Math 461 Spring 2024

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University of Illinois Urbana-Champaign

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Outline

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HW2 is due today at the end of the class. Please submit your HW2 as ONE pdf file via the HW2 folder in the course Moodle page. Make sure that the quality of your pdf file is good enough.

I will post the Solutions to HW2 on my homepage later this afternoon.

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Example 9 (The problem of points)

Independent trials, each results in a success with probability p and a failure with probability $1 - p$, are performed. Find the probability that n (not necessarily consecutive) successes occur before m (not necessarily consecutive) failures.

Let E be the event that n (not necessarily consecutive) successes occur before m (not necessarily consecutive) failures. Then E is equal to the event that there are at least n successes in the first $n + m - 1$ trials. So the answer is

$$\sum_{k=n}^{n+m-1} \binom{n+m-1}{k} p^k (1-p)^{n+m-1-k}.$$

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Conditioning is a useful technique in finding probability. Let's illustrate this technique with two examples. This first is a homework problem from Chap. 2.

Example 10

Independent trials, consisting of rolling a pair of fair dice, are performed, Find the probability that an outcome of 5 appears before an outcome of 7, where outcome is the sum of the two dice.

Let E be the event that an outcome of 5 appears before an outcome of 7, let F be the event that the first trial results in an outcome of 5, G be the event that the first trial results in an outcome of 7, and H be the event that the first trial results in neither an outcome of 5 nor an outcome of 7.

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$$\begin{aligned} P(E) &= P((F \cap E) + P(G \cap E) + P(H \cap E) \\ &= P(F)P(E|F) + P(G)P(E|G) + P(H)P(E|H) \\ &= \frac{4}{36} \cdot 1 + \frac{6}{36} \cdot 0 + \frac{26}{36} P(E). \end{aligned}$$

Thus

$$\frac{10}{36} P(E) = \frac{4}{36},$$

and so $P(E) = \frac{2}{5}$.

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Example 11 (Gambler's ruin)

Two gamblers, A and B, bet on the outcomes of successive coin flips. On each flip, if the coin comes up Heads, A gets \$1 from B, otherwise, B gets \$1 from A. They continue to do so until one of them is out of money. If the successive flips are independent and each flip is Heads with probability p, what is the probability that A ends up with all the money if A starts with \$i and B with \$(N-i)?

Let E be the event that A ends up with all the money. Let P_i be the probability of E when A starts with \$i and B with \$(N-i). Then $P_0 = 0$ and $P_N = 1$. Let H be the event that the first flip results in Heads. Then for $i = 1, 2, \ldots, N-1$,

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$$P_i = P(H)P(E|H) + P(H^c)P(E|H^c) = pP_{i+1} + qP_{i-1},$$

where $q = 1 - p$.

This can be rewritten as

$$(p+q)P_i = pP_{i+1} + qP_{i-1}, \quad i = 1, \dots, N-1$$

which is the same as

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}), \quad i = 1, \dots, N-1.$$

List them all out, we get

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List them all out, we get

$$P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p}P_1$$

$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$$

$$\ldots$$

$$P_i - P_{i-1} = \frac{q}{p}(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1$$

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$$P_N - P_{N-1} = \frac{q}{p}(P_{N-1} - P_{N-2}) = \left(\frac{q}{p}\right)^{N-1} P_1.$$

Adding up the first $i-1$ equations, we get

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Adding up the first $i - 1$ equations, we get

$$P_i - P_1 = P_1 \left(\frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{i-1} \right).$$

Thus

$$P_i = \left(1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{i-1} \right) P_1$$
$$= \begin{cases} \frac{1-(q/p)^i}{1-q/p} P_1, & \text{if } p \neq q \\ iP_1, & \text{if } p = q. \end{cases}$$

Definition

Using the fact $P_N = 1$, we get

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Using the fact $P_N = 1$, we get

$$P_1 = \begin{cases} \frac{1-q/p}{1-(q/p)^N}, & \text{if } q \neq p \\ \frac{1}{N}, & \text{if } p = q. \end{cases}$$

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It is often the case that when a random experiment is performed, we are mainly interested in some function of the outcome, as opposed the actual outcome itself. In general, "any" real-valued function on the sample space is called a random variable.

Example 1

Independent trials, each results in a success with probability p and a failure with probability $1 - p$, are performed 5 times. For each success, you win \$1 and for each failure, you lose \$1. Obviously, you are interested in your net winning.

Let X be your net winning, then X is a function on the sample space and thus it is a random variable.

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Let X be your net winning, then X is a function on the sample space and thus it is a random variable.

The possible values of X are: ± 1, ± 3, ± 5. The probabilities that it takes each of these values are

$$P(X=5)=\binom{5}{5}p^5, \quad P(X=3)=\binom{5}{4}p^4(1-p)$$
$$P(X=1)=\binom{5}{3}p^3(1-p)^2, \quad P(X=-1)=\binom{5}{2}p^2(1-p)^3$$
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Example 2

3 balls are randomly selected, without replacement, from a box containing 20 balls labeled $1, \ldots, 20$. Let X be the smallest number selected.

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3 balls are randomly selected, without replacement, from a box containing 20 balls labeled $1, \ldots, 20$. Let X be the smallest number selected.

X is a random variable. The possible values of X are $1, \dots, 18$ and

$$P(X = i) = \frac{\binom{20-i}{2}}{\binom{20}{3}}, \quad , i = 1, \dots, 18.$$

Example 3

Independent trials, each results in a success with probability p and a failure with probability $1 - p$, are performed. Let X be the number of trials needed in order to get a success.

X is a random variable. Its possible values are $1, 2, \dots$ and

$$P(X = i) = (1 - p)^{i-1} p, \quad i = 1, 2, \dots.$$

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Example 4

Independent trials, each results in a success with probability p and a failure with probability $1 - p$, are performed until a success occurs or a total of n trials are performed. Let X be the number of trials needed.

X is a random variable. Its possible values are $1, 2, \ldots, n$ and

$$P(X = i) = (1 - p)^{i-1} p, \quad i = 1, 2, \ldots, n - 1$$
$$P(X = n) = (1 - p)^{n-1}.$$

Example 4

Independent trials, each results in a success with probability p and a failure with probability $1-p$, are performed until a success occurs or a total of n trials are performed. Let X be the number of trials needed.

X is a random variable. Its possible values are $1, 2, \ldots, n$ and

$$P(X=i) = (1-p)^{i-1}p, \quad i = 1, 2, \ldots, n-1$$
$$P(X=n) = (1-p)^{n-1}.$$

Example 5

For all the examples above, we describe the random variables by listing all their possible values and the probability they take these values. This does not always work.

A number is chosen randomly from $(0, 1)$. Let X be the value of the number.

X is a random variable. Its possible values are in $(0, 1)$. The probability that it takes any value in $(0, 1)$ is 0. For any sub-interval A of $(0., 1)$,

$$P(X \in A) = |A|,$$

when A denotes the length of the interval A.

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For a random variable X, the function

$$F(x) = P(X \leq x), \quad x \in \mathbb{R},$$

is called the (cumulative) distribution function of X.

It is a non-deceasing, right-continuous function with

$$\lim_{x \to \infty} F(x) = 1, \quad \lim_{x \to -\infty} F(x) = 0.$$

If we know the distribution function F of a random variable X, then we can find the probability of any event defined in terms of X. For instance, for any $a < b$,

$$P(X \in (a, b]) = F(b) - F(a).$$

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