Solutions to Math 461 Test 1, Spring 2024

1. (16 points) ((a) Suppose that E_1, E_2, E_3, E_4 are independent events and that $\mathbb{P}(E_1) = \mathbb{P}(E_2) = \mathbb{P}(E_3) = \mathbb{P}(E_4) = \frac{1}{2}$. Find $\mathbb{P}((E_1 \cup E_2) \cap (E_3 \cup E_4))$.
(b) Suppose that X is a discrete random variable with $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 2$. Find $\mathbb{E}[(2-3X)^2]$.

Solution. (a)

$$
\begin{aligned}
&\mathbb{P}((E_1 \cup E_2) \cap (E_3 \cup E_4)) = \mathbb{P}(E_1 \cup E_2)\mathbb{P}(E_3 \cup E_4) \\
&= (\mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2))\,(\mathbb{P}(E_3) + \mathbb{P}(E_4) - \mathbb{P}(E_3 \cap E_4)) \\
&= \left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\frac{1}{2}\right) = \frac{9}{16}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\mathbb{E}[(2-3X)^2] = \mathbb{E}[4 - 12X + 9X^2] \\
&= 4 - 12\mathbb{E}[X] + 9\mathbb{E}[X^2] = 4 - 12\mathbb{E}[X] + 9(\mathrm{Var}(X) + (\mathbb{E}[X])^2) \\
&= 4 - 12 + 9 \cdot (2+1) = 19.
\end{aligned}
$$

2. (16 points) A box contains 7 red and 5 black balls. Players A and B choose balls randomly from the box consecutively (and without replacement) until a red ball is selected; whoever get a red ball is declared the winner. Assume that A chooses first. Find the probability that A is the winner.

Solution.

$$
\begin{aligned}
\mathbb{P}(A \text{ wins after 1 selction}) &= \frac{7}{12} \\
\mathbb{P}(A \text{ wins after 3 selctions}) &= \frac{5}{12} \cdot \frac{4}{11} \cdot \frac{7}{10} \\
\mathbb{P}(A \text{ wins after 5 selctions}) &= \frac{5}{12} \cdot \frac{4}{11} \cdot \frac{3}{10} \cdot \frac{2}{9} \cdot \frac{7}{8} \\
\mathbb{P}(A \text{ is the winner}) &= \frac{7}{12} + \frac{5}{12} \cdot \frac{4}{11} \cdot \frac{7}{10} + \frac{5}{12} \cdot \frac{4}{11} \cdot \frac{3}{10} \cdot \frac{2}{9} \cdot \frac{7}{8}.
\end{aligned}
$$

3. (16 points) Box A contains 5 orange and 7 blue balls. Box B contains 3 orange and 9 blue balls. We flip a fair coin. If the coin comes up heads, we randomly select a ball from Box A, whereas if the coin comes up tails, we randomly select a ball from Box B. Suppose the selected ball is orange, what is the probability that the coin came up tails?

Solution. Let E be the event that the selected ball is orange, let T be the event that the coin came up tails, and let H be the event that the coin came up heads. Then

$$\begin{aligned}\mathbb{P}(T|E) &= \frac{\mathbb{P}(T)P(E|T)}{\mathbb{P}(T)P(E|T)+\mathbb{P}(H)P(E|H)}\\ &= \frac{\frac{1}{2}\frac{3}{12}}{\frac{1}{2}\frac{3}{12}+\frac{1}{2}\frac{5}{12}} = \frac{3}{8}.\end{aligned}$$

4. (18 points) A box contains 15 orange, 15 blue, 15 green and 15 red balls. 10 balls are randomly selected from the box without replacement. (a) Find the probability that the color orange is missing from the 10 selected balls; (b) Find the probability that the colors orange and blue are missing from the 10 selected balls; (c) Find the probability that the 10 selected balls are all red; (d) Find the probability that at least one color is missing from the 10 selected balls.

Solution. Let A_1 be the event that orange is missing from the 10 selected balls; A_2 the event that blue is missing from the 10 selected balls; A_3 the event that green is missing from the 10 selected balls; A_4 the event that red is missing from the 10 selected balls. Then

(a)

$$\mathbb{P}(A_1) = \frac{\binom{45}{10}}{\binom{60}{10}};$$

(b)

$$\mathbb{P}(A_1 \cap A_2) = \frac{\binom{30}{10}}{\binom{60}{10}};$$

(c)

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{\binom{15}{10}}{\binom{60}{10}};$$

(d)

$$\begin{aligned}&P(A_1 \cup A_2 \cup A_3 \cup A_4)\\ &= \sum_{i=1}^{4} P(A_i) - \sum_{1\le i<j\le 4} P(A_i \cap A_j) + \sum_{1\le i<j<k\le 4} P(A_i \cap A_j \cap A_k)\\ &= 4 \cdot \frac{\binom{45}{10}}{\binom{60}{10}} - \binom{4}{2} \cdot \frac{\binom{30}{10}}{\binom{60}{10}} + \binom{4}{3} \cdot \frac{\binom{15}{10}}{\binom{60}{10}}.\end{aligned}$$

5. (16 points) Suppose that a biased coin that lands on heads with probability p is flipped 10 times.. Given that there are exactly 7 heads in these 10 flips, find the conditional probability that there are exactly 4 heads in the first 5 flips. Simplify your answer so that p does not appear in your answer.

Solution. Let E be the event that there are exactly 7 heads in the 10 flips, and let F be the event there are exactly 4 heads in the first 5 flips. Then

$$\mathbb{P}(F|E) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)} = \frac{\binom{5}{4} p^4(1-p) \binom{5}{3} p^3(1-p)^2}{\binom{10}{7} p^7(1-p)^3} = \frac{\binom{5}{4}\binom{5}{3}}{\binom{10}{7}}.$$

6. (18 points) 30 balls are randomly distributed into 20 boxes so that all 20^{30} possibilities are equally likely. Let X be the number of boxes that have exactly 1 ball. Find $\mathbb{E}[X]$. (Hint: Write X as the sum of some simpler random variables.)

Solution. For $i = 1, 2, \ldots, 20$, let $X_i = 1$ if box number i has exactly 1 ball and $X_i = 0$ otherwise. Then $X = X_1 + X_2 + \cdots + X_{20}$. For $i = 1, 2, \ldots, 20$,

$$\mathbb{P}(X_i = 1) = \frac{30 \cdot 19^{29}}{20^{30}}.$$

Thus

$$\mathbb{E}[X_1 + X_2 + \cdots + X_{20}] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_{20}] = 20 \cdot \frac{30 \cdot 19^{29}}{20^{30}}.$$