Math 461 Spring 2024

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University of Illinois Urbana-Champaign

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Outline

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HW5 is due today before the end of class.

Test 1 is next Friday. There is no homework due next Friday. Topics covered in Test 1 include everything we covered in the first 4 Chapters.

I will do a brief review next Wed and spend most of the lecture time next Wed answering questions.

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The function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

is a probability density. It is called the standard normal density.

A random variable is called a standard normal random variable if it is an absolutely continuous random variable with density given by the function ϕ above.

The distribution function of a standard normal random variable is given by

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt,$$

and there is no explicit expression for this function Φ.

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and there is no explicit expression for this function Φ.

To get the value of Φ, you can either use a calculator or the table in the book. The table in the book only gives the values of $\Phi(x)$ for some positive x. To get the value of $\Phi(x)$ for negative x, we can use the formula

$$\Phi(-x) = 1 - \Phi(x), \quad x \in \mathbb{R},$$

which is due to the symmetry of the density ϕ.

$$1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-\frac{s^2}{2}}\, ds = \Phi(-x).$$

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If X is a standard normal random variable, then

$$E[X] = 0, \quad \mathrm{Var}(X) = 1.$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx = 0.$$

$$\mathrm{Var}(X) = E[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x d(-e^{-\frac{x^2}{2}})$$
$$= \frac{1}{\sqrt{2\pi}} \left(-x e^{-\frac{x^2}{2}}\right)\Big|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1.$$

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Suppose X is a standard normal random variable, μ and $\sigma > 0$ are constants. Let $Y = \mu + \sigma X$. Y is an absolutely continuous random variable with density given by

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, \quad y \in \mathbb{R}.$$

The distribution of Y is given by

$$F_Y(y) = P(Y \leq y) = P(\mu + \sigma X \leq y) = P(X \leq \frac{y-\mu}{\sigma})$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(y-\mu)/\sigma} e^{-\frac{x^2}{2}} dx.$$

Differentiating wrt y, we get the density of Y given above.

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The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

is called a normal density with parameters (μ, σ^2).

A random variable is called a normal random variable with parameters (μ, σ^2) if it is an absolutely continuous random variable with density given by the function f above.

If X is a normal random variable with parameters (μ, σ^2), then the random variable

$$Z = \frac{X - \mu}{\sigma}$$

is a standard normal random variable.

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Thus any normal random variable X with parameters (μ, σ^2) can be written as

$$X = \mu + \sigma Z$$

with Z being a standard normal random variable.

If X is a normal random variable with parameters (μ, σ^2), then

$$E[X] = \mu, \quad \mathrm{Var}(X) = \sigma^2.$$

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Example 1

Suppose X is a normal random variable with parameters $(3, 9)$. Find (a) $P(2 < X < 5)$; (b) $P(X > 0)$; (c) $P(|X - 3| > 6)$.

Let $Z = (X - 3)/3$. Then Z is a standard normal random variable.
(a)

$$
\begin{aligned}
P(2 < X < 5) &= P\left(\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right) \\
&= P\left(-\frac{1}{3} < Z < \frac{2}{3}\right) = \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) \\
&= \Phi\left(\frac{2}{3}\right) - \left(1 - \Phi\left(\frac{1}{3}\right)\right) \approx 0.3779.
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$$P(2 < X < 5) = P(\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3})$$
$$= P(-\frac{1}{3} < Z < \frac{2}{3}) = \Phi(\frac{2}{3}) - \Phi(-\frac{1}{3})$$
$$= \Phi(\frac{2}{3}) - (1 - \Phi(\frac{1}{3})) \approx 0.3779.$$

(b)

$$P(X > 0) = P(\frac{X-3}{3} > \frac{0-3}{3}) = P(Z > -1)$$
$$= 1 - \Phi(-1) = \Phi(1) \approx 0.8413.$$

(c)

$$P(|X-3| > 6) = P(|\frac{X-3}{3}| > 2) = P(|Z| > 2)$$
$$= P(Z > 2) + P(Z < -2) = 2P(Z > 2)$$
$$= 2(1 - \Phi(2)) \approx 0.0456.$$

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Example 2

A test is often regarded as being good if the test scores can be approximated by a normal distribution. The instructor often uses the test scores to get the mean μ and variance σ^2. Then the instructor assigns the grade A those whose score is greater than $\mu + \sigma$; B to those whose score is between μ and $\mu + \sigma$; C to those whose score is between $\mu - \sigma$ and μ; D to those whose score is between $\mu - 2\sigma$ and $\mu - \sigma$; and F to those whose score is below $\mu - 2\sigma$.

Let X be the score of a randomly chosen student in the course. Then

$$P(X > \mu + \sigma) = P(\frac{X - \mu}{\sigma} > 1) = 1 - \Phi(1) \approx 0.1587$$

$$\begin{aligned} P(\mu < X < \mu + \sigma) &= P(0 < \frac{X - \mu}{\sigma} < 1) \\ &= \Phi(1) - \Phi(0) \approx 0.3413 \end{aligned}$$

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$$P(\mu < X < \mu + \sigma) = P(0 < \frac{X - \mu}{\sigma} < 1)$$
$$= \Phi(1) - \Phi(0) \approx 0.3413$$

$$
\begin{aligned}
&P(\mu-\sigma<X<\mu)=P(-1<\frac{X-\mu}{\sigma}<0)\\
&\quad=\Phi(0)-\Phi(-1)\approx 0.3413\\
&P(\mu-2\sigma<X<\mu-\sigma)=P(-2<\frac{X-\mu}{\sigma}<-1)\\
&\quad=\Phi(-1)-\Phi(-2)=\Phi(2)-\Phi(1)\approx 0.1359\\
&P(X<\mu-2\sigma)=P(\frac{X-\mu}{\sigma}<-2)=\Phi(-2)=1-\Phi(2)\approx 0.0228.
\end{aligned}
$$