Math 461 Spring 2024

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University of Illinois Urbana-Champaign

January 22, 2024

Outline

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I have setup a HW1 folder in the Moodle page. Please submit your HW1 in ONE pdf file via that folder. Make sure the quality of your file is good enough. The deadline for submitting HW1 is next Friday, 01/26, before the end of our lecture.

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There are lots of phenomena in nature whose outcome cannot be predicted with certainty in advance, but the set of all the possible outcomes is known. For instance, when you toss a coin, you do not know whether “Heads” or “Tails” will appear, but you do know the outcome will be either ‘Heads” or “Tails”. These are what we call random phenomena or random experiments. Probability theory is concerned with such random experiments.

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Tossing a coin. $S = \{H, T\}$. H stands for "Heads", and T stands for "Tails".

Tossing a (6-sided) die. $S = \{1, 2, 3, 4, 5, 6\}$.

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Keeping on tossing a coin until an H appears.
$S = \{H, TH, TTH, TTTH, \dots\}$.

Measuring the lifetime of a light-bulb. $S = [0, \infty)$.

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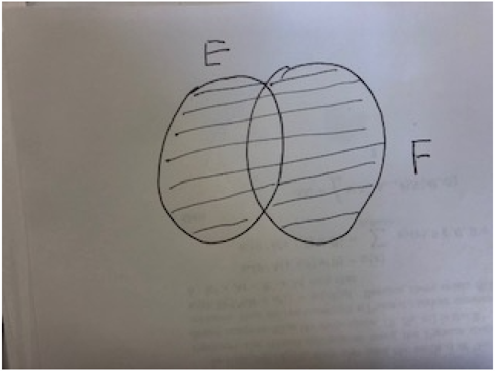
Events are simply subsets of the sample space, so we can talk about various set theoretical operations of events.

Union: $E \cup F$ occurs if and only if E or F occurs.



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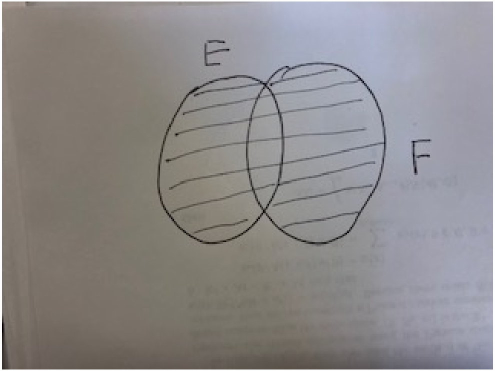

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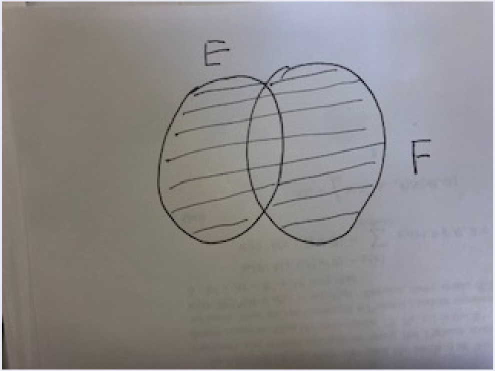

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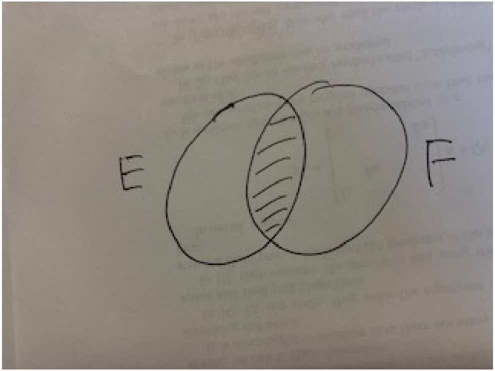

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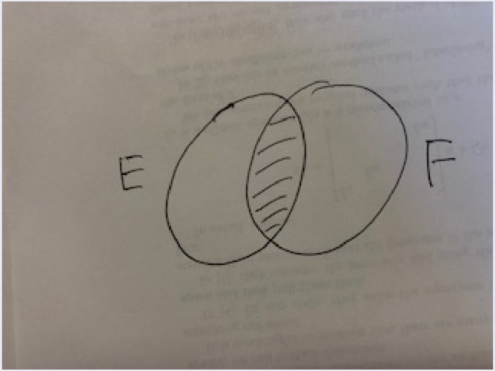
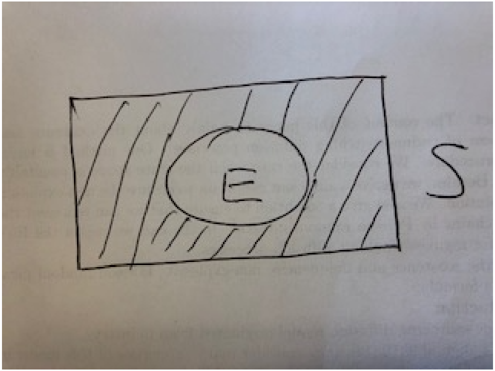


The complement of E, denoted as E^c, consists of all the elements of S which are not in E.



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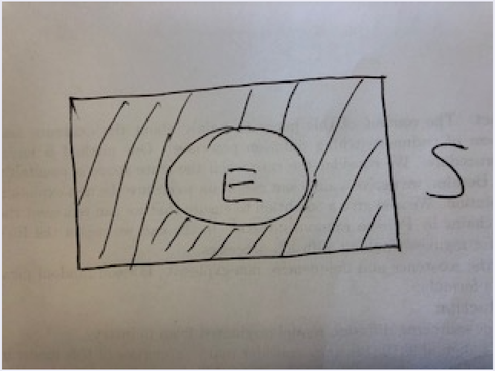

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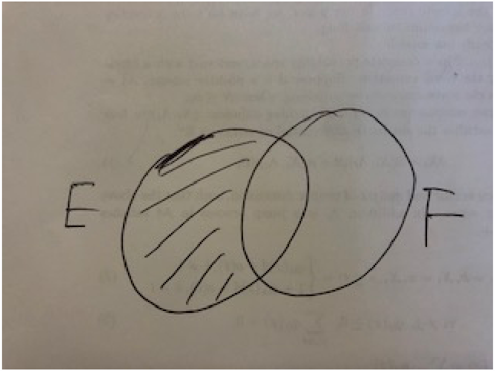

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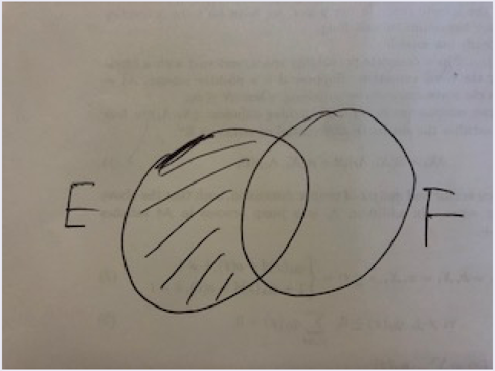

$E \subset F$ if every element of E is an element of F.

If $E \cap F = \emptyset$, then we say that E and F are disjoint, or mutually exclusive.

Similarly, we can define the union and intersection of more than 2 events

$$\cup_{i=1}^{n} E_i, \quad \cup_{i=1}^{\infty} E_i$$

and

$$\cap_{i=1}^{n} E_i, \quad \cap_{i=1}^{\infty} E_i.$$

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Properties of set theoretical operations

Commutativity: $E \cup F = F \cup E$ and $E \cap F = F \cap E$;
Associativity: $(E \cup F) \cup G = E \cup (F \cup G)$ and $(E \cap F) \cap G = E \cap (F \cap G)$
Distributivity: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$ and $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$.

De Morgan's law

$$(\cup_{i=1}^{n} E_i)^c = \cap_{i=1}^{n} E_i^c \qquad (\cup_{i=1}^{\infty} E_i)^c = \cap_{i=1}^{\infty} E_i^c$$

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Consider a random experiment whose sample space is S. A real-valued function P on the space of all events of the experiment is called a probability (measure) if

(1) for all event E, $0 \leq P(E) \leq 1$;

(2) $P(S) = 1$;

(3) for any sequence $E_1, E_2, \ldots$ of mutually disjoint events,

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i).$$

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Tossing a coin for which Heads is twice likely as Tails. $P(H) = \frac{2}{3}$, $P(T) = \frac{1}{3}$.

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Tossing a fair die twice. $P((i,j)) = \frac{1}{36}$, $i,j = 1,\ldots,6$.

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Measuring the lifetime of a light-bulb. $P(A) = \int_A e^{-t}dt$ for any subset A of $\mathbb{R}_+$.

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Outline

Properties of Probability Measures

Suppose that P is a probability measure. Then

(1) $P(\emptyset) = 0$;

(2) if $E_1, \ldots, E_n$ are disjoint, then

$$P(\cup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i);$$

(3) if $E \subset F$, the $P(E) \leq P(F)$;

(4) $P(E^c) = 1 - P(E)$;

(5) $P(\cup_{i=1}^n E_i) = 1 - P(\cap_{i=1}^n E_i^c)$;

(6) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Proof

(1) Take $E_1 = E_2 = \cdots = \emptyset$, then

$$P(\emptyset) = P(\cup_{i=1}^{\infty} E_i) = P(\emptyset) + P(\emptyset) + \cdots ,$$

so $P(\emptyset) = 0$.
(2) Take $E_{n+1} = E_{n+2} = \cdots = \emptyset$, then $E_1, E_2, \cdots$ is a sequence of disjoint events, thus by countable additivity,

$$P(\cup_{i=1}^{n} E_i) = P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{n} P(E_i).$$

(3) $P(F) = P(E) + P(F \setminus E) \geq P(E)$.
(4) $1 = P(E \cup E^c) = P(E) + P(E^c)$.

Proof (cont)

(5) Follows immediately from (4),
(6) Let $I = E \setminus F$, $II = F \setminus E$ and $III = E \cap F$. Then
$P(E \cup F) = P(I \cup II \cup III) = P(I) + P(II) + P(III)$ and
$P(E) = P(I) + P(III)$, $P(F) = P(II) + P(III)$ and $P(E \cap F) = P(III)$.
Thus $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Example

A fair die is tossed 100 times. Find the probability that there is at least one 5.

The complement of "at least one 5" is "there is no 5". So the answer is

$$1 - \left(\frac{5}{6}\right)^{100}.$$

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A generalization of (6) to the case of the union of n events is the following inclusion-exclusion formula, which can be proved by induction.

Inclusion-exclusion formula

If $E_1, E_2, \ldots, E_n$ are events, then

$$P(\cup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3})$$
$$+ \cdots + (-1)^{k+1} \sum_{i_1<\cdots<i_k} P(\cap_{j=1}^k E_{i_j})$$
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