Math 461 Spring 2024

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Outline

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A random variable X is said to be a Bernoulli random variable with parameter $p \in (0, 1)$ if $P(X = 1) = p$ and $P(X = 0) = 1 - p$.

Suppose that an experiment whose outcome can be classified as either success or failure is performed. If we let $X = 1$ when the outcome is a success and $X = 0$ when it is failure, then X is a Bernoulli random variable and the parameter is the probability of success.

If X is a Bernoulli random variable with parameter p, then $E[X] = E[X^2] = p$ and $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = p(1 - p)$.

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Suppose that independent trials, each results in a success with probability p and a failure with probability $1 - p$, are performed n times. Let X be the total number of successes, then X is said to be a binomial random variable with parameters (n, p). A Bernoulli random variable with parameter p is a binomial random variable with parameters $(1, p)$.

If X is a binomial random variable with parameters (n, p), then

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

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Example 1

It is known that screws produced by a certain company will be defective with probability 0.01, independent of each other. The company sells the screws in packages of 10 and offers a money back guarantee that at most 1 of the 10 screws will be defective. What is the proportion of sold packages must the company give the money back?

Let X be the number of defectives in a package. Then X is a binomial random variable with parameters $(10, 0.01)$.

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - (0.99)^{10} - 10 \cdot (0.01)(0.99)^9 \\ &\approx 0.004. \end{aligned}$$

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Example 2

A communication system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one half of its components function. For what values of p is a 5-component system more likely to operate effectively than a 3-component system?

The probability that a 5-component system operates effectively is

$$\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5.$$

The probability that a 3-component system operates effectively is

$$\binom{3}{2}p^2(1-p) + p^3.$$

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Solving the inequality

$$\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5 > \binom{3}{2}p^2(1-p) + p^3,$$

we get $p > \frac{1}{2}$.

If X is a binomial random variable with parameters (n, p), then

$$E[X] = np, \quad \mathrm{Var}(X) = np(1-p).$$

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\begin{aligned}
E[X] &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
&= np \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} p^j (1-p)^{n-1-j} = np.
\end{aligned}
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Similarly, we can find that

$$E[X^2] = n(n-1)p^2 + np.$$

Thus

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = np(1-p).$$

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

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Proposition

If X is a binomial random variable with parameters (n, p), then as k goes from 0 to n, $P(X = k)$ first increases and then decreases, reaching its largest value when k is the largest integer less than or equal to $(n+1)p$.

$$\frac{P(X=k)}{P(X=k-1)} = \frac{\binom{n}{k}p^k(1-p)^{n-k}}{\binom{n}{k-1}p^{k-1}(1-p)^{n-k+1}} = \frac{(n-k+1)p}{k(1-p)}$$

which is ≥ 1 iff $(n-k+1)p \geq k(1-p)$ iff $k \leq (n+1)p$.

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Let $\lambda > 0$ be a constant. A non-negative integer-valued random variable X is said to be a Poisson random variable with parameter λ if

$$p(x) = P(X = x) = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!}, & x = 0, 1, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

It is indeed a probability mass function since

$$\sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}e^{\lambda} = 1.$$

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$$E[X] = \mathrm{Var}(X) = \lambda.$$

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Similarly, we get $E[X^2] = \lambda(\lambda+1)$. Thus $\mathrm{Var}(X) = \lambda$.

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Similarly, we get $E[X^2] = \lambda(\lambda + 1)$. Thus $\mathrm{Var}(X) = \lambda$.

Poisson random variables are widely used in applications since they may be used as approximations of binomial random variables with parameters (n, p) when n is big, p is small so np is of moderate size.

Suppose X is a binomial random variable with parameters (n, p), where n is big, p is small so that np is of moderate size. Let $\lambda = np$.

$$P(X = k) = \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} = \frac{n!}{k!(n-k)!}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{\lambda^k}{k!}\frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^k} \approx e^{-\lambda}\frac{\lambda^k}{k!}.$$

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Therefore, if n independent trials, each results in a success with probability p, are performed, when n is big, p is small so that np is of moderate size, the number of successes in the n trials is approximately a Poisson random variable with parameter $\lambda = np$.

Examples

1. Number of misprints on a page of a book.
2. Number of people in a community over the age of 90.

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Examples

1. Number of misprints on a page of a book.
2. Number of people in a community over the age of 90.

Example

A machine produces screws, 1% of which are defective. Find the probability that in a box of 100 screws there are at most 3 defective ones. Assume independence.

The number of defectives in the box is a binomial random variable with parameters $(100, 0.01)$. So the exact answer is

$$P(X \leq 3) = (0.99)^{100} + 100 \cdot (0.01)(0.99)^{99} + \binom{100}{2}(0.01)^2(0.99)^{98} + \binom{100}{3}(0.01)^3(0.99)^{97}.$$

X is approximately a Poisson random variable with parameter 1, so

$$P(X \leq 3) \approx e^{-1} + e^{-1} + e^{-1}\frac{1}{2} + e^{-1}\frac{1}{6}.$$

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