Math 461 Spring 2024

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Outline

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1. General Info
2. 6.1 Joint distribution functions
3. 6.2 Independent random variables

HW7 is due Friday, 03/22, before the end of class time .

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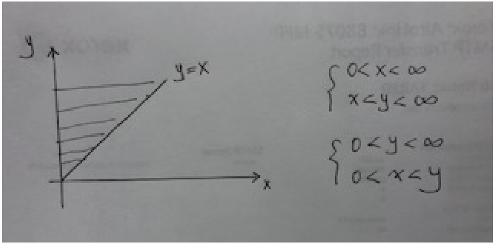
Example 3

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find $P(X < Y)$.



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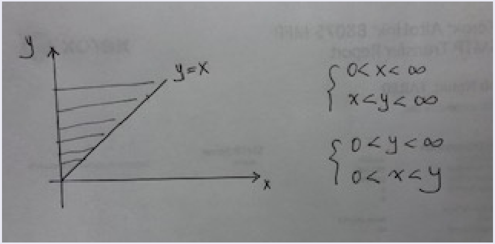

$$P(X < Y) = \int_0^\infty \int_x^\infty 6e^{-2x}e^{-3y}dydx$$
$$= \int_0^\infty 2e^{-2x} \int_x^\infty 3e^{-3y}dydx$$
$$= \int_0^\infty 2e^{-5x}dx = \frac{2}{5}.$$

Example 4

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-(x+2y)}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the density of $Z = X/Y$.

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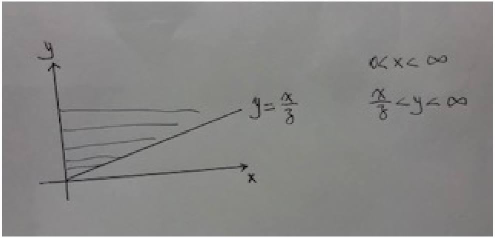
Find the density of $Z = X/Y$.

Z is a positive random variable. To find the density of Z, we need to find

$$P(Z \leq z) = P(\frac{X}{Y} \leq z) = P(Y \geq \frac{X}{z}), \quad z > 0.$$



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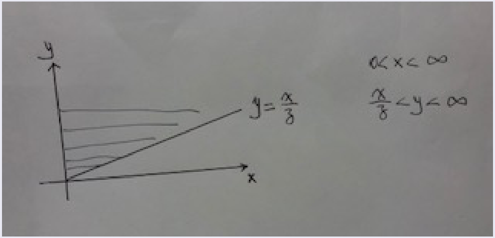

$$P(Z \leq z) = \int_0^\infty e^{-x} \int_{x/z}^\infty 2e^{-2y} dydx$$
$$= \int_0^\infty e^{-(1+\frac{2}{z})x} dx = \frac{z}{z+2}.$$

Thus the density of Z is

$$f_Z(z) = \begin{cases} \frac{2}{(z+2)^2}, & z > 0 \\ 0, & z \leq 0. \end{cases}$$

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Example 5

Consider the disk of radius R centered at the origin. A point is random chosen from this disk. Let X and Y be the x and y coordinates of the chosen point. Then the joint density of X and Y is

$$f(x,y) = \begin{cases} c, & x^2+y^2 < R^2 \\ 0, & x^2+y^2 \geq R^2. \end{cases}$$

(a) Find the value of c. (b) Find the marginal densities of X and Y. (c) Find the density of Z, the distance between the chosen point and the origin. (d) Find $E[Z]$.

(a) $c = 1/(\pi R^2)$.

(b) X takes values in $(-R, R)$. For $x \in (-R, R)$,

$$f_X(x) = \frac{1}{\pi R^2} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2\sqrt{R^2 - x^2}}{\pi R^2}.$$

Thus the density of X is

$$f_X(x) = \begin{cases} \frac{2\sqrt{R^2-x^2}}{\pi R^2}, & x \in (-R, R) \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the density of Y is

$$f_Y(y) = \begin{cases} \frac{2\sqrt{R^2-y^2}}{\pi R^2}, & y \in (-R, R) \\ 0, & \text{otherwise.} \end{cases}$$

(c) Z takes values in $(0, R)$. For $z \in (0, R)$,

$$P(Z \leq z) = \frac{z^2}{R^2}.$$

Thus the density of Z is

$$f_Z(z) = \begin{cases} \frac{2z}{R^2}, & z \in (0, R) \\ 0, & \text{otherwise.} \end{cases}$$

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$$E[Z] = \int_0^R \frac{2z^2}{R^2} dz = \frac{2R}{3}.$$

If X and Y are absolutely continuous with joint distribution F, then the joint density is

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y}(x, y).$$

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If X and Y are absolutely continuous with joint distribution F, then the joint density is

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We can also define the joint distribution function of n random variables $X_1, \ldots, X_n$ in exactly the same manner as we did for $n = 2$: The joint distribution function of $X_1, \ldots, X_n$ is defined by

$$F(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

The distribution function F_{X_i} of X_i, $i = 1, \ldots, n$, is called the marginal distribution function of X_i:

$$F_{X_1}(x_1) = P(X_1 \leq x_1) = F(x_1, \infty, \ldots, \infty), \quad x_1 \in \mathbb{R}$$

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The joint mass function of n discrete random variables $X_1, \ldots, X_n$ is defined by

$$p(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

The mass function p_{X_i} of X_i, $i = 1, \ldots, n$, is called the marginal mass function of X_i:

$$p_{X_1}(x_1) = P(X_1 = x_1) = \sum_{x_2, \ldots, x_n} p(x_1, \ldots, x_n), \quad x_1 \in \mathbb{R}$$

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n random variables $X_1, \ldots, X_n$ are said to be jointly absolutely continuous if there is a non-negative function f on $\mathbb{R}^n$ such that for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y_1, \ldots, y_n) dy_n \cdots dy_1.$$

f is called the joint density of $X_1, \ldots, X_n$.

If $X_1, \ldots, X_n$ are jointly absolutely continuous with joint density f, then for any region C of $\mathbb{R}^n$,

$$P((X_1, \ldots, X_n) \in C) = \int \cdots \int_C f(x_1, \ldots, x_n) dx_1 \cdots dx_n.$$

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If $X_1, \dots, X_n$ are jointly absolutely continuous with joint density f, then $X_1, \dots, X_n$ are also absolutely continuous with densities

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_n) dx_2 \dots dx_n, \quad x_1 \in \mathbb{R}$$

$$\dots$$

$$f_{X_n}(x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, , \dots, x_{n-1}, x_n) dx_1 \dots dx_{n-1}, \quad x_n \in \mathbb{R}.$$

$f_{X_i}(x_i)$ is called the marginal density of X_i.

Example: Multinomial distribution

A sequence n independent trails are performed. Suppose that each trial can result in any one of r possible outcomes with respective probabilities $p_1, \ldots, p_r, \sum_{i=1}^r p_i = 1$. If we let X_i denote the number of the n trials that result in outcome i, $i = 1, \ldots, r$, then

$$P(X_1 = n_1, \ldots, X_r = n_r) = \binom{n}{n_1, \ldots, n_r} p_1^{n_1} \cdots p_r^{n_r},$$

whenever $n_1, \ldots, n_r$ are non-negative integers such that $\sum_{i=1}^r n_i = n$.

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Two random variables X and Y are said to be independent if for any two subsets A and B of $\mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

It can be shown that X and Y are independent if and only if

$$F(x, y) = F_X(x)F_Y(y), \quad (x, y) \in \mathbb{R}^2.$$

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If X and Y are discrete random variables with joint mass function $p(\cdot,\cdot)$, then X and Y are independent if and only if

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Example 1

Independent trails, each results in a success with probability p, are performed $n + m$ times. Let X be the number of successes in the first n trials; Y be the number of successes in the last m trials and Z the total number of successes.

X and Y are independent, but X and Z are not independent.

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