Math 461 Spring 2024

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Outline

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1. General Info
2. 8.4 The strong law of large numbers

HW10 is due today, before the end of class.

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1 General Info

2 8.4 The strong law of large numbers

In Section 8.2, we discussed the weak law of large numbers.

Weak law of large numbers

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common (finite) mean $E[X_1] = \mu$. Then, for any $\epsilon > 0$,

$$P\left(\left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \epsilon \right) \to 0 \quad \text{as} \quad n \to \infty.$$

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We say that a sequence of random variables Z_n converge to a random variable Z in probability if, for any $\epsilon > 0$,

$$\lim_{n \to \infty} P(|Z_n - Z| \geq \epsilon) = 0.$$

Using this concept, the weak law of large numbers can be stated

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In this section we give the following

strong law of large numbers

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common (finite) mean $E[X_1] = \mu$. Then, with probability 1,

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Suppose the sample space is (0, 1] and the probability of an interval is its length. Define

$$
\begin{aligned}
X_1(x) &= 1_{(0,1/2]}(x); & X_2(x) &= 1_{(1/2,1]}(x),\\
X_3(x) &= 1_{(0,1/4]}(x), & X_4(x) &= 1_{(1/4,1/2]}(x),\\
X_5(x) &= 1_{(1/2,3/4]}(x), & X_6(x) &= 1_{(3/4,1]}(x),\\
X_7(x) &= 1_{(0,1/8]}(x), & X_8(x) &= 1_{(1/8,1/4]}(x),\\
X_9(x) &= 1_{(1/4,3/8]}(x), & X_{10}(x) &= 1_{(3/8,1/2]}(x),\\
X_{11}(x) &= 1_{(1/2,5/8]}(x), & X_{12}(x) &= 1_{(5/8,3/4]}(x),\\
X_{13}(x) &= 1_{(3/4,7/8]}(x), & X_{14}(x) &= 1_{(7/8,1]}(x),\\
&\cdots & &\cdots
\end{aligned}
$$

Then obviously X_n converges to 0 in probability. But for for any $x \in (0, 1]$, $X_n(x)$ does not converge.

Proof of the strong law of large numbers

I am going to give a proof under the additional assumption that $E[X_1^4] = K < \infty$.

By considering $X_n' = X_n - \mu$ if necessary, we may and do assume that $\mu = 0$. We now show $(X_1 + \cdots + X_n)/n$ tend to 0 with probability 1.

Let $S_n = X_1 + \cdots + X_n$. Consider

$$E[S_n^4] = E[(X_1 + \cdots + X_n)^4]$$

Expanding the right side will results in terms of the form

$$X_i^4, \quad X_i^3 X_j, \quad X_i^2 X_j^2, \quad X_i^2 X_j X_k, \quad X_i X_j X_k X_l$$

where i, j, k, l are all different. Since all the X_i have mean 0, it follows by independence that

Proof of the strong law of large numbers (cont)

$$E[X_i^3 X_j] = E[X_i^3]E[X_j] = 0,$$
$$E[X_i^2 X_j X_k] = E[X_i^2]E[X_j]E[X_k] = 0,$$
$$E[X_i X_j X_k X_l] = E[X_i]E[X_j]E[X_k]E[X_l] = 0.$$

For, for a given pair i and j, there are $\binom{4}{2} = 6$ terms in the expansion that equal to $X_i^2 X_j^2$. Hence

$$\begin{aligned} E[S_n^4] &= nE[X_1^4] + 6\binom{n}{2}E[X_1^2 X_2^2] \\ &= nK + 3n(n-1)E[X_1^2]E[X_2^2] \\ &= nK + 3n(n-1)(E[X_1^2])^2. \end{aligned}$$

Proof of the strong law of large numbers (cont)

Now, since

$$0 \leq \mathrm{Var}(X_1^2) = E[X_1^4] - (E[X_1^2])^2,$$

we have

$$(E[X_1^2])^2 \leq E[X_1^4] = K.$$

Therefore,

$$E[S_n^4] \leq nK + 3n(n-1)K$$

which implies

$$E\left[\frac{S_n^4}{n^4}\right] \leq \frac{K}{n^3} + \frac{3K}{n^2}.$$

Consequently,

$$E\left[\sum_{n=1}^{\infty} \frac{S_n^4}{n^4}\right] \leq \sum_{n=1}^{\infty} E\left[\frac{S_n^4}{n^4}\right] < \infty.$$

Proof of the strong law of large numbers (cont)

Thus, with probability 1, $\sum_{n=1}^{\infty} \frac{S_n^4}{n^4} < \infty$, which implies that with probability 1, $\frac{S_n^4}{n^4} \to 0$, and hence $\frac{S_n}{n} \to 0$. The proof is now complete.