Math 461 Spring 2024

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Outline

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HW11 is due on Friday, 04/06, before the end of class.

In the rest of the semester, I will review and answer questions. If there are certain topics or questions you want me to go over in the lecture, please send me emails.

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Chapter 1: the multiplication rule; permutations; combinations; binomial theorem; multinomial theorem.

Chapter 2: basic properties of probability measures; sample spaces with equally likely outcomes; inclusion-exclusion formula. In some problems, when ordering is not explicitly mentioned, you maybe able to solve the problems by either thinking that order is relevant, or by thinking that order is irrelevant. But you need to be consistent. If order is relevant for the numerator, order must be relevant for the denominator also. Do not mix-match them.

Chapter 3: conditional probability; using conditional probability to find probabilities of intersections; Bayes' formula; independence; independent trials; using conditioning as a tool to find probabilities of (complicated) events.

Chapter 4: random variables, distribution functions; discrete random variables; probability mass functions; expectations; variances; find the expectation of a function of a discrete random variable X given the mass function of X; basic properties of expectations and variances; Bernoulli random variables; binomial random variables; Poisson random variables; Poisson approximation to binomial random variables, geometric random variables, negative binomial random variables.

memorize basic facts, like mass function, expectation and variance; about Bernoulli random variables; binomial random variables; Poisson random variables; geometric random variables, negative binomial random variables.

Basic relations between various classes of random variables: between Bernoulli and binomial; between geometric and negative binomial.

Expectation of sums of random variables.

Section 5.1: Absolutely continuous random variables, density.

Density to distribution: integration

If X is absolutely continuous, to go from distribution to density: differentiation.

Section 5.2: Expectation of absolutely continuous random variables

X is an absolutely continuous random variable with density f and ϕ is a function on $\mathbb{R}$. Find $E[\phi(X)]$.

Variance of an absolutely continuous random variable. Properties of expectation and variance

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Section 5.3: Uniform random variables. Expectation and variance.

Section 5.4: normal random variables. density, expectation and variance

Normal approximation to binomial (the DeMoivre-Laplace central limit theorem, rounding correctly when using normal to approximate integer-valued random variables.

Section 5.5 Exponential random variables: density, distribution, expectation, variance, memoryless property.

Section 5.6 Gamma random variables: density, expectation and variance.

exp is the continuous counterpart of geometric, Gamma is the continuous counterpart of negative binomial

Section 5.7: X is an absolutely continuous random variable with density and ϕ is a function on $\mathbb{R}$. Find the density of $Y = \phi(X)$.

First find the distribution of Y, and then differentiate.

relation between normal and Gamma

Section 6.1: Joint distribution functions, marginal distribution functions.

Joint mass functions, marginal mass functions

Jointly abs. cont random variables, joint density, marginal density

If X and Y are jointly abs. cont with joint density f, to find the probability of an event defined in terms of X and Y, simply integrate the joint density in an appropriate region (double integral).

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Section 6.2: Independent random variables. When finding probability involving two independent discrete random variable, try to break things up appropriately and then use independence

Section 6.3 Sums of independent random variables.

Sums of independent binomial random variables; Sums of independent Poisson random variables, Sums of independent negtaive binomial random variables; Sums of independent normal random variables, Sums of independent Gamma random variables.

Section 6.4: Conditional mass function

Section 6.5: Conditional density, find conditional probabilities

Section 6.6: the joint density of the min and max of n iid abs cont random variables. The density of the min, the density of the max

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Sections 4.9 and 7.2: Expectation of sums of random variables

Section 7.4: covariance, correlation, properties of covariance, variance of sums of random variables.

Section 7.5: Conditional expectation, computing expectations by conditioning.

Section 7.7: Moment generating functions. I will not give a problem on the final that you have to use moment generating functions. But knowing the basic properties of moment generating functions does not hurt.

Basic facts about important random variables are summarized in two tables in Chap. 7 .

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Suppose that $X_1, \ldots X_n$ are independent and identically distributed absolutely continuous random variables with common density f and common distribution F. Let $U = X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $V = X_{(n)} = \max\{X_1, \ldots, X_n\}$. Let's find the joint density of U and V. To find the joint density, we first look for the joint distribution of U and V, and then we take the mixed partial derivative.

For $u < v$,

$$\begin{aligned} P(U \le u, V \le v) &= P(V \le v) - P(U > u, V \le v) \\ &= P(X_1 \le v, \ldots, X_n \le v) - P(u < X_1 \le v, \ldots, u < X_n \le v) \\ &= [F(v)]^n - [F(v) - F(u)]^n. \end{aligned}$$

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For $u \geq v$,

$$P(U \leq u, V \leq v) = P(V \leq v) = [F(v)]^n.$$

Thus the joint density of U and V is

$$f_{U,V}(u, v) = \begin{cases} n(n-1)[F(v) - F(u)]^{n-2} f(u) f(v), & u < v, \\ 0, & \text{otherwise.} \end{cases}$$

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From the derivations above, we have

$$P(U \le u) = P(U \le u, V < \infty) = 1 - [1 - F(u)]^n$$
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Thus,

$$f_U(u) = n[1 - F(u)]^{n-1} f(u),$$
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Example 1

Let $X_1, \ldots, X_n$ be independent and identically distributed on $(0,1)$. Find the joint density of $U = \min\{X_1, \ldots, X_n\}$ and $V = \max\{X_1, \ldots, X_n\}$.

The joint density is

$$f_{U,V}(u,v) = \begin{cases} n(n-1)(v-u)^{n-2}, & 0 < u < v < 1, \\ 0, & \text{otherwise.} \end{cases}$$

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A Bernoulli random variable with parameter p is a random variable taking only 0 and 1 as its values, the parameter p is equal to $P(X = 1)$.

Suppose that independent trials, each resulting in a success with probability p, are performed n times. If X is the number of successes in these n trials, then X is a binomial random variable with parameters (n, p). The mass function of X is

$$p(x) = \begin{cases} \binom{n}{x} p^x (1 - 0^{n-x}, & x = 0, 1, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

A Bernoulli random variable with parameter p is a binomial random variable with parameter $(1, p)$.

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A Poisson random variable with parameter λ is a non-negative integer-valued random variable with mass function

$$p(x) = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!}, & x = 0, 1, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that X is a binomial random variable with parameter (n, p). If n is large and p is small so that np is of moderate size, then X is approximately a Poisson random variable with parameter np.

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