Math 461 Spring 2024

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University of Illinois Urbana-Champaign

March 20, 2024

Outline

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HW7 is due Friday, 03/22, before the end of class time . Please submit your HW7 via the course Moodle page.

Solution to HW6 is on my homepage now.

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Two random variables X and Y are said to be independent if for any two subsets A and B of $\mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

It can be shown that X and Y are independent if and only if

$$F(x, y) = F_X(x)F_Y(y), \quad (x, y) \in \mathbb{R}^2.$$

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$$f(x,y) = f_X(x)f_Y(y), \quad (x,y) \in \mathbb{R}^2.$$

Example 2

Suppose that the number of people entering a certain post office on a given day is a Poisson random variable with parameter $\lambda > 0$. Assume that each person entering the post office is male with probability p and female with probability $1 - p$, independent of all others. Show that the number of males and the number of females entering the post office on a given day are independent Poisson random variables with parameters λp and $\lambda(1 - p)$ respectively.

Let X and Y be the number of males and the number of females entering the post office on a given day respectively. X and Y are non-negative integer-valued random variables.

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Let X and Y be the number of males and the number of females entering the post office on a given day respectively. X and Y are non-negative integer-valued random variables.

For any non-negative integers i and j,

$$P(X=i,Y=j)=P(X=i,Y=j|X+Y=i+j)P(X+Y=i+j)$$
$$=\binom{i+j}{i}p^i(1-p)^je^{-\lambda}\frac{\lambda^{i+j}}{(i+j)!}$$
$$=e^{-\lambda p}\frac{(\lambda p)^i}{i!}e^{-\lambda(1-p)}\frac{(\lambda(1-p))^j}{j!}.$$

Hence

$$P(X=i)=e^{-\lambda p}\frac{(\lambda p)^i}{i!}\sum_{j=0}^{\infty}e^{-\lambda(1-p)}\frac{(\lambda(1-p))^j}{j!}=e^{-\lambda p}\frac{(\lambda p)^i}{i!}.$$

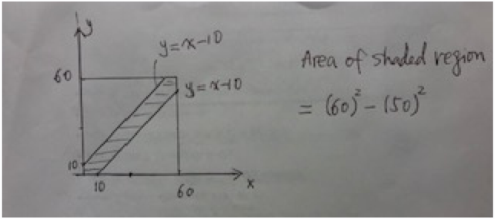
Similarly

$$P(Y=j)=e^{-\lambda(1-p)}\frac{(\lambda(1-p))^j}{j!}.$$

Therefore X and Y are independent Poisson random variables with parameters λp and $\lambda(1-p)$ respectively.

Example 3

A man and a woman decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between noon and 1 pm. Find the probability that the first to arrive needs to wait no more than 10 minutes.



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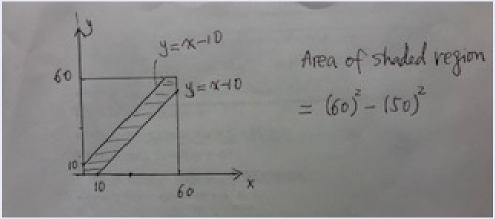

The answer is

$$\frac{60^2 - 50^2}{60^2} = \frac{11}{36}.$$

Proposition

(i) Suppose that X and Y are discrete with joint mass function $p(\cdot,\cdot)$. Then X and Y are independent if and only if

$$p(x,y) = g(x)h(y), \quad (x,y) \in \mathbb{R}^2$$

for some functions g and h on $\mathbb{R}$.

(ii) Suppose that X and Y are jointly abs. cont. with joint density $f(\cdot,\cdot)$. Then X and Y are independent if and only if

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Example 4

The joint density of X and Y is

$$f(x,y) = \begin{cases} 10e^{-(2x+5y)}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

If

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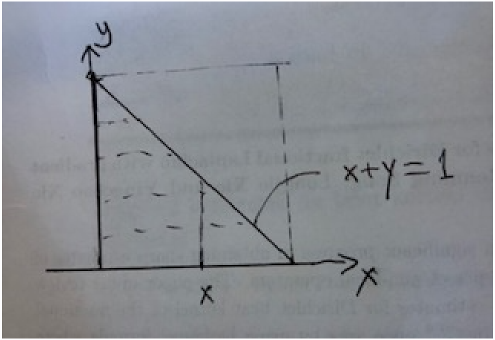
Example 5

The joint density of X and Y is

$$f(x,y) = \begin{cases} 24xy, & x \in (0,1), y \in (0,1), x+y \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$



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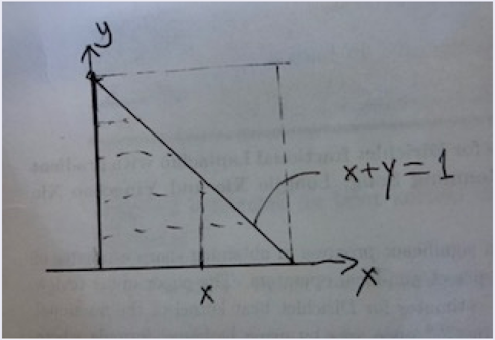

Both X and Y take values in $(0,1)$. For $x \in (0,1)$,

$$f_X(x) = \int_0^{1-x} 24xydy = 12x(1-x)^2.$$

Similarly, for $y \in (0,1)$,

$$f_Y(y) = 12y(1-y)^2.$$

X and Y are not independent!

The concept of independent random variables can be extended to more than 2 random variables.

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n random variables $X_1, \dots, X_n$ are said to be independent if for any subsets $A_1, \dots, A_n$ of $\mathbb{R}$,

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It can be shown that n random variables $X_1, \dots, X_n$ with joint distribution function $F(\cdot, \dots, \cdot)$ are independent if and only if

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Example 6

Suppose that $X_1, \ldots, X_n$ are independent absolutely continuous random random variables with a common density f. . Define

$$U = \min\{X_1, \ldots, X_n\}, \quad V = \max\{X_1, \ldots, X_n\}.$$

Find the densities of U and V respectively.

Let's deal with V first. Let F be the common distribution. For any $v \in \mathbb{R}$,

$$P(V \le v) = P(X_1 \le v, \ldots, X_n \le v) = (F(v))^n.$$

Thus the density of V is $f_V(v) = n(F(v))^{n-1} f(v)$.

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Now let's deal with U. For any $u \in \mathbb{R}$,

$$\begin{aligned} P(U \leq u) &= 1 - P(U > u) = 1 - P(X_1 > u, \ldots, X_n > u) \\ &= 1 - (1 - F(u))^n. \end{aligned}$$

Thus the density of U is

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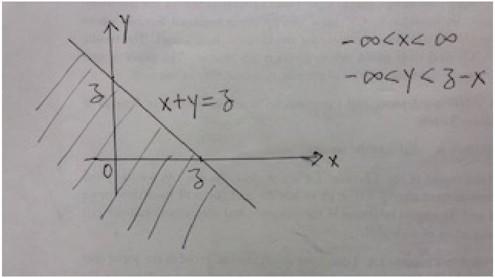
Suppose X and Y are independent abs. cont. random variables with density f_X and f_Y respectively. Find the density of $Z = X + Y$.



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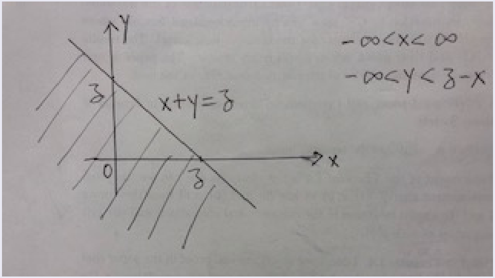

For any $z \in \mathbb{R}$,

$$\begin{aligned} F_Z(z) &= P(X + Y \leq z) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f_X(x) f_Y(y) dy \right) dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z} f_X(x) f_Y(v - x) dv \right) dx, \quad (y = v - x) \\ &= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_X(x) f_Y(v - x) dx dv \end{aligned}$$

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