# Math 461 Spring 2024

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# Outline

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1. General Info
2. 5.5 Exponential Random Variables
3. 5.6 Other Absolutely Continuous Random Variables

HW6 is due Friday, 03/08, before the end of the class.

Solution to Test 1 is on my homepage. The distribution of the score of Test 1 is available on my homepage.

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For any $\lambda > 0$, the function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

is a probability density. It is called an exponential density with parameter $\lambda$.

A random variable $X$ is called an exponential random variable with parameter $\lambda > 0$ if it is an absolutely continuous random variable whose density is an exponential density with parameter $\lambda$.

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If $X$ is an exponential random variable with parameter $\lambda > 0$, then for any $x \geq 0$,

$$P(X \leq x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x},$$

$$P(X > x) = e^{-\lambda x}.$$

Thus the distribution function of $X$ is

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

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\begin{aligned}
E[X] &= \int_0^\infty x\lambda e^{-\lambda x} dx = \int_0^\infty x d(-e^{-\lambda x}) \\
&= - xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx \\
&= 0 - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda},
\end{aligned}
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For $n > 1$,

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E[X^n] &= \int_0^\infty x^n \lambda e^{-\lambda x} dx = \int_0^\infty x^n d(-e^{-\lambda x}) \\
&= - x^n e^{-\lambda x}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-\lambda x} dx \\
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**Example**

Suppose that the length $X$ of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/5$. Find the probability that the phone call will (a) last more than 5 minutes; (b) last between 5 and 10 minutes.

$$P(X > 5) = \int_5^\infty \frac{1}{5} e^{-x/5} dx = e^{-1}.$$

$$P(5 \le X \le 10) = \int_5^{10} \frac{1}{5} e^{-x/5} dx = e^{-1} - e^{-2}.$$

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Suppose $X$ is an exponential random variable with parameter $\lambda > 0$. For any $s, t > 0$,

$$P(X > s + t | X > t) = \frac{P(X > s + t)}{P(X > t)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s).$$

This property is called the memoryless property. Any exponential random variable satisfies the memoryless property.

The memoryless property is equivalent to

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It can be shown that if $g$ is a non-negative right continuous function on $(0, \infty)$ taking values in $(0, 1)$ such that

$$g(s + t) = g(s)g(t), \quad s, t > 0,$$

then there exists $\lambda > 0$ such that

$$g(t) = e^{-\lambda t}, \quad t > 0.$$

Thus if a random variable satisfies the memoryless property, it must be an exponential random variable. Thus exponential random variables are very important in applications.

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If the lifetime of a certain object, like a light bulb or computer chip, has the memoryless property, then we can use the exponential distribution to model the lifetime. The lifetime of a car usually does not satisfy the memoryless property, thus it is not reasonable to use an exponential random variable to model the lifetime of a car.

**Example 2**

Suppose that $X$ an exponential random variable with parameter $\lambda > 0$. Define a new random variable $Y$ as follows: $Y = n$ when $X \in (n-1, n]$, $n = 1, 2, \ldots$. Find the mass function of $Y$.

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For $n = 1, 2, \ldots,$

$$\begin{aligned} P(Y = n) =& P(n - 1 < X \leq n) = e^{-\lambda(n-1)} - e^{-\lambda n} \\ =& e^{-\lambda(n-1)}(1 - e^{-\lambda}). \end{aligned}$$

Thus $Y$ is a geometric random variable with parameter $p = 1 - e^{-\lambda}$.

Geometric random variables are the discrete counterpart of exponential random variables. Exponential random variables are the continuous counterpart of geometric random variables.

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**Example 3**

Suppose that the lifetime $X$ of a light bulb in months is an exponential random variable with parameter $\lambda = 1/12$. If the light bulb has been working for 12 months, find the probability that it will work for another 12 months.

By the meomoryless property

$$P(X > 24 | X > 12) = P(X > 12) = e^{-1}.$$

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For any $\alpha > 0$, we define

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy.$$

For any $\alpha > 0$, $\Gamma(\alpha) \in (0, \infty)$. But we do not know the value of $\Gamma(\alpha)$ in general. We do know that $\Gamma(1) = 1$.

We claim that, for any $\alpha > 0$, $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$. Combining this with $\Gamma(1) = 1$, we immediately get $\Gamma(n) = (n-1)!$ for all $n \geq 1$.

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$$\Gamma(\alpha+1) = \int_0^\infty y^\alpha e^{-y} dy = \int_0^\infty y^\alpha d(-e^{-y})$$
$$= - y^\alpha e^{-y}\Big|_0^\infty + \int_0^\infty \alpha y^{\alpha-1} e^{-y} dy$$
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By using a simple change of variables, one can check that, for any $\alpha > 0$ and $\lambda > 0$, the function

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}\lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

is a probability density. It is called a Gamma density with parameters $(\alpha, \lambda)$.

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