# Math 461 Spring 2024

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# Outline

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Solution to HW9 is is on my homepage now.

Test 2 is this Friday. On Wed, I will do a brief review and then use the rest of the time to answer questions.

Materials covered on Test 2: Section 4.9, Section 5.1, Section 5.2, Section 5.3, Section 5.4, Section 5.5, Section 5.6, Section 5.7, Section 6.1, Section 6.2, Section 6.3, Section 6.4, Section 6.5, Section 6.6, Section 7.2, Section 7.4

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The moment generating function of a random variable $X$ is defined to be the function

$$M_X(t) = E[e^{tX}], \quad t \in \mathbb{R}.$$

$M_X(t)$ may not be defined for all $t \in \mathbb{R}$, but it is always defined for $t = 0$. In fact, $M_X(0) = 1$. We will concentrate on random variables $X$ for which $M_X(t)$ is defined at least in an interval around the origin. All the important random variables we learned in the course satisfy this property.

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## Proposition

**(i)** If $X$ is a binomial random variable with parameters $(n, p)$, then

$$M_X(t) = (pe^t + 1 - p)^n, \qquad \text{for all } t \in \mathbb{R}.$$

**(ii)** If $X$ is a Poisson random variable with parameter $\lambda$, then

$$M_X(t) = e^{\lambda(e^t - 1)}, \qquad \text{for all } t \in \mathbb{R}.$$

**(iii)** If $X$ is a geometric random variable with parameter $p$, then

$$M_X(t) = \frac{pe^t}{1 - (1 - p)e^t}, \qquad \text{for all } t < -\ln(1 - p).$$

**(iv)** If $X$ is a negative binomial random variable with parameters $(r, p)$, then

$$M_X(t) = \left(\frac{pe^t}{1 - (1 - p)e^t}\right)^r, \qquad \text{for all } t < -\ln(1 - p).$$

## Proposition (cont)

**(v)** If $X$ is uniformly distributed in the interval $(a, b)$, then

$$M_X(t) = \frac{e^{tb} - e^{ta}}{b - a}, \quad \text{for all } t \in \mathbb{R}.$$

**(vi)** If $X$ is an exponential random variable with parameter $\lambda$, then

$$M_X(t) = \frac{\lambda}{\lambda - t}, \quad \text{for all } t < \lambda.$$

**(vii)** If $X$ is a Gamma random variable with parameters $(\alpha, \lambda)$, then

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}, \quad \text{for all } t < \lambda.$$

**(viii)** If $X$ is a normal random variable with parameters $(\mu, \sigma^2)$, then

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right), \quad \text{for all } t \in \mathbb{R}.$$

**Theorem**

If $X$ and $Y$ are two random variables with $M_X(t) = M_Y(t)$ for all $t$, then $X$ and $Y$ have the same distribution.

This theorem says that the moment generating function $M_X(t)$ of $X$ also contains all the statistical information about $X$.

**Theorem**

If $X$ and $Y$ are independent random variables, then

$$M_{X+Y}(t) = M_X(t)M_Y(t), \qquad \text{for all } t.$$

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**Theorem**

Suppose that $X$ and $Y$ are independent random variables.

**(i)** If $X$ is a binomial random variable with parameters $(m, p)$ and $Y$ is a binomial random variable with parameters $(n, p)$, then $X + Y$ is a binomial random variable with parameters $(m + n, p)$.

**(ii)** If $X$ is a Poisson random variable with parameter $\lambda_1$ and $Y$ is a Poisson random variable with parameter $\lambda_2$, then $X + Y$ is a Poisson random variable with parameters $\lambda_1 + \lambda_2$.

**(iii)** If $X$ is a negative binomial random variable with parameters $(r, p)$ and $Y$ is a negative binomial random variable with parameters $(s, p)$, then $X + Y$ is a negative binomial random variable with parameters $(r + s, p)$.

**Theorem (cont)**

**(iv)** If $X$ is a normal random variable with parameters $(\mu_1, \sigma_1^2)$ and $Y$ is a normal random variable with parameters $(\mu_2, \sigma_2^2)$, then $X + Y$ is a normal random variable with parameters $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

**(v)** If $X$ is a Gamma random variable with parameters $(\alpha_1, \lambda)$ and $Y$ is a Gamma random variable with parameters $(\alpha_2, \lambda)$, then $X + Y$ is a Gamma random variable with parameters $(\alpha_1 + \alpha_2, \lambda)$.

The proofs of all five parts are the same. Let's write out the details for the case of normal random variables.

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We know that

$$M_X(t) = \exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right), \quad M_Y(t) = \exp\left(\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right).$$

Since $X$ and $Y$ are independent,

$$\begin{aligned} M_{X+Y}(t) &= M_X(t)M_Y(t) \\ &= \exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right)\exp\left(\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right) \\ &= \exp\left((\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}\right). \end{aligned}$$

Hence $X + Y$ is a normal random variable with parameters $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

**Example**

Suppose that $X_1, X_2, \ldots$ are independent Bernoulli random variables with parameter $p \in (0, 1)$, and that $N$ is a Poisson random variable with parameter $\lambda$ independent of all the $X_i$'s. Define

$$Y = \sum_{i=1}^{N} X_i$$

with the convention $\sum_{i=1}^{0} = 0$. Find $E[Y]$ and $\mathrm{Var}(Y)$.

Let try to find out the moment generating function of $Y$.

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Let try to find out the moment generating function of $Y$.

$$
\begin{aligned}
M_Y(t) &= E\left[\exp\left(t\sum_{i=1}^{N} X_i\right)\right] = E\left[E\left[\exp\left(t\sum_{i=1}^{N} X_i\right)|N\right]\right] \\
&= \sum_{n=0}^{\infty} E\left[\exp\left(t\sum_{i=1}^{N} X_i\right)|N=n\right] P(N=n) \\
&= \sum_{n=0}^{\infty} E\left[\exp\left(t\sum_{i=1}^{n} X_i\right)|N=n\right] P(N=n) \\
&= \sum_{n=0}^{\infty} E\left[\exp\left(t\sum_{i=1}^{n} X_i\right)\right] P(N=n) \\
&= \sum_{n=0}^{\infty} (pe^t + 1 - p)^n e^{-\lambda} \frac{\lambda^n}{n!} \\
&= e^{\lambda p(e^t - 1)}
\end{aligned}
$$

Thus $Y$ is a Poisson random variable with parameter $\lambda p$. Hence

$$E[Y] = \mathrm{Var}(Y) = \lambda p.$$

Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables and that $N$ is a non-negative integer-valued random variable independent of all the $X_i$'s. Define

$$Y = \sum_{i=1}^{N} X_i$$

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Let $M_X(t)$ be the common moment generating function of $X_1, X_2, \ldots$.

$$
\begin{aligned}
M_Y(t) &= E\left[\exp\left(t\sum_{i=1}^{N} X_i\right)\right] = E\left[E\left[\exp\left(t\sum_{i=1}^{N} X_i\right)|N\right]\right] \\
&= \sum_{n=0}^{\infty} E\left[\exp\left(t\sum_{i=1}^{N} X_i\right)|N=n\right] P(N=n) \\
&= \sum_{n=0}^{\infty} E\left[\exp\left(t\sum_{i=1}^{n} X_i\right)|N=n\right] P(N=n) \\
&= \sum_{n=0}^{\infty} E\left[\exp\left(t\sum_{i=1}^{n} X_i\right)\right] P(N=n) \\
&= \sum_{n=0}^{\infty} (M_X(t))^n P(N=n) \\
&= E[(M_X(t))^N] = E[e^{(\ln M_X(t))N}] = M_N(\ln M_X(t)).
\end{aligned}
$$

Taking derivative, we get

$$M_Y'(t) = E[N\,(M_X(t))^{N-1}\,M_X'(t)].$$

Hence

$$M_Y'(0) = E[N\,(M_X(0))^{N-1}\,M_X'(0)] = E[NE[X]] = E[N]E[X].$$

Taking derivative again, we get

$$M_Y''(t) = E[N(N-1)\,(M_X(t))^{N-2}\,(M_X'(t))^2 + N\,(M_X(t))^{N-1}\,M_X''(t)].$$

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Thus

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\begin{aligned}
E[Y^2] = M_Y''(0) &= E[N(N-1)\,(E[X])^2 + NE[X^2]] \\
&= (E[X])^2\,(E[N^2] - E[N]) + E[N]E[X^2] \\
&= E[N](E[X^2] - (E[X])^2) + (E[X])^2\,E[N^2] \\
&= E[N]\mathrm{Var}(X) + (E[X])^2\,E[N^2].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{Var}(Y) &= E[N]\mathrm{Var}(X) + (E[X])^2\,E[N^2] - (E[X])^2\,(E[N])^2 \\
&= E[N]\mathrm{Var}(X) + (E[X])^2\,\mathrm{Var}(N).
\end{aligned}
$$

Thus

$$E[Y^2] = M_Y''(0) = E[N(N-1)\,(E[X])^2 + NE[X^2]]$$
$$= (E[X])^2\,(E[N^2] - E[N]) + E[N]E[X^2]$$
$$= E[N](E[X^2] - (E[X])^2) + (E[X])^2\,E[N^2]$$
$$= E[N]\mathrm{Var}(X) + (E[X])^2\,E[N^2].$$

Therefore

$$\mathrm{Var}(Y) = E[N]\mathrm{Var}(X) + (E[X])^2\,E[N^2] - (E[X])^2\,(E[N])^2$$
$$= E[N]\mathrm{Var}(X) + (E[X])^2\,\mathrm{Var}(N).$$