Math 461 Spring 2024

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University of Illinois at Urbana-Champaign

January 31, 2024

Outline

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HW2 is due on Friday, 02/02, before the end of the class.

Please submit your HW2 as ONE pdf file via the HW2 folder in the course Moodle page.

Solutions to HW1 is on my homepage.

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In general, $P(E|F) \neq P(E)$. In the case when $P(E|F) = P(E)$, we say that E is independent of F. Here we have to require $P(F) > 0$ and the roles of E and F are not symmetric.

When $P(E|F) = P(E)$, we have

$$P(E \cap F) = P(E)P(F).$$

In this relation, we do not need to assume $P(F) > 0$ and the roles of E and F are symmetric. So we take this as our definition.

Definition

Two events E and F are said to be independent if

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If they are not independent, we say they are dependent.

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(i) The null event $\emptyset$ is independent of any event.
(ii) The sure event S is independent of any event.

Do not confuse "independence" with "disjointness"! In fact, if E and F are disjoint, E and F are not independent in general.

Independence is not always intuitive. You have to check independence by the definition, as we will show in an example soon.

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Example 1

A card is chosen at random from an ordinary deck of 52 cards. Let E be the event that the card is an ace, and let F be the event that the card is a spade. Then E and F are independent.

Example 2

A fair die is rolled twice. (i) Let E be the event that the first toss is 3 and F the event that the second is even; (ii) Let A be the event that the sum is 6 and B the event that the first toss is 4; (iii) Let C be the event that the sum is 7 and D the event that the first toss is 2.

E and F are independent; A and B are dependent; C and D are independent.

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Proposition

If E and F are independent, then so are (i) E^c and F; (ii) E and F^c; (iii) E^c and F^c.

Proof

Since E and F are independent, we have $P(E \cap F) = P(E)P(F)$. Now

$$P(F) = P(E \cap F) + P(E^c \cap F) = P(E)P(F) + P(E^c \cap F),$$

Hence,

$$P(E^c \cap F) = (1 - P(E))P(F) = P(E^c)P(F),$$

which gives (i). (ii) and (iii) follow from (i).

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If E and F are independent, and we know $P(E)$ and $P(F)$, then we can find the probability of any combination of E and F.

Definition

3 events E, F and G are said to be independent if

$$P(E \cap F \cap G) = P(E)P(F)P(G), \quad P(E \cap F) = P(E)P(F)$$
$$P(E \cap G) = P(E)P(G), \quad P(F \cap G) = P(F)P(G).$$

Just requiring the 1st identity is not enough. Just requiring pairwise independence is not enough.

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Example 3

$S = \{1, 2, 3, 4\}$, $P(1) = P(2) = P(3) = P(4) = \frac{1}{4}$. $E = \{1, 2\}$, $F = \{1, 3\}$, $G = \{1, 4\}$. Pairwise independent, but

$$\frac{1}{4} = P(E \cap F \cap G) \neq P(E)P(F)P(G) = \frac{1}{8}.$$

Example 4

Suppose F and G are dependent and $E = \emptyset$. Then

$$P(E \cap F \cap G) = 0 = P(E)P(F)P(G),$$

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but E, F, G are not independent.

If E, F, G are independent, then E is independent of any combination of F and G. For instance, E is independent of $F \cup G$.

If E, F, G are independent, then

$$P(E \cup F \cup G) = 1 - P(E^c \cap F^c \cap G^c) = 1 - P(E^c)P(F^c)P(G^c).$$

Definition

n events $E_1, \ldots, E_n$ are said to be independent if, for any subcollection $E_{i_1}, \ldots, E_{i_k}$, $k \leq n$, of these events,

$$P(\cap_{j=1}^{k} E_{i_j}) = \prod_{j=1}^{k} P(E_{i_j}).$$

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An infinite family of events is said to be independent if every finite subfamily of these events is independent.

Checking the independence of an infinite family of events is not easy.

Sometimes, the random experiment under consideration consists of of performing a sequence of sub-experiments. For instance, keeping on tossing a die. In many cases, the outcomes of any group of the sub-experiments have no effect on the probabilities of the outcomes of any sub-experiments outside this subgroup. In this case, we say the sub-experiments are independent. If this is the case and $(E_i : i = 1, 2, \ldots)$ is a sequence of events such that E_i only depends the i-th sub-experiment, then $(E_i : i = 1, 2, \ldots)$ is independent.

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If each of the sub-experiment is identical, then we call these sub-experiments trials. “Independent trials” means that the sub-experiments are identical and independent.

Example 5

An infinite sequence of independent trials is to be performed. Each trial results in a success with probability p and a failure with probability $1 - p$. Find the probability that (a) at least one success in the first n trials; (b) exactly k successes in the first n trials.

(a) $1 - (1 - p)^n$; (b) $\binom{n}{k} p^k (1 - p)^{n-k}$.

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Example 6

Experience shows that 20% of the people reserving tables at a certain restaurant never show up. If the restaurant has 50 tables and takes 52 reservations, find the probability that it will be able to accommodate everyone showing up with a reservation. Assume independence between reservations.

The answer is equal to

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\begin{aligned}
&1 - P(\text{exactly 51 show up }) - P(\text{ all 52 show up})\\
=&1 - \binom{52}{51}(0.8)^{51}(0.2) - (0.8)^{52}.
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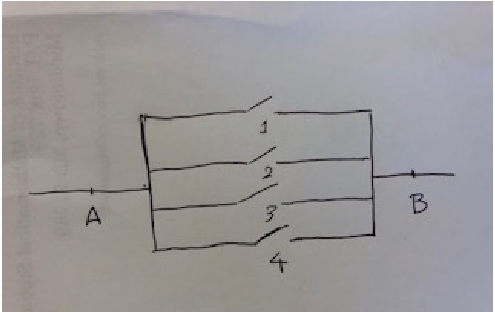
Example 7

A system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if components i, independent of all other components, functions with probability p_i, then the probability that the system functions is

$$1 - \prod_{i=1}^{n}(1 - p_i).$$



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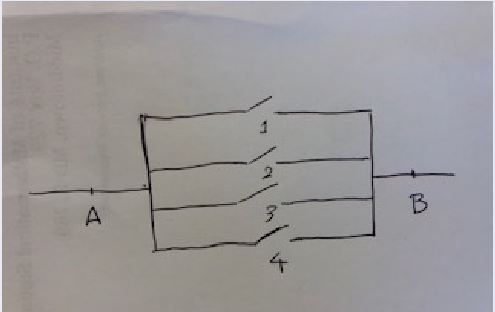
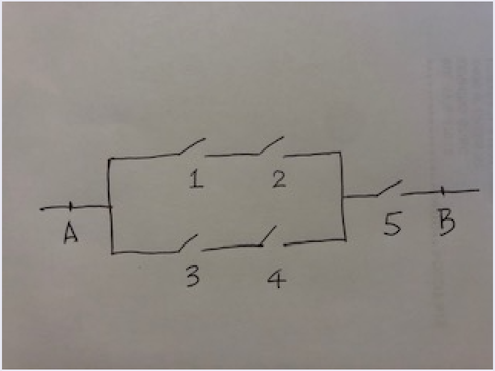


Example 8

The probability that the i-th switch is on in the circuit below is p_i, $i = 1, 2, 3, 4, 5$. If the switches are independent, find the probability that electricity can flow between A and B.



For $i = 1, 2, 3, 4, 5$, let E_i be the event that the i-the switch is on. Then we are are looking for

$$
\begin{aligned}
&P(((E_1 \cap E_2) \cup (E_3 \cap E_4)) \cap E_5) = P((E_1 \cap E_2) \cup (E_3 \cap E_4))P(E_5) \\
&= [P(E_1 \cap E_2) + P(E_3 \cap E_4) - P(E_1 \cap E_2 \cap E_3 \cap E_4)]P(E_5) \\
&= (p_1 p_2 + p_3 p_4 - p_1 p_2 p_3 p_4) p_5.
\end{aligned}
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