Math 461 Spring 2024

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University of Illinois Urbana-Champaign

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Outline

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HW5 is due Friday, 02/23, before the end of class.

Solutions to HW4 is on my homepage.

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A non-negative function f on $\mathbb{R}$ is called a probability density function if

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

A random variable X is said to be absolutely continuous if there is a non-negative function f on $\mathbb{R}$ such that

$$P(X \leq x) = \int_{-\infty}^{x} f(t)dt, \quad x \in \mathbb{R}.$$

f must be a probability density and it is called the density of X.

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If we know the density f of an absolutely continuous random variable X, then

$$P(a < X < b) = P(a \leq X \leq b) = \int_a^b f(x)dx.$$

The density function of an absolutely continuous random variable X contains all the statistical info about X. If we know the density f, we can get the distribution F of X by

$$F(x) = \int_{-\infty}^{x} f(t)dt, \quad x \in \mathbb{R}.$$

If we know the the distribution F of an absolutely continuous random variable X, we can simply differentiate F to get the density f. For points where F is not differentiable, we simply let f equal 0 there.

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Example 2

Suppose that X is an absolutely continuous random variable with density given by

$$f(x) = \begin{cases} cx^2, & x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Find $P(\frac{1}{3} < X < \frac{2}{3})$.

Since f is a density, we have

$$1 = \int_0^1 cx^2 dx = \frac{c}{3},$$

Thus $c = 3$. Consequently

$$P(\frac{1}{3} < X < \frac{2}{3}) = \int_{1/3}^{2/3} 3x^2 dx = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}.$$

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Example 3

Suppose that X is an absolutely continuous random variable with density f. Find the density of $Y = 2X$.

The distribution function F of $Y = 2X$ is

$$F_Y(y) = P(Y \leq y) = P(2X \leq y) = P(X \leq y/2) = \int_{-\infty}^{y/2} f(x)dx.$$

Differentiating (by the second fundamental theorem of calculus), we get the density function of Y is $f_Y(y) = \frac{1}{2} f(\frac{y}{2})$.

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Any non-negative function g on $\mathbb{R}$ with

$$\int_{-\infty}^{\infty} g(x)dx \in (0, \infty)$$

can be normalized into a probability density.

In fact, let $c = \int_{-\infty}^{\infty} g(x)dx$. Then the function

$$f(x) = \frac{1}{c} g(x), \quad x \in \mathbb{R}$$

is a non-negative function on $\mathbb{R}$ with $\int_{-\infty}^{\infty} f(x)dx = 1$, and thus is a density.

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Example 4

The function

$$g(x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

satisfies

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

Thus

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}$$

is a probability density.

The corresponding distribution function is

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)} dt = \frac{1}{2} + \frac{1}{\pi} \arctan x, \quad x \in \mathbb{R}.$$

This distribution is called a Cauchy distribution and a random variable with this distribution is called a Cauchy random variable.

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Definition

Suppose that X is an absolutely continuous random variable with density f. If

$$\int_{-\infty}^{\infty} |x| f(x) dx < \infty,$$

then X has finite expectation and we define the expectation of X to be

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx.$$

Example 1

Suppose that X is an absolutely continuous random variable with density given by

$$f(x) = \begin{cases} 3x^2, & x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Find $E[X]$.

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$$E[X] = \int_0^1 x \cdot 3x^2 dx = \frac{3}{4}.$$

Example 2

Suppose that X is an absolutely continuous random variable with density given by

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

Then

$$\int_{-\infty}^{\infty} \frac{|x| dx}{\pi(1+x^2)} = 2\int_0^{\infty} \frac{x dx}{\pi(1+x^2)} = \infty.$$

Thus X does not finite expectation.

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Thus X does not finite expectation.

Example 3

Suppose that X is an absolutely continuous random variable with density given by

$$f(x) = \begin{cases} \frac{1}{2}, & x \in (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

Find $E[X^2]$.

Put $Y = X^2$. Then Y takes values in $[0,1)$. For $y \in [0,1)$,

$$P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \sqrt{y}.$$

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Thus the distribution function of Y is

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ \sqrt{y}, & y \in [0, 1), \\ 1, & y > 1. \end{cases}$$

Thus the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{2} y^{-1/2}, & y \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$E[X^2] = E[Y] = \int_0^1 y \frac{1}{2} y^{-1/2} dy = \frac{1}{3}.$$

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Hence

$$E[X^2] = E[Y] = \int_0^1 y \frac{1}{2} y^{-1/2} dy = \frac{1}{3}.$$

Theorem

Suppose that X is an absolutely continuous random variable with density f and that ϕ is a function on $\mathbb{R}$. If

$$\int_{-\infty}^{\infty} |\phi(x)| f(x) dx < \infty,$$

then the random variable $\phi(X)$ has finite expectation and

$$E[\phi(X)] = \int_{-\infty}^{\infty} \phi(x) f(x) dx.$$

Example 3 (revisit)

$$E[X^2] = \int_{-1}^{1} x^2 \frac{1}{2} dx = \int_{0}^{1} x^2 dx = \frac{1}{3}.$$

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