Math 461 Spring 2024

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Outline

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Example

Let $U_1, U_2, \ldots$ be independent random variables uniformly distributed in the interval $(0, 1)$. Define

$$N = \min\{n : \sum_{i=1}^{n} U_i > 1\}.$$

Find $E[N]$.

We try to find $E[N]$ by getting a more general result. For $x \in [0, 1]$, let

$$N(x) = \min\{n : \sum_{i=1}^{n} U_i > x\}$$

and

$$m(x) = E[N(x)].$$

We now derive an equation for $m(x)$ by conditioning on U_1.

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We now derive an equation for $m(x)$ by conditioning on U_1.

$$m(x) = E[N(x)] = E[E[N(x)|U_1]] = \int_0^1 E[N(x)|U_1 = y]dy.$$

Now we have

$$E[N(x)|U_1 = y] = \begin{cases} 1, & y > x, \\ 1 + m(x-y), & y \le x. \end{cases}$$

The formula above is obvious when $y > x$. It is also true for $y \le x$, since, if $U_1 = y$, then at that point, the remaining number of uniform random variables needed is the same as if we were starting and were going to add independent uniform random variables until their sum exceeded $x - y$. Thus

$$\begin{aligned} m(x) =& 1 + \int_0^x m(x-y)dy \\ =& 1 + \int_0^x m(u)du. \end{aligned}$$

Differentiating with respect to x, we get

$$m'(x) = m(x),$$

or equivalently

$$\frac{m'(x)}{m(x)} = 1.$$

Integrating this equation we get

$$\log(m(x)) = x + c$$

or

$$m(x) = ke^x.$$

Since $m(0) = 1$, we have $k = 1$ and hence $m(x) = e^x$. Therefore $m(1) = e$.

Example

Consider a gambling situation in which there are r players, with player i initially having n_i units, $n_i > 0$, $i = 1, \ldots, r$. At each stage, two of the players are chosen to play a game, with the winner of the game receiving 1 unit from the loser. Any player whose fortune drops to 0 is eliminated, and this continues until a single player has all $n = \sum_{i=1}^{r} n_i$ units, with that player designated the victor. Assuming the results of successive games are independent and each game is equally likely to be won by either of its two players, find the expected number of stages until one of the players has all n units.

We first deal wit the case there are only two players, with players 1 and 2 initially having j and $n - j$ units, respectively. Let X_j denote the number of stages that will be played, and $m_j = E[X_j]$.

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Let $Y = 1$ if player 1 wins the first game and $Y = 0$ if player 2 wins the first game. Conditioning on Y, we get that for $j = 1, \ldots, n-1$,

$$m_j = E[X_j] = E[E[X_j|Y]] = \frac{1}{2}E[X_j|Y=1] + \frac{1}{2}E[X_j|Y=0]$$
$$= 1 + \frac{1}{2}m_{j+1} + \frac{1}{2}m_{j-1}.$$

Thus, for $j = 1, \ldots, n-1$,

$$m_{j+1} = 2m_j - m_{j-1} - 2.$$

Using $m_0 = 0$, we get

$$m_2 = 2m_1 - 2$$
$$m_3 = 2m_2 - m_1 - 2 = 3(m_1 - 2)$$
$$m_4 = 2m_3 - m_2 - 2 = 4(m_1 - 3).$$

By induction, we get for $i = 1, \dots, n$,

$$m_i = i(m_1 - i + 1)$$

Since $m_n = 0$, we know that $m_1 = n - 1$. Hence for $j = 1, \dots, n - 1$,

$$m_j = j(n - j).$$

Now let deal with the general case of r players with initial amounts n_i, $i = 1, \dots, r$, $\sum_{i=1}^{r} n_i = n$. Let X be the number of stages needed to obtain a victor and let X_i denote the number of stages involving player i. Now, from the viewpoint of player i, starting with n_i, he will continue to play stages until his fortune is n or 0. Thus, the number of stages he plays is exactly the same as when he has a single opponent with an initial fortune $n - n_i$. Consequently

$$E[X_i] = n_i(n - n_i).$$

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So

$$E[\sum_{i=1}^{r} X_i] = \sum_{i=1}^{r} n_i(n - n_i) = n^2 - \sum_{i=1}^{r} n_i^2.$$

But each stage involves 2 players, and so

$$X = \frac{1}{2}\sum_{i=1}^{r} X_i.$$

Taking expectation, we get

$$E[X] = \frac{1}{2}\left(n^2 - \sum_{i=1}^{r} n_i^2\right).$$