

# Math 461 Spring 2024

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University of Illinois Urbana-Champaign

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# Outline

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- 1 General Info
- 2 2.5 Sample Spaces Having Equally likely Outcomes
- 3 3.2 Conditional Probabilities

HW1 is due today before the end of class.

Please turn in your HW1 as ONE pdf file via the HW1 folder in the course Moodle page. Make the quality of you pdf file is good enough.

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### Example 6

$n$  people are present in a room, what is the probability that no 2 of them celebrate their birthdays on the same day of the year? (Assume that there are 365 days in each year, and that for any person the birthday is equally likely to be any one of these 365 days.)

Solution.

$$p = \frac{365 \cdot 364 \cdot \dots \cdot (365 - n + 1)}{365^n}$$

When  $n \geq 23$ ,  $p \leq \frac{1}{2}$ . When  $n = 50$ ,  $p \approx 0.03$ .

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### Example 7

Suppose that  $n$  distinct balls are randomly distributed into  $r$  distinct boxes (so that a ball is equally likely to be distributed into any one of the  $r$  boxes). Find the probability that there are exactly  $k$  balls in the first  $r_1$  boxes.

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$$\frac{\binom{n}{k} r_1^k (r - r_1)^{n-k}}{r^n}.$$

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### Example 8

Suppose that we have a box containing  $r$  balls labeled  $1, 2, \dots, r$ . A random sample of size  $n$  is drawn from the box without replacement and the numbers on the balls are noted. These balls are returned to the box and a second sample of size  $n$  is randomly drawn without replacement. Find the probability that these 2 samples have exactly  $k$  balls in common.

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$$\frac{\binom{n}{k} \binom{r-n}{n-k}}{\binom{r}{n}}.$$

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### Example 9

10 married couples are randomly seated at a round table. Find the probability that at least 1 wife sits next to her husband.

For  $i = 1, \dots, 10$ , let  $E_i$  be the event that the  $i$ -th couple sits together. Then  $\cup_{i=1}^{10} E_i$  is the event that at least 1 wife sits next to her husband.

We know that, for any  $i = 1, \dots, 10$ ,

$$P(E_i) = \frac{18!2}{19!}.$$

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For  $1 \leq i_1 < i_2 \leq 10$ ,

$$P(E_{i_1} \cap E_{i_2}) = \frac{17!2^2}{19!}.$$

More generally, for  $1 \leq n \leq 10$  and  $1 \leq i_1 < i_2 < \cdots < i_n \leq 10$ ,

$$P(\cap_{j=1}^n E_{i_j}) = \frac{(19-n)!2^n}{19!}.$$

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By the inclusion-exclusion formula

$$\begin{aligned}
 P(\cup_{i=1}^{10} E_i) &= \sum_{i=1}^{10} P(E_i) + \cdots + (-1)^{n+1} \sum_{i_1 < \cdots < i_n} P(\cap_{j=1}^n E_{i_j}) \\
 &\quad + \cdots - P(\cap_{i=1}^{10} E_i) \\
 &= \binom{10}{1} \frac{18!2}{19!} - \binom{10}{2} \frac{17!2^2}{19!} + \binom{10}{3} \frac{16!2^3}{19!} \\
 &\quad - \binom{10}{4} \frac{15!2^4}{19!} + \cdots - \binom{10}{10} \frac{9!2^{10}}{19!}
 \end{aligned}$$

### Example 10

Suppose that each of  $N$  men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. Find the probability that none of the men gets his own hat.

For  $i = 1, \dots, N$ , let  $E_i$  be the event that the  $i$ -th man gets his own hat. Then  $\cup_{i=1}^N E_i$  is the event that at least one of the men gets his own hat, the complement of the event we are interested in.

For each  $i = 1, \dots, N$ ,

$$P(E_i) = \frac{(N-1)!}{N!} = \frac{1}{N}.$$

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For  $1 \leq i_1 < i_2 \leq N$ ,

$$P(E_{i_1} \cap E_{i_2}) = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)}.$$

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 &\quad + \cdots + (-1)^{N+1} P(\cap_{i=1}^N E_i) \\
 &= \binom{N}{1} \frac{1}{N} - \binom{N}{2} \frac{(N-2)!}{N!} + \binom{N}{3} \frac{(N-3)!}{N!} \\
 &\quad - \binom{N}{4} \frac{(N-4)!}{N!} + \cdots + (-1)^{N+1} \binom{N}{N} \frac{1}{N!} \\
 &= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{N+1} \frac{1}{N!}.
 \end{aligned}$$

So the answer is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^N \frac{1}{N!}$$

When  $N$  is large, the answer is approximately equal to  $e^{-1}$ .

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Suppose that a fair die is tossed twice. Suppose further that we are told that the first toss results in a 3. Given this info, what is the probability that the sum is 6?

We can argue as follows: given the first toss is a 3, there are 6 possible outcomes: (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), and each of these outcomes are equally likely. So the desired result is  $\frac{1}{6}$ .

If we use  $E$  to denote the event that the sum is 6, and  $F$  the event that the first toss results in a 3, then the probability just obtained is called the conditional probability of  $E$  given  $F$  and is denoted by  $P(E|F)$ .

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A formula for  $P(E|F)$  that is valid for all events  $E$  and  $F$  can be derived in the same manner.

If  $F$  occurs, then in order for  $E$  to occur, the outcome must be in  $E \cap F$ . Now as  $F$  has occurred,  $F$  becomes the new sample space. Hence the conditional probability of  $E$  given  $F$  is equal to the probability of  $E \cap F$  relative to the probability of  $F$ . Thus

### Definition

If  $P(F) > 0$ , we define

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

If  $P(F) = 0$ ,  $P(E|F)$  is undefined.



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### Example 1

A box contains 10 orange, 5 white and 10 blue balls. A ball is randomly chosen and it is noted that it is not blue. What is the probability that it is white?

Answer:  $\frac{1}{3}$ .

### Example 2

A fair coin is tossed twice. Given that there is at least one H, what is the probability that both are H?

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Using the definition of conditional probability, one can easily check

$$P(E \cap F) = P(F)P(E|F).$$

More generally, we have

$$P(\cap_{i=1}^n E_i) = P(E_1)P(E_2|E_1) \cdots P(E_n | \cap_{i=1}^{n-1} E_i).$$

These formulas are very useful in finding the probability of intersections.

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These formulas are very useful in finding the probability of intersections.

### Example 3

Suppose that a box contains 8 red balls and 4 white balls. We randomly draw two balls from the box without replacement. Find the probability that (a) both balls are red; (b) the second ball is red.

(a) Using the obvious notation,

$$P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) = \frac{8}{12} \frac{7}{11}.$$

$$P(R_2) = P(R_1 \cap R_2) + P(W_1 \cap R_2) = \frac{8}{12} \frac{7}{11} + \frac{4}{12} \frac{8}{11}.$$

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Suppose that in the previous example. 3 balls are randomly selected from the box without replacement. Find the probability that all 3 are red.

$$\begin{aligned}
 P(R_1 \cap R_2 \cap R_3) &= P(R_1)P(R_2|R_1)P(R_3|R_1 \cap R_2) \\
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