Math 461 Spring 2024

Renming Song

University of Illinois Urbana-Champaign

January 29, 2024

Outline

Outline

Solutions to HW1 are available on my hmepage.

HW2 is due this Friday at the end of the class.

Solutions to HW1 are available on my hmepage.

HW2 is due this Friday at the end of the class.

Outline

Definition

If $P(F) > 0$, we define

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

If $P(F) = 0$, $P(E|F)$ is undefined.

Using the definition of conditional probability, one can easily check

$$P(E \cap F) = P(F)P(E|F).$$

More generally, we have

$$P(\cap_{i=1}^{n} E_i) = P(E_1)P(E_2|E_1) \cdots P(E_n| \cap_{i=1}^{n-1} E_i).$$

These formulas are very useful in finding the probability of intersections.

Definition

If $P(F) > 0$, we define

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

If $P(F) = 0$, $P(E|F)$ is undefined.

Using the definition of conditional probability, one can easily check

$$P(E \cap F) = P(F)P(E|F).$$

More generally, we have

$$P(\cap_{i=1}^{n} E_i) = P(E_1)P(E_2|E_1) \cdots P(E_n| \cap_{i=1}^{n-1} E_i).$$

These formulas are very useful in finding the probability of intersections.

Definition

If $P(F) > 0$, we define

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

If $P(F) = 0$, $P(E|F)$ is undefined.

Using the definition of conditional probability, one can easily check

$$P(E \cap F) = P(F)P(E|F).$$

More generally, we have

$$P(\cap_{i=1}^{n} E_i) = P(E_1)P(E_2|E_1) \cdots P(E_n| \cap_{i=1}^{n-1} E_i).$$

These formulas are very useful in finding the probability of intersections.

Definition

If $P(F) > 0$, we define

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

If $P(F) = 0$, $P(E|F)$ is undefined.

Using the definition of conditional probability, one can easily check

$$P(E \cap F) = P(F)P(E|F).$$

More generally, we have

$$P(\cap_{i=1}^{n} E_i) = P(E_1)P(E_2|E_1) \cdots P(E_n| \cap_{i=1}^{n-1} E_i).$$

These formulas are very useful in finding the probability of intersections.

Example 3

Suppose that a box contains 8 red balls and 4 white balls. We randomly draw two balls from the box without replacement. Find the probability that (a) both balls are red; (b) the second ball is red.

(a) Using the obvious notation,

$$P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) = \frac{8}{12}\frac{7}{11}.$$

$$P(R_2) = P(R_1 \cap R_2) + P(W_1 \cap R_2) = \frac{8}{12}\frac{7}{11} + \frac{4}{12}\frac{8}{11}.$$

Example 3

Suppose that a box contains 8 red balls and 4 white balls. We randomly draw two balls from the box without replacement. Find the probability that (a) both balls are red; (b) the second ball is red.

(a) Using the obvious notation,

$$P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) = \frac{8}{12}\frac{7}{11}.$$

$$P(R_2) = P(R_1 \cap R_2) + P(W_1 \cap R_2) = \frac{8}{12}\frac{7}{11} + \frac{4}{12}\frac{8}{11}.$$

Example 3

Suppose that a box contains 8 red balls and 4 white balls. We randomly draw two balls from the box without replacement. Find the probability that (a) both balls are red; (b) the second ball is red.

(a) Using the obvious notation,

$$P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) = \frac{8}{12}\frac{7}{11}.$$

$$P(R_2) = P(R_1 \cap R_2) + P(W_1 \cap R_2) = \frac{8}{12}\frac{7}{11} + \frac{4}{12}\frac{8}{11}.$$

Example 3

Suppose that in the previous example. 3 balls are randomly selected from the box without replacement. Find the probability that all 3 are red.

$$P(R_1 \cap R_2 \cap R_3) = P(R_1)P(R_2|R_1)P(R_3|R_1 \cap R_2)$$
$$= \frac{8}{12}\frac{7}{11}\frac{6}{10}.$$

Example 3

Suppose that in the previous example. 3 balls are randomly selected from the box without replacement. Find the probability that all 3 are red.

$$P(R_1 \cap R_2 \cap R_3) = P(R_1)P(R_2|R_1)P(R_3|R_1 \cap R_2)$$
$$= \frac{8}{12}\frac{7}{11}\frac{6}{10}.$$

Example 4

An ordinary deck of 52 cards is randomly divided into 4 distinct piles of 13 each. Find the probability that each pile has exactly 1 ace.

Solution. From an example in Section 2.5, we know that the answer is

$$\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}}$$

Example 4

An ordinary deck of 52 cards is randomly divided into 4 distinct piles of 13 each. Find the probability that each pile has exactly 1 ace.

Solution. From an example in Section 2.5, we know that the answer is

$$\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}}$$

Solution by conditional probability. For $i = 1, 2, 3, 4$, let E_i be the event that the i-th pile has exactly 1 ace. Then

$$P(E_1) = \frac{4\binom{48}{12}}{\binom{52}{13}}, \quad P(E_2|E_1) = \frac{3\binom{36}{12}}{\binom{39}{13}},$$

$$P(E_3|E_1 \cap E_2) = \frac{2\binom{24}{12}}{\binom{26}{13}}, \quad P(E_4|E_1 \cap E_2 \cap E_3) = \frac{\binom{12}{12}}{\binom{13}{13}} = 1.$$

So the answer is

$$\frac{4\binom{48}{12}}{\binom{52}{13}} \frac{3\binom{36}{12}}{\binom{39}{13}} \frac{2\binom{24}{12}}{\binom{26}{13}} \frac{\binom{12}{12}}{\binom{13}{13}}.$$

The 2 answers are the same. See the book for yet another solution via conditional probability.

Solution by conditional probability. For $i = 1, 2, 3, 4$, let E_i be the event that the i-th pile has exactly 1 ace. Then

$$P(E_1) = \frac{4\binom{48}{12}}{\binom{52}{13}}, \quad P(E_2|E_1) = \frac{3\binom{36}{12}}{\binom{39}{13}},$$

$$P(E_3|E_1 \cap E_2) = \frac{2\binom{24}{12}}{\binom{26}{13}}, \quad P(E_4|E_1 \cap E_2 \cap E_3) = \frac{\binom{12}{12}}{\binom{13}{13}} = 1.$$

So the answer is

$$\frac{4\binom{48}{12}}{\binom{52}{13}} \frac{3\binom{36}{12}}{\binom{39}{13}} \frac{2\binom{24}{12}}{\binom{26}{13}} \frac{\binom{12}{12}}{\binom{13}{13}}.$$

The 2 answers are the same. See the book for yet another solution via conditional probability.

Outline

Example 1

A certain blood test is 95% effective in detecting a certain disease when it is in fact present. However, the test also yields a "false positive" result for 1% of the healthy people tested. If 0.5% of the population has the disease, what is the probability that a person has the disease given that the person's test result is positive?

Solution. Let E be the event that the person has the disease, and F the event that the person's test result is positive. We are looking for $P(E|F)$, which is equal to

$$\frac{P(E \cap F)}{P(F)}.$$

Example 1

A certain blood test is 95% effective in detecting a certain disease when it is in fact present. However, the test also yields a "false positive" result for 1% of the healthy people tested. If 0.5% of the population has the disease, what is the probability that a person has the disease given that the person's test result is positive?

Solution. Let E be the event that the person has the disease, and F the event that the person's test result is positive. We are looking for $P(E|F)$, which is equal to

$$\frac{P(E \cap F)}{P(F)}.$$

We are given

$$P(E) = 0.005, \quad P(E^c) = 0.995$$

and

$$P(F|E) = .95 \quad P(F|E^c) = .01.$$

Thus

$$P(E \cap F) = P(E)P(F|E) = (0.005) \cdot (0.95)$$

and

$$\begin{aligned} P(F) = P(E \cap F) + P(E^c \cap F) &= P(E)P(F|E) + P(E^c)P(F|E^c) \\ &= (0.005) \cdot (0.95) + (0.995) \cdot (0.01). \end{aligned}$$

The answer is

$$\frac{(0.005) \cdot (0.95)}{(0.005) \cdot (0.95) + (0.995) \cdot (0.01)} \approx 0.323.$$

The example above is a special case of the following general situation. Suppose $A_1, A_2, \ldots, A_n$ are n disjoint events with their union being the whole sample space and with $P(A_i) > 0$ for each $i = 1, \ldots, n$. Let B be an event with $P(B) > 0$. Suppose that $P(A_i), P(B|A_i), i = 1, \ldots, n$ are given. Find $P(A_i|B)$.

$$B = B \cap \left(\cup_{j=1}^{n} A_j\right) = \cup_{j=1}^{n} (B \cap A_j).$$

So

$$P(B) = \sum_{j=1}^{n} P(A_j) P(B|A_j).$$

Thus

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}.$$

The example above is a special case of the following general situation. Suppose $A_1, A_2, \ldots, A_n$ are n disjoint events with their union being the whole sample space and with $P(A_i) > 0$ for each $i = 1, \ldots, n$. Let B be an event with $P(B) > 0$. Suppose that $P(A_i), P(B|A_i), i = 1, \ldots, n$ are given. Find $P(A_i|B)$.

$$B = B \cap \left(\cup_{j=1}^{n} A_j\right) = \cup_{j=1}^{n} (B \cap A_j).$$

So

$$P(B) = \sum_{j=1}^{n} P(A_j)P(B|A_j).$$

Thus

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}.$$

The example above is a special case of the following general situation. Suppose $A_1, A_2, \ldots, A_n$ are n disjoint events with their union being the whole sample space and with $P(A_i) > 0$ for each $i = 1, \ldots, n$. Let B be an event with $P(B) > 0$. Suppose that $P(A_i), P(B|A_i), i = 1, \ldots, n$ are given. Find $P(A_i|B)$.

$$B = B \cap \left(\cup_{j=1}^{n} A_j\right) = \cup_{j=1}^{n}(B \cap A_j).$$

So

$$P(B) = \sum_{j=1}^{n} P(A_j)P(B|A_j).$$

Thus

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}.$$

The formula above is known as the Bayes' formula. You do not need to memorize this formula. It is much easier to remember the short derivation of it.

Example 2

In answering a certain multiple choice question with 5 possible answers, a student either knows the answer or guesses. Assume that a student knows the answer with probability 0.8. Assume that, when not knowing the answer, the student guesses the 5 answers with equal probability. Find the probability that the student knows the answer given that the student answered it correctly.

The formula above is known as the Bayes' formula. You do not need to memorize this formula. It is much easier to remember the short derivation of it.

Example 2

In answering a certain multiple choice question with 5 possible answers, a student either knows the answer or guesses. Assume that a student knows the answer with probability 0.8. Assume that, when not knowing the answer, the student guesses the 5 answers with equal probability. Find the probability that the student knows the answer given that the student answered it correctly.

Solution. Let K be the event that the student knows the answer, and C the event that the student answered it correctly. Then

$$P(K) = 0.8, \quad P(K^c) = 0.2$$

and

$$P(C|K) = 1 \quad P(C|K^c) = 0.2.$$

So

$$P(K|C) = \frac{P(K \cap C)}{P(C)} = \frac{P(K)P(C|K)}{P(K)P(C|K) + P(K^c)P(C|K^c)}$$
$$= \frac{(0.8) \cdot 1}{(0.8) \cdot 1 + (0.2) \cdot (0.2)}.$$

Example 3

Suppose that there are 3 chests of drawers and each chest has 2 drawers. The first chest has a gold coin in each drawer; the second chest has a gold in one drawer and a silver coin in the other; the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer is randomly opened. If the drawer has a gold coin, what is the probability that the other drawer also has a glod coin?

Solution. For $i = 1, 2, 3$, let E_i be the event that the i-th chest is chosen, and let G be the event that the drawer opened has a gold coin. We are looking for $P(E_1|G)$.

Example 3

Suppose that there are 3 chests of drawers and each chest has 2 drawers. The first chest has a gold coin in each drawer; the second chest has a gold in one drawer and a silver coin in the other; the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer is randomly opened. If the drawer has a gold coin, what is the probability that the other drawer also has a glod coin?

Solution. For $i = 1, 2, 3$, let E_i be the event that the i-th chest is chosen, and let G be the event that the drawer opened has a gold coin. We are looking for $P(E_1|G)$.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3};$$

and

$$P(G|E_1) = 1, \quad P(G|E_2) = \frac{1}{2}, \quad P(G|E_3) = 0.$$

So

$$\begin{aligned} P(E_1|G) &= \frac{P(E_1 \cap G)}{P(E_1 \cap G) + P(E_2 \cap G) + P(E_3 \cap G)} \\ &= \frac{P(E_1)P(G|E_1)}{P(E_1)P(G|E_1) + P(E_2)P(G|E_2) + P(E_3)P(G|E_3)} \\ &= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}\frac{1}{2}} = \frac{2}{3}. \end{aligned}$$

A plane is missing, and it is presumed that it is equally likely to have gone down in any of 3 possible regions. Let $1-\beta_i$ be the probability that the plane will be found upon a search of the region when the plane is, in fact, in that region, $i =, 2, 3$. Find the probability that the plane is in the i-th region given that a search of region 1 did not locate the plane.

Solution. For $i = 1, 2, 3$, let E_i be the event that the plane is the i-th region. Let F be the event that a search of region 1 did not locate the plane. We are looking for $P(E_1|F)$, $P(E_2|F)$ and $P(E_3|F)$.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3},$$

and

$$P(F|E_1) = \beta_1, \quad P(F|E_2) = P(F|E_3) = 1.$$

A plane is missing, and it is presumed that it is equally likely to have gone down in any of 3 possible regions. Let $1-\beta_i$ be the probability that the plane will be found upon a search of the region when the plane is, in fact, in that region, $i =, 2, 3$. Find the probability that the plane is in the i-th region given that a search of region 1 did not locate the plane.

Solution. For $i = 1, 2, 3$, let E_i be the event that the plane is the i-th region. Let F be the event that a search of region 1 did not locate the plane. We are looking for $P(E_1|F)$, $P(E_2|F)$ and $P(E_3|F)$.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3},$$

and

$$P(F|E_1) = \beta_1, \quad P(F|E_2) = P(F|E_3) = 1.$$

A plane is missing, and it is presumed that it is equally likely to have gone down in any of 3 possible regions. Let $1-\beta_i$ be the probability that the plane will be found upon a search of the region when the plane is, in fact, in that region, $i =, 2, 3$. Find the probability that the plane is in the i-th region given that a search of region 1 did not locate the plane.

Solution. For $i = 1, 2, 3$, let E_i be the event that the plane is the i-th region. Let F be the event that a search of region 1 did not locate the plane. We are looking for $P(E_1|F)$, $P(E_2|F)$ and $P(E_3|F)$.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3},$$

and

$$P(F|E_1) = \beta_1, \quad P(F|E_2) = P(F|E_3) = 1.$$

So

$$P(E_1|F) = \frac{P(E_1 \cap F)}{P(E_1 \cap F) + P(E_2 \cap F) + P(E_3 \cap F)}$$
$$= \frac{P(E_1)P(F|E_1)}{P(E_1)P(F|E_1) + P(E_2)P(F|E_2) + P(E_3)P(F|E_3)}$$
$$= \frac{\beta_1}{\beta_1 + 2}.$$

Similarly,

$$P(E_2|F) = P(E_3|F) = \frac{1}{\beta_1 + 2}.$$

So

$$P(E_1|F) = \frac{P(E_1 \cap F)}{P(E_1 \cap F) + P(E_2 \cap F) + P(E_3 \cap F)}$$
$$= \frac{P(E_1)P(F|E_1)}{P(E_1)P(F|E_1) + P(E_2)P(F|E_2) + P(E_3)P(F|E_3)}$$
$$= \frac{\beta_1}{\beta_1 + 2}.$$

Similarly,

$$P(E_2|F) = P(E_3|F) = \frac{1}{\beta_1 + 2}.$$