X+Y=100)}{P(X+Y=100)}=\frac{P(X=k,Y=100-k)}{P(X+Y=100)}$$

$$=\frac{P(X=k(P(Y=100-k)}{P(X+Y=100)}=\frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{100-k}}{(100-k)!}}{e^{-\lambda_1-\lambda_2}\frac{(\lambda_1+\lambda_2)^{100}}{100!}}$$

$$=\left(\begin{array}{c}100\\k\end{array}\right)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{100-k}.$$