# Math 461 Spring 2024

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University of Illinois Urbana-Champaign

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# Outline

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HW4 is due Friday, 02/16, before the end of class. Please submit your HW4 as ONE pdf file via the HW4 folder in the course Moodle page.

Solutions to HW3 is on my homepage.

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If $n$ independent trials, each results in a success with probability $p$, are performed, when $n$ is big, $p$ is small so that $np$ is of moderate size, the number of successes in the $n$ trials is approximately a Poisson random variable with parameter $\lambda = np$.

**Examples**

1. Number of misprints on a page of a book.
2. Number of people in a community over the age of 95.

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A machine produces screws, 1% of which are defective. Find the probability that in a box of 100 screws there are at most 3 defective ones. Assume independence.

The number of defectives in the box is a binomial random variable with parameters $(100, 0.01)$. So the exact answer is

$$P(X \leq 3) = (0.99)^{100} + 100 \cdot (0.01)(0.99)^{99} + \binom{100}{2}(0.01)^2(0.99)^{98} + \binom{100}{3}(0.01)^3(0.99)^{97}.$$

$X$ is approximately a Poisson random variable with parameter 1, so

$$P(X \leq 3) \approx e^{-1} + e^{-1} + e^{-1}\frac{1}{2} + e^{-1}\frac{1}{6}.$$

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Poisson random variables also arise in situations where "incidents" occur at certain points in time, like earthquakes, people entering a certain establishment.

In a lot of situations, the following assumptions are (approximately) satisfied: For some $\lambda > 0$, the following hold:

1. The probability of exactly 1 incident occurs in a given interval of length $h$ is $\lambda h + o(h)$,
2. The probability of 2 or more incidents occur in an interval of length $h$ is $o(h)$.
3. For any integer $n \geq 1$, any non-negative integers $j_1, \ldots, j_n$, and any set of $n$ non-overlapping intervals, if $E_i$ denotes the event that exactly $j_i$ incidents occur in the $i$-th interval, $i = 1, \ldots, n$, then $E_1, \ldots, E_n$ are independent.

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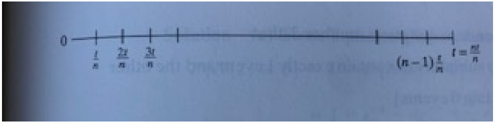
Under the assumptions above, the number of incidents occurring in any interval of length $t$ is a Poisson random variable with parameter $\lambda t$. It suffices to deal with the case when the interval is $[0, t]$.

Let $N(t)$ be the number of incidents occurring in $[0, t]$. For any $n \geq 1$, we divide $[0, t]$ into $n$ sub-intervals of equal length:

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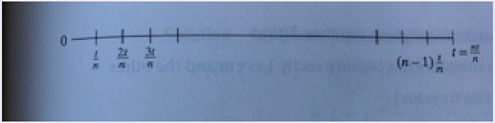


The event $\{N(t) = k\}$ can be written as the disjoint union of 2 events $A$ and $B$ where

$A$ is the event that "$k$ of the $n$ sub-intervals contains exactly 1 incident each and the other $n-k$ sub-intervals contains 0 incident", and $B$ is the event that "$N(t)=k$ and at least one of the sub-intervals contain 2 or more incidents". Thus $P(N(t)=k)=P(A)+P(B)$.

$$\begin{aligned} P(B) &\leq P(\text{at least 1 subinterval contain 2 or more incidents}) \\ &= P(\cup_{i=1}^{n}\{\text{ the } i\text{-th subinterval contain 2 or more incidents})\\ &\leq \sum_{i=1}^{n} P(\text{ the } i\text{-th subinterval contain 2 or more incidents}) \\ &= \sum_{i=1}^{n} o(\frac{t}{n}) = n\cdot o(\frac{t}{n}) \to 0 \end{aligned}$$

as $n\to\infty$.

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$$P(A) = \binom{n}{k} \left( \frac{\lambda t}{n} + o(\frac{\lambda t}{n}) \right)^k \left( 1 - \frac{\lambda t}{n} - o(\frac{\lambda t}{n}) \right)^{n-k}.$$

Since

$$n \left( \frac{\lambda t}{n} + o(\frac{\lambda t}{n}) \right) \to \lambda t,$$

we have

$$P(A) \to e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

Thus

$$P(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

**Examples**

**(a)** The number of earthquakes during some fixed time interval.

**(b)** The number of $\alpha$-particles discharged from some radioactive material in a fixed period of time.

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Suppose that independent trials, each results in a success with probability $p \in (0,1)$ and a failure with probability $1-p$, are performed until a success occurs. Let $X$ be the number of trials need, then

$$P(X = n) = (1-p)^{n-1}p, \quad n = 1, 2, \ldots.$$

Such a random variable is called a geometric random variable with parameter $p$.

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**Example 1**

Cards are randomly selected from an ordinary deck, one at a time, until a spade is obtained. If we assume that each card is returned to the deck before the next one is selected, find the probability that (a) exactly 10 cards are needed; (b) at least 10 cards are needed.

**Solution**

The number of cards needed is a geometric random variable with parameter $\frac{1}{4}$. Thus (a) $(\frac{3}{4})^9(\frac{1}{4})$; (b) $(\frac{3}{4})^9$.

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$X$ is a geometric random variable with parameter $p$, then

$$E[X] = \frac{1}{p}, \quad \mathrm{Var}(X) = \frac{1-p}{p^2}.$$

Let $q = 1 - p$. Then

$$E[X] = \sum_{n=1}^{\infty} nq^{n-1}p = p\sum_{n=0}^{\infty} \frac{d}{dq}(q^n)$$
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$$\begin{aligned} E[X^2] &= \sum_{n=1}^{\infty} n^2 q^{n-1} p = p \sum_{n=0}^{\infty} \frac{d}{dq}(nq^n) \\ &= p \frac{d}{dq}\left(\sum_{n=0}^{\infty} nq^n\right) = p \frac{d}{dq}\left(\frac{q}{1-q} E[X]\right) \\ &= p \frac{d}{dq}\left(\frac{q}{(1-q)^2}\right) = \frac{2}{p^2} - \frac{1}{p}. \end{aligned}$$

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