Math 461 Spring 2024

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University of Illinois Urbana-Champaign

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Outline

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HW6 is due Friday, 03/08, before the end of the class.

Solution to Test 1 is on my homepage.

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For any $\alpha > 0$, the Gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy.$$

By using a simple change of variables, one can check that, for any $\alpha > 0$ and $\lambda > 0$, the function

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}\lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

is a probability density. It is called a Gamma density with parameters (α, λ).

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A random variable *X* is called a Gamma random variable with parameters (α, λ) if it is an absolutely continuous random variable whose density is a Gamma density with parameters (α, λ).

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$$E[X] = \frac{\alpha}{\lambda}, \quad \mathrm{Var}(X) = \frac{\alpha}{\lambda^2}.$$

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$$E[X] = \frac{1}{\Gamma(\alpha)} \int_0^\infty x(\lambda x)^{\alpha-1} \lambda e^{-\lambda x} dx$$
$$= \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty (\lambda x)^{\alpha} \lambda e^{-\lambda x} dx$$
$$= \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} = \frac{\alpha}{\lambda}.$$

$$E[X^2] = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^2(\lambda x)^{\alpha-1} \lambda e^{-\lambda x} dx$$
$$= \frac{1}{\lambda^2\Gamma(\alpha)} \int_0^\infty (\lambda x)^{\alpha+1} \lambda e^{-\lambda x} dx$$
$$= \frac{\Gamma(\alpha+2)}{\lambda^2\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\lambda^2}.$$

By using the same argument, we can find that, for any $n = 1, 2, \ldots,$

$$E[X^n] = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{\lambda^n}.$$

Example 1

Suppose that X is an absolutely continuous random variable with density f. Find the density of $Y = X^2$.

Y is a positive random variable. For any $y > 0$,

$$P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx.$$

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Differentiate with respect to y and using the second fundamental theorem of calculus, we find that the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}(f(\sqrt{y}) + f(-\sqrt{y})), & y > 0 \\ 0, & y \leq 0. \end{cases}$$

In the particular case when X is a standard normal random variable, the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-\frac{y}{2}}, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

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By comparing with the Gamma density with parameters $(\frac{1}{2}, \frac{1}{2})$

$$g(y) = \begin{cases} \frac{1}{\Gamma(1/2)}(\frac{y}{2})^{1/2-1}\frac{1}{2}e^{-\frac{y}{2}}, & y > 0 \\ 0, & y \leq 0, \end{cases}$$

we get that, when X is a standard normal random variable, X^2 is a Gamma random variable with parameters $(\frac{1}{2}, \frac{1}{2})$.

Furthermore, we can get

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

Combining this with $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, we get

$$\Gamma(\frac{n}{2}) = \frac{\sqrt{\pi}(n-1)!}{2^{n-1}(\frac{n-1}{2})!}, \quad n \text{ odd}.$$

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Similarly, we get that, when X is a normal random variable with parameters $(0, \sigma^2)$, X^2 is a Gamma random variable with parameters $(\frac{1}{2}, \frac{1}{2\sigma^2})$.

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Thus, for any $a, b > 0$, the function

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

is a probability density. It is called a Beta density with parameters (a, b).

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The Beta function $B(a,b)$ and the Gamma function are related by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Combining this with the fact that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, we can get

If X is a Beta random variable with parameters (a,b), then

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Suppose that X is an absolutely continuous random variable with density f and $Y = \phi(X)$ for some function ϕ. Find the density of Y.

Although there is a theorem in the book, I recommend that you follow the following procedure:
(1) Find the distribution of Y;
(2) Differentiate.

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Example 1

Suppose that X is a uniform random variable on $(0, 1)$, and $\lambda > 0$. Find the density of $Y = -\frac{1}{\lambda}\ln(1 - X)$.

Y is a non-negative random variable. For any $y > 0$,

$$P(Y \leq y) = P(-\frac{1}{\lambda}\ln(1 - X) \leq y) = P(\ln(1 - X) \geq -\lambda y)$$
$$= P(1 - X \geq e^{-\lambda y}) = P(X \leq 1 - e^{-\lambda y}) = 1 - e^{-\lambda y}.$$

Thus the density of Y is

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Y is an exponential random variable with parameter λ.

Example 2

Suppose that X is an exponential random variable with parameter λ and $\beta \neq 0$. Find the density of $Y = X^{1/\beta}$.

Y is a non-negative random variable. Let consider the case $\beta < 0$. For any $y > 0$,

$$P(Y \leq y) = P(X^{1/\beta} \leq y) = P(X \geq y^{\beta}) = e^{-\lambda y^{\beta}}.$$

Thus the density of $Y = X^{1/\beta}$ is

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Using a similar derivation, we can find that in the case $\beta > 0$, the density of $Y = X^{1/\beta}$ is

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