Math 461 Spring 2024

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University of Illinois Urbana-Champaign

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Outline

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HW5 is due Friday, 02/23, before the end of class.

Solutions to HW4 is on my homepage.

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Theorem

Suppose that X is an absolutely continuous random variable with density f and that ϕ is a function on $\mathbb{R}$. If

$$\int_{-\infty}^{\infty} |\phi(x)| f(x) dx < \infty,$$

then the random variable $\phi(X)$ has finite expectation and

$$E[\phi(X)] = \int_{-\infty}^{\infty} \phi(x) f(x) dx.$$

Suppose X is an absolutely continuous random variable with finite expectation. For any $a, b \in \mathbb{R}$,

$$E[aX + b] = aE[X] + b.$$

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Example 4

Suppose that X is an absolutely continuous random variable with density

$$f(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Find $E[e^X]$.

$$E[e^X] = \int_0^1 e^x \, dx = e - 1.$$

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Example 5

A stick of length 1 is split at a random point U with density

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Find the expected length of the piece that contains the point p, $p \in (0,1)$.

The length of the piece containing the point p is

$$L_p(U) = \begin{cases} 1 - U, & U \leq p \\ U, & U > p. \end{cases}$$

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$$E[L_p(U)] = \int_0^1 L_p(u)du = \int_0^p (1-u)du + \int_p^1 udu$$
$$= \frac{1}{2} + p(1-p).$$

Definition

Suppose that X is an absolutely continuous random variable with finite expectation $\mu = E[X]$. The variance of X is defined to be

$$\mathrm{Var}(X) = E[(X-\mu)^2].$$

One can easily check that

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2.$$

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Let f be the density of X. Then

$$\begin{aligned}\mathrm{Var}(X) &= \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty}(x^2-2\mu x+\mu^2)f(x)dx \\ &= \int_{-\infty}^{\infty} x^2 f(x)dx - 2\mu\int_{-\infty}^{\infty} xf(x)dx + \mu^2\int_{-\infty}^{\infty} f(x)dx \\ &= E[X^2]-2\mu E[X]+\mu^2 = E[X^2]-(E[X])^2.\end{aligned}$$

Suppose that X is an absolutely continuous random variable with finite variance, and a, b are real numbers. Then

$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X).$$

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Example 6

Suppose that X is an absolutely continuous random variable with density

$$f(x) = \begin{cases} 3x^2, & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathrm{Var}(X)$.

$$E[X] = \int_0^1 x3x^2 dx = \frac{3}{4}, \quad E[X^2] = \int_0^1 x^2 3x^2 dx = \frac{4}{5}.$$

So

$$\mathrm{Var}(X) = \frac{4}{5} - (\frac{3}{4})^2.$$

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Definition

A random variable X is said to be uniformly distributed over the interval (a, b) if its density is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in (a, b) \\ 0, & \text{otherwise.} \end{cases}$$

If X is uniformly distributed in (a, b), then

$$E[X] = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$$

and

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a} dx = \frac{b^2 + ab + a^2}{3}$$

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Example 1

Buses arrive at a specified bus stop at 15 minute intervals starting at 7 am. If a passenger arrives at the stop at a time that is uniformly distributed between 7 an 7:30, find the probability that he waits (a) less than 5 minutes; (b) more than 10 minutes.

Let X be the passenger's arrival time in minutes, after 7 am. Then the answer for (a) is

$$P(10 < X \leq 15) + P(25 < X \leq 30) = \frac{1}{3}.$$

The answer for (b) is

$$P(0 < X \leq 5) + P(15 < X \leq 20) = \frac{1}{3}.$$

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Example 2

A point is chosen at random on a line segment of length L. Find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

Imagine that the line segment is the interval $(0, L)$. Let X the coordinate of the random chosen point. Then X is uniformly distributed in $(0, L)$. The answer is

$$P\left(\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < \frac{1}{4}\right) = 1 - P\left(\frac{L}{5} < X < \frac{4L}{5}\right) = \frac{2}{5}.$$

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Before we introduce the concept of normal random variables, let us look at the function

$$g(x) = e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

The function g is strictly positive, and goes to zero very fast near ∞ and $-\infty$, and so

$$c = \int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$$

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\begin{aligned}
c^2 &= \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy \\
&= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-\frac{r^2}{2}} rdrd\theta = 2\pi \int_{0}^{\infty} re^{-\frac{r^2}{2}} dr = 2\pi.
\end{aligned}
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Thus $c = \sqrt{2\pi}$ and hence the function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

is a density function. It is called the standard normal density.

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