Math 461 Spring 2024

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University of Illinois Urbana-Champaign

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Outline

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HW8 is due Friday, 03/29, before the end of class.

Solution to HW7 is on my homepage now.

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Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed absolutely continuous random variables with common density f and common distribution F. Define

$$\begin{aligned} X_{(1)} &= \min\{X_1, \ldots, X_n\} = \text{ smallest of } X_1, \ldots, X_n \\ X_{(2)} &= \text{ second smallest of } X_1, \ldots, X_n \\ &\ldots \\ X_{(n)} &= \max\{X_1, \ldots, X_n\} = \text{ largest of } X_1, \ldots, X_n. \end{aligned}$$

$X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are called the order statistics of $X_1, X_2, \ldots, X_n$. We could find the joint density of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. I am only going to present the joint density of $X_{(1)}$ and $X_{(n)}$.

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Let $U = X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $V = X_{(n)} = \max\{X_1, \ldots, X_n\}$. Let's find the joint density of U and V. To find the joint density, we first look for the joint distribution of U and V, and then we take the mixed partial derivative.

For $u < v$,

$$P(U \le u, V \le v) = P(V \le v) - P(U > u, V \le v)$$
$$= P(X_1 \le v, \ldots, X_n \le v) - P(u < X_1 \le v, \ldots, u < X_n \le v)$$
$$= [F(v)]^n - [F(v) - F(u)]^n.$$

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For $u \geq v$,

$$P(U \leq u, V \leq v) = P(V \leq v) = [F(v)]^n.$$

Thus the joint density of U and V is

$$f_{U,V}(u,v) = \begin{cases} n(n-1)[F(v) - F(u)]^{n-2} f(u) f(v), & u < v, \\ 0, & \text{otherwise.} \end{cases}$$

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From the derivations above, we have

$$P(U \le u) = P(U \le u, V < \infty) = 1 - [1 - F(u)]^n$$
$$P(V \le v) = P(U \le v, V \le v) = [F(v)]^n.$$

Thus,

$$f_U(u) = n[1 - F(u)]^{n-1} f(u),$$
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Example 1

Let $X_1, \ldots, X_n$ be independent and identically uniformly distributed on $(0, 1)$. Find the joint density of $U = \min\{X_1, \ldots, X_n\}$ and $V = \max\{X_1, \ldots, X_n\}$.

The joint density is

$$f_{U,V}(u, v) = \begin{cases} n(n-1)(v-u)^{n-2}, & 0 < u < v < 1, \\ 0, & \text{otherwise.} \end{cases}$$

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Proposition

Suppose that X and Y are discrete random variables with joint mass function $p(\cdot,\cdot)$, and that g is a function on $\mathbb{R}^2$. If

$$\sum_{x,y} |g(x,y)| p(x,y) < \infty,$$

then

$$E[g(X,Y)] = \sum_{x,y} g(x,y)p(x,y).$$

Suppose that X and Y are jointly abs. cont. random variables with joint density $f(\cdot,\cdot)$, and that g is a function on $\mathbb{R}^2$. If

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$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy.$$

By taking $g(x, y) = x$, we get that in the discrete case,

$$E[X] = \sum_{x,y} xp(x, y),$$

and the absolutely continuous case,

$$E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x, y)dxdy.$$

By taking $g(x, y) = y$, we get that in the discrete case,

$$E[Y] = \sum_{x,y} yp(x, y),$$

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By taking $g(x, y) = x + y$, we get that in the discrete case,

$$E[X + Y] = \sum_{x,y} (x + y)p(x, y) = E[X] + E[Y].$$

and the absolutely continuous case,

$$E[X + Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y)f(x, y)dxdy = E[X] + E[Y].$$

Using induction, we can show that

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i].$$

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Example 1

An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of the accident, an ambulance is at a location Y that is also uniformly distributed on that stretch of the road. Assume that X and Y are independent. Find the expected distance between the ambulance and the accident.

The joint density of X and Y is

$$f(x,y) = \begin{cases} \frac{1}{L^2}, & 0 < x, y < L, \\ 0, & \text{otherwsie.} \end{cases}$$

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$$E[|X-Y|] = \frac{1}{L^2}\int_0^L\int_0^L |x-y|dxdy.$$

$$\int_0^L |x-y|dx = \int_0^y (y-x)dx + \int_y^L (x-y)dx$$
$$= y^2 - \frac{y^2}{2} + \frac{L^2}{2} - \frac{y^2}{2} - y(L-y)$$
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$$E[|X-Y|] = \frac{1}{L^2}\int_0^L (\frac{L^2}{2} + y^2 - y)dy = \frac{L}{3}.$$

Now we are going to use the formula

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

to find the expectation of some complicated random variables. In Section 4.9, we have seen quite a few examples of this nature. Please review these examples. Now I am going to give more examples of this nature.

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Example 2

Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability p, find the expected number of ducks that escape unhurt when a flock of size 10 flies overhead.

For $i = 1, \ldots, 10$, let $X_i = 1$ if the i-th duck escapes unhurt and $X_i = 0$ otherwise. We are looking for $E[X_1 + \cdots + X_{10}]$. Since each hunter will, independently, hit the i-th duck with probability $p/10$,

$$E[X_i] = P(X_i = 1) = \left(1 - \frac{p}{10}\right)^{10}.$$

Hence

$$E[X_1 + \cdots + X_{10}] = 10\left(1 - \frac{p}{10}\right)^{10}.$$

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Example 3

Suppose that n balls are randomly distributed into r boxes (so that all r^n possibilities are equally likely). Find the expected number of boxes that has exactly 1 ball.

For $i = 1, \ldots, r$, Let $X_i = 1$ if the i-th box has exactly 1 ball and $X_i = 0$ otherwise. We are looking for $E[X_1 + \cdots + X_r]$. For $= 1, \ldots, r$,

$$E[X_i] = P(X_i = 1) = n\frac{1}{r}\left(\frac{r-1}{r}\right)^{n-1}.$$

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