Math 461 Spring 2024

Renming Song

University of Illinois Urbana-Champaign

January 24, 2022

Outline

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HW1 is due this Friday, 01/26, before the end of class.

Please submit HW1 in ONE pdf file via the HW1 folder in the course Moodle page. Make sure that the quality of your file is good enough.

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In a lot of cases, it is natural to assume that all the outcomes in the sample space are equally likely. In this case, if the sample space S contains N elements, say, $1, 2, \ldots, N$, then

$$P(\{1\}) = P(\{2\}) = \cdots P(\{N\}) = \frac{1}{N}.$$

Then for any event E,

$$P(E) = \frac{\#\text{ of points in } E}{N} = \frac{\#\text{ of points in } E}{\#\text{ of points in } S}.$$

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Example 1

A fair die is rolled twice. Find the probability that the sum is even.

Solution. Let E be the event that the sum is even, E_2 be the event that sum is 2, E_4 be the event that sum is 4, . . . , E_{12} be the event that sum is 12. Then $E = E_2 \cup E_4 \cup \cdots \cup E_{12}$.

$E_2 = \{(1,1)\}$ and so $P(E_2) = \frac{1}{36}$. $E_4 = \{(1,3),(2,2),(3,1)\}$ and so $P(E_4) = \frac{3}{36}$, $E_6 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ and so $P(E_6) = \frac{5}{36}$, ... Adding things up, we get $P(E) = \frac{1}{2}$.

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Example 2

A committee of 7 is randomly selected from a group of 10 men and 10 women. Find the probability that the committee consists of 4 men and 3 women. (Randomly means that all the possible outcomes are equally likely.)

Solution.

$$\frac{\binom{10}{4}\binom{10}{3}}{\binom{20}{7}}.$$

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Example 3

A closet contains 10 distinct pairs of shoes. If 8 shoes are randomly selected, find the probability that there will be (a) no complete pair; (b) exactly 1 complete pair.

Solution. We are not told if the order is relevant or not. It should not make a difference. (a) If we think that order is irrelevant, then the answer is

$$\frac{\binom{10}{8}2^8}{\binom{20}{8}}.$$

If we think that order is relevant then the answer is

$$\frac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}.$$

The 2 answers are the same. **Be consistent!**

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(b) The answer is

$$\frac{\binom{10}{1}\binom{9}{6}2^6}{\binom{20}{8}} = \frac{\binom{10}{6}\binom{4}{1}2^6}{\binom{20}{8}}.$$

Example 4

In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. Assume that the cards are dealt randomly. Find the probability that (a) one of the players gets all 13 spades; (b) each player gets 1 king.

(a) The answer is

$$\frac{4\binom{39}{13,13,13}}{\binom{52}{13,13,13,13}}.$$

(b) The answer is

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$$\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}}.$$

Example 5

A hand of 5 cards is randomly selected from an ordinary deck of 52 cards. Find the probability that (a) the hand is straight (5 cards with distinct consecutive values, and are not all of the same suit. Ace is both high and low.); (b) a flush (all 5 cards of the same suit); (c) the hand contains exactly one pair; (d) the hand contains exactly 2 pairs; (e) the hand contains a 3-of-a-kind and no pairs; (f) the hand contains a 4-of-kind.

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(a)

$$\frac{10(4^5 - 4)}{\binom{52}{5}}.$$

(b)

$$\frac{4\binom{13}{5}}{\binom{52}{5}}.$$

(c)

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}} = \frac{\binom{13}{3}4^3\binom{10}{1}\binom{4}{2}}{\binom{52}{5}}$$

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$$\frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{\binom{13}{1}\binom{4}{1}\binom{12}{2}\binom{4}{2}^2}{\binom{52}{5}}.$$

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