Solutions to Math 461 Test 2, Spring 2024

1. (12 points) The joint density of X and Y is

$$f(x,y) = \begin{cases} x+y, & 0<x<1, 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathrm{Cov}(X,Y)$.

Solution.

$$\mathbb{E}[X] = \int_0^1 \int_0^1 x(x+y)dxdy = \int_0^1 \left(\frac{1}{3}+\frac{y}{2}\right)dy = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}.$$

By symmetry, we also have $\mathbb{E}[Y] = \frac{7}{12}$.

$$\mathbb{E}[XY] = \int_0^1 \int_0^1 xy(x+y)dxdy = \int_0^1 \left(\frac{y}{3}+\frac{y^2}{2}\right)dy = \frac{1}{6}\frac{1}{6} = \frac{1}{3}.$$

Thus $\mathrm{Cov}(X,Y) = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}$.

2. (12 points) Suppose that X is an absolutely continuous random variable with density

$$f(x) = \begin{cases} \frac{3}{2}x^2, & x \in (-1,1) \\ 0, & \text{otherwise} \end{cases}$$

Find the density of $Y = X^4$.

Solution. For $y \in (0,1)$,

$$\mathbb{P}(Y \le y) = \mathbb{P}(X^4 \le y) = \mathbb{P}(-y^{1/4} \le X \le y^{1/4}) = 2\mathbb{P}(0 < X \le y^{1/4}) = \int_0^{y^{1/4}} 3x^2 dx = y^{3/4}.$$

Taking derivatives, we get that the density of Y is

$$f_Y(y) = \begin{cases} \frac{3}{4}y^{-1/4}, & 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

3. (18 points) Suppose that X is an exponential random variable with parameter $\lambda_1 = 1$, Y is an exponential random variable with parameter $\lambda_2 = 2$, and that X and Y are independent. Find the density of $Z = X/Y$.

Solution. For $z > 0$, we have

$$\mathbb{P}(Z \le z) = \mathbb{P}(X/Y \le z) = \mathbb{P}(X \le zY) = \int_0^\infty \int_{x/z}^\infty e^{-x} 2e^{-2y} dydx$$
$$= \int_0^\infty e^{-(1+\frac{2}{z})x} dx = \frac{z}{2+z}.$$

Thus

$$f_Z(z) = \begin{cases} 2(2+z)^{-2}, & z > 0 \\ 0, & \text{otherwise} \end{cases}$$

4. (18 points) Suppose that U and V are independent random variables and both are uniformly distributed in $(0,2)$. Define $X = \min(U,V)$ and $Y = \max(U,V)$. (a) Find (or write down) the joint density of X and Y. (b) For $y \in (0,2)$, find the conditional density $f_{X|Y}(x|y)$.

Solution. (a) The joint density of X and Y is

$$f(x,y) = \begin{cases} \frac{1}{2}, & 0 < x < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

(b) For $y \in (0,2)$,

$$f_Y(y) = \int_0^y \frac{1}{2} dx = \frac{y}{2}.$$

Thus

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & x \in (0,y) \\ 0, & \text{otherwise} \end{cases}$$

5. (20 points) According to a certain survey, 10% of 9th grade boys and 20% of 9th grade girls never eat breakfast. Assume that the breakfast habits of all 9th graders are independent. Suppose that random samples of 400 9th grade boys and 400 9th grade girls are chosen. Use normal approximation to find the probability that at least 130 of the 800 9th graders never eat breakfast. (For this problem, you can use the following values of the distribution function Φ of the standard normal distribution: $\Phi(0.9) = 0.8159$, $\Phi(0.95) = 0.8289$ and $\Phi(1) = 0.8413$.)

Solution. By normal approximation, the number X of the 400 9th grade boys never eat breakfast is approximately normal with mean 40 and variance 36, and the number Y of the 400 9th grade girls never eat breakfast is approximately normal with mean 80 and variance 64,. So the total number $X+Y$ of the 800 9th graders never eat breakfast is approximately normal with mean 120 and variance 100. Thus

$$\mathbb{P}(X+Y \geq 130) = \mathbb{P}(X+Y \geq 129.5) = \mathbb{P}(\frac{X+Y-120}{10} \geq \frac{9.5}{10}) = 1 - \Phi(.95) = .1711.$$

6. (20 points) A box contains 20 balls labeled 1, 2, . . . , 20. Suppose that A randomly chooses 3 balls, without replacement, from the box. All 3 balls are then returned to the box. Then B randomly chooses 3 balls, without replacement, from the box. Let X be the number of balls that are chosen by neither A nor B. Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. (Hint: Write X as the sum of some simple random variables. No need to simplify your answers.)

Solution. Let X_i be one if neither A nor B chooses the i-th ball, for $i = 1, \ldots, 20$. Then $X = X_1 + \cdots + X_{20}$. For any $i = 1, \ldots, 20$,

$$\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = \left(\frac{17}{20}\right)^2 = \frac{17^2}{20^2},$$

and

$$\operatorname{Var}(X_i) = \frac{17^2}{20^2} \cdot \frac{20^2 - 17^2}{20^2}.$$

Thus

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_{20}] = 20\mathbb{E}[X_1] = \frac{17^2}{20}.$$

For $i, j = 1, \ldots, 20$, $i \neq j$,

$$\begin{aligned}
\mathbb{E}[X_i X_j] &= \mathbb{P}(X_i X_j = 1) = \mathbb{P}(X_i = 1, X_j = 1) \\
&= \left(\frac{\binom{18}{3}}{\binom{20}{3}} \right)^2 = \frac{68^2}{95^2}.
\end{aligned}$$

Thus $\operatorname{Cov}(X_i, X_j) = \frac{68^2}{95^2} - \frac{17^4}{20^4}$. Therefore

$$\begin{aligned}
\operatorname{Var}(X) &= \sum_{i=1}^{20} \operatorname{Var}(X_i) + \sum_{i \neq j} \operatorname{Cov}(X_i, X_j) \\
&= 20 \cdot \frac{17^2}{20^2} \cdot \frac{20^2 - 17^2}{20^2} + 380 \cdot \left(\frac{68^2}{95^2} - \frac{17^4}{20^4} \right).
\end{aligned}$$