Math 461 Spring 2024

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University of Illinois Urbana-Champaign

February 05, 2024

Outline

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Solutions to HW2 is on my homepage.

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Last time, we saw a few examples of random variables that can take at most countably many values. We described these random variables by listing all their possible values and the probability they take these values. This does not always work.

Example 5

A number is chosen randomly from $(0, 1)$. Let X be the value of the number.

X is a random variable. Its possible values are in $(0, 1)$. The probability that it takes any value in $(0, 1)$ is 0. For any sub-interval A of $(0, 1)$,

$$P(X \in A) = |A|,$$

when A denotes the length of the interval A.

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For a random variable X, the function

$$F(x) = P(X \leq x), \quad x \in \mathbb{R},$$

is called the (cumulative) distribution function of X.

It is a non-decreasing, right-continuous function with

$$\lim_{x \to \infty} F(x) = 1, \quad \lim_{x \to -\infty} F(x) = 0.$$

The distribution function F of a random variable X contains all the statistical info about X. If we know F , then we can find the probability of any event defined in terms of X. For instance, for any $a < b$,

$$P(X \in (a, b]) = F(b) - F(a).$$

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A random variable that can take at most countably many values is called a discrete random variable. For a discrete random variable X, the function

$$p(x) = P(X = x), \quad x \in \mathbb{R},$$

is called the probability mass function of X.

If X can only take the values $x_1, x_2, \ldots$, then

$$p(x_i) > 0, \quad i = 1, 2, \ldots,$$

$$p(x) = 0 \quad \text{for all other values of } x$$

and

$$\sum_i p(x_i) = 1.$$

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The probability mass function p of a discrete random variable X also contains all the statistical info about X. If we know p, then we can find the probability of any event defined in terms of X. For instance,

$$F(x) = \sum_{x_i \leq x} p(x_i).$$

If X is a discrete random variable whose possible values are $x_1, x_2, \ldots$, then in general, its distribution function is a step function. One can read off the probability mass function from the distribution function.

For instance, if X is a discrete random variable with probability mass function

$$p(1) = \frac{1}{4}, \quad p(2) = \frac{1}{2}, \quad p(3) = \frac{1}{8}, \quad p(4) = \frac{1}{8},$$

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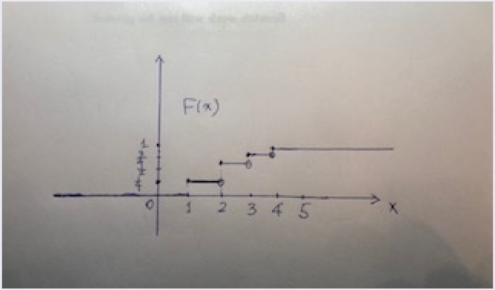

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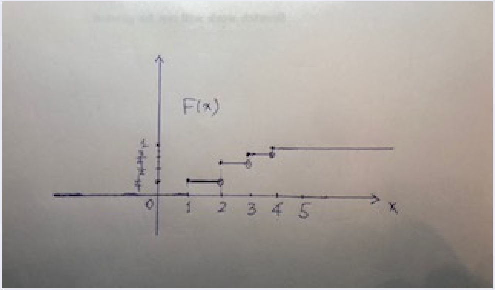

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Suppose that F is the distribution function of a random variable X, then

(1) F is non-decreasing, i.e, $a \leq b$ implies $F(a) \leq F(b)$;

(2) $\lim_{x \to \infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$;

(3) F is right-continuous, i.e, for any $b \in \mathbb{R}$, $\lim_{x \downarrow b} F(x) = F(b)$.

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Any function on $\mathbb{R}$ satisfying the three properties above is the distribution function of some random variable.

Once we know the distribution function F of a random variable X, we can find the probability of any event defined in terms of X. Here are some examples:

$$P(a < X \le b) = F(b) - F(a), \quad P(a \le X \le b) = F(b) - F(a-),$$
$$P(a \le X < b) = F(b-) - F(a-), \quad P(a < X < b) = F(b-) - F(a)$$
$$P(X > a) = 1 - F(a), \quad P(X = a) = F(a) - F(a-).$$

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Example

The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/3, & 0 \leq x < 1, \\ x/2, & 1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

Find (a) $P(\frac{1}{2} \leq X \leq \frac{3}{2})$; (b) $P(\frac{1}{2} \leq X \leq 1)$; (c) $P(\frac{1}{2} \leq X < 1)$; (d) $P(1 \leq X \leq \frac{3}{2})$; (e) $P(1 < X < 2)$.

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$$P(\frac{1}{2} \le X \le \frac{3}{2}) = F(\frac{3}{2}) - F(\frac{1}{2}-) = \frac{3}{4} - \frac{1}{6},$$
$$P(\frac{1}{2} \le X \le 1) = F(1) - F(\frac{1}{2}-) = \frac{1}{2} - \frac{1}{6},$$
$$P(\frac{1}{2} \le X < 1) = F(1-) - F(\frac{1}{2}-) = \frac{1}{3} - \frac{1}{6},$$
$$P(1 \le X \le \frac{3}{2}) = F(\frac{3}{2}) - F(1-) = \frac{3}{4} - \frac{1}{3},$$
$$P(1 < X < 2) = F(2-) - F(1) = 1 - \frac{1}{2}.$$