Math 461 Spring 2024

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University of Illinois Urbana-Champaign

March 25, 2024

Outline

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HW8 is due Friday, 03/29, before the end of class. Please submit your HW8 as a pdf file via the course Moodle page.

Solution to HW7 is on my homepage now.

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Let X and Y be discrete random variables with joint mass function $p(\cdot,\cdot)$. If y is a possible value of Y (i.e, $p_Y(y) > 0$), then

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x,y)}{p_Y(y)}.$$

The function $x \mapsto \frac{p(x,y)}{p_Y(y)}$ is a mass function. It is called the conditional mass function of X given $Y = y$.

The function

$$p_{X|Y}(x|y) = \begin{cases} \frac{p(x,y)}{p_Y(y)}, & p_Y(y) > 0, \\ 0, & \text{otherwise} \end{cases}$$

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is called the conditional mass function of X given Y.

If X and Y are independent, then for any possible value y of Y,

$$p_{X|Y}(x|y) = p_X(x), \quad x \in \mathbb{R}.$$

We always have

$$p(x,y) = p_Y(y)p_{X|Y}(x|y), \quad x, y \in \mathbb{R}.$$

We can similarly define the conditional mass function of Y given X:

$$p_{Y|X}(y|x) = \begin{cases} \frac{p(x,y)}{p_X(x)}, & p_X(x) > 0, \\ 0, & \text{otherwise} \end{cases}$$

We also have

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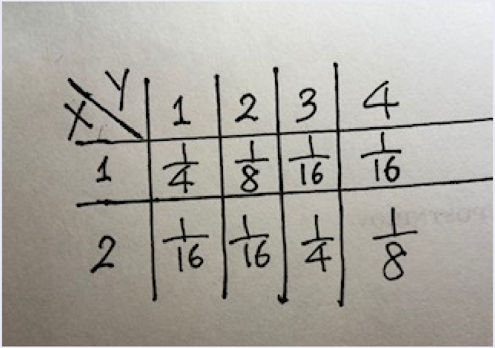
Example 1

The joint mass function of X and Y is given below. Find $p_{X|Y}(x|2)$.



X \ Y	1	2	3	4
1	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{8}$

$p_Y(2) = 3/16$. So

$$p_{X|Y}(x|2) = \begin{cases} 2/3, & x = 1, \\ 1/3, & x = 2, \\ 0, & \text{otherwsie.} \end{cases}$$

Example 1

Suppose X and Y are independent, X is a Poisson random variable with parameter λ_1, Y is a Poisson random variable with parameter λ_2. For $n \geq 1$, find the conditional mass function of X given $X + Y = n$.

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Example 1

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We know that $X+Y$ a Poisson random variable with parameter $\lambda_1+\lambda_2$. If $X+Y=n$, then X can only take values $0,1,\ldots,n$. For any $x=0,1,\ldots,n$,

$$
\begin{aligned}
p_{X|X+Y}(x|n) &= \frac{P(X=x, X+Y=n)}{P(X+Y=n)} = \frac{P(X=x, Y=n-x)}{P(X+Y=n)} \\
&= \frac{P(X=x)P(Y=n-x)}{P(X+Y=n)} = \frac{e^{-\lambda_1}\frac{\lambda_1^x}{x!}e^{-\lambda_2}\frac{\lambda_2^{n-x}}{(n-x)!}}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}} \\
&= \binom{n}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-x}.
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Thus, given $X+Y=n$, X is a binomial random variable with parameters $(n, \lambda_1/(\lambda_1+\lambda_2))$.

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$$\begin{aligned} p_{X|X+Y}(x|n) &= \frac{P(X = x, X + Y = n)}{P(X + Y = n)} = \frac{P(X = x, Y = n - x)}{P(X + Y = n)} \\ &= \frac{P(X = x)P(Y = n - x)}{P(X + Y = n)} = \frac{e^{-\lambda_1} \frac{\lambda_1^x}{x!} e^{-\lambda_2} \frac{\lambda_2^{n-x}}{(n-x)!}}{e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!}} \\ &= \binom{n}{x} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-x}. \end{aligned}$$

Thus, given $X + Y = n$, X is a binomial random variable with parameters $(n, \lambda_1/(\lambda_1 + \lambda_2))$.

Example 2

X and Y are independent geometric random variables with parameter p. For $n \geq 2$, find the conditional mass function of X given $X + Y = n$.

$X + Y$ is a negative binomial random variable with parameters $(2, p)$:

$$p_{X+Y}(n) = \binom{n-1}{1} p^2 (1-p)^{n-2}, \quad n = 2, 3, \ldots.$$

Give $X + Y = n$, X can only take values $1, \ldots, n-1$. For, $x = 1, \ldots, n-1$,

$$\begin{aligned} p_{X|X+Y}(x|n) &= \frac{P(X = x, X + Y = n)}{P(X + Y = n)} = \frac{P(X = x, Y = n - x)}{P(X + Y = n)} \\ &= \frac{P(X = x)P(Y = n - x)}{P(X + Y = n)} = \frac{p(1-p)^{x-1} p(1-p)^{n-x-1}}{(n-1)p^2(1-p)^{n-2}} \\ &= \frac{1}{n-1}. \end{aligned}$$

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Thus the conditional mass function of X given $X + Y = n$ is

$$p_{X|X+Y}(x|n) = \begin{cases} \frac{1}{n-1}, & x = 1, \ldots, n-1, \\ 0, & \text{otherwise.} \end{cases}$$

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A number Y is chosen randomly from $\{1, 2, \ldots, 100\}$ and then another number X is randomly chosen from $\{1, 2, \ldots, Y\}$. Find the joint mass function of X and Y.

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For any $y = 1, \ldots, 100$,

$$p_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & x = 1, \ldots, y, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the joint mass function of X and Y is

$$p(x, y) = p_Y(y) p_{X|Y}(x|y) = \begin{cases} \frac{1}{100y}, & y = 1, \ldots, 100; x = 1, \ldots, y, \\ 0, & \text{otherwise.} \end{cases}$$

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Outline

Suppose that X and Y are jointly absolutely continuous with joint density $f(\cdot,\cdot)$. For any y with $f_Y(y) > 0$, the function

$$x \mapsto \frac{f(x,y)}{f_Y(y)}, \qquad x \in \mathbb{R}$$

is a probability density function. It is called the conditional density of X given $Y = y$.

More generally, the function

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For any y with $f_Y(y) > 0$, the conditional density $f_{X|Y}(x|y)$ allows us to define the conditional probability $P(X \in A|Y = y)$. For example, for any $a < b$,

$$P(X \in (a,b)|Y = y) = \int_a^b f_{X|Y}x|y)dx.$$

$$P(X \in (a,b)|Y = y) = \lim_{h \downarrow 0} P(X \in (a,b)|Y \in (y-h, y+h))$$

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Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 < x < y, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{Y|X}(y|x)$ for $0 < x < y$.

For $x > 0$,

$$f_X(x) = \int_x^{\infty} \lambda^2 e^{-\lambda y} dy = \lambda e^{-\lambda x}.$$

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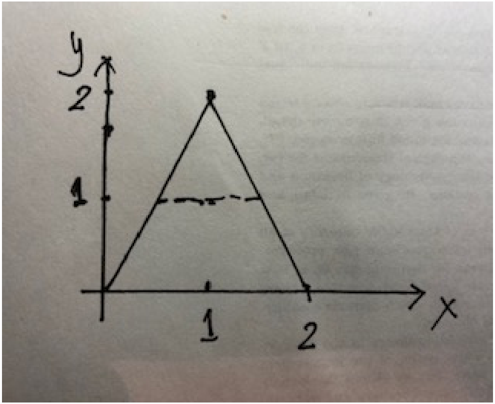
Example 2

Let X and Y be uniformly distributed in the triangle with vertices at $(0,0),(2,0),(1,2)$. Find $P(X \leq 1 | Y = 1)$.



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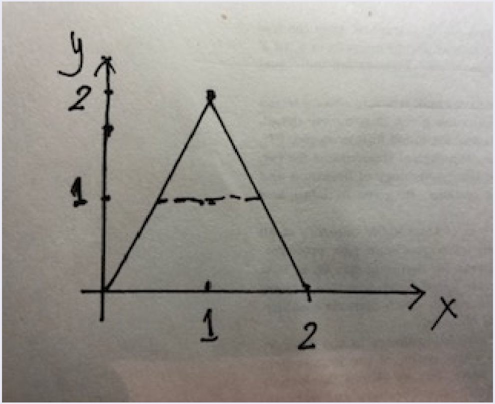

$f_Y(1) = \frac{1}{2}\int_{1/2}^{3/2} dx = \frac{1}{2}$. Thus

$$f_{X|Y}(x|1) = \begin{cases} 1, & x \in (1/2, 3/2), \\ 0, & \text{otherwise.} \end{cases}$$

Thus $P(X \leq 1|Y = 1) = \frac{1}{2}$.

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Suppose that a point X is randomly chosen from the interval $(0, 1)$, and the a point Y is chosen randomly from $(0, X)$. Find the joint density of X and Y.

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